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DERIVATIVE-FREE CHARACTERIZATIONS OF Q_K SPACES II

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ABSTRACT. The Q_K spaces on the open unit disk are characterized by some oscillations in the Bergman metric without the use of derivatives. Our results are new even in the case of Q_p spaces.

1. INTRODUCTION

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk of complex plane $\mathbb{C}$. A particular class of Möbius invariant function spaces, the so-called Q_p spaces, has attracted a lot of attention in recent years. Denote by $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$. For $a \in \mathbb{D}$, $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$ is the Green function in $\mathbb{D}$, where $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ is the Möbius map of $\mathbb{D}$. For $0 \le p < \infty$, the space Q_p consists of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{Q_p}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g^p(z,a)\, dA(z) < \infty, \tag{1}$$

where dA is an area measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$. We denote $d\tau(z) = \frac{dA}{(1-|z|^2)^2}$ the Möbius invariant measure. We know that the Green function $g(a,z)$ in (1) can be replaced by the expression $1 - |\varphi_a(z)|^2$ (cf. [1]). It is well known that $Q_1 = BMOA$ and Q_0 is the classical Dirichlet space $\mathcal{D}$ with

$$\|f\|_{\mathcal{D}}^2 = \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$

For $1 < p < \infty$, $Q_p = \mathcal{B}$. Here $\mathcal{B}$ is the Bloch space defined as follows.

$$\mathcal{B} = \{f : f \in H(\mathbb{D}), \|f\|_{\mathcal{B}} = \sup_{a\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty\}.$$

For more about Q_p spaces see [1], [2] and [7].

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For any nonnegative, nondecreasing and Lebesgue measurable function K on $(0,1]$, we say that f belongs to the space Q_K if

$$\|f\|_{Q_K}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(1-|\varphi_a(z)|^2)\, dA(z) < \infty. \tag{2}$$

The space $Q_{K,0}$ consists of analytic functions f on $\mathbb{D}$ for which

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^2 K(1-|\varphi_a(z)|^2) dA(z) = 0.$$

Modulo constants, Q_K is a Banach space under the norm $|f(0)| + \|f\|_{Q_K}$ and Q_K is Möbius invariant; that is, $\|f\circ\varphi_a\|_{Q_K} = \|f\|_{Q_K}$ whenever $f \in Q_K$ and $a \in \mathbb{D}$. It is easy to see that $Q_{K,0}$ is a closed subspace in Q_K. For $0 < p < \infty$, $K(t) = t^p$ gives the space Q_p. $K(t) = 1$ gives the Dirichlet space $\mathcal{D}$. More results on Q_K spaces can be found in [3], [4] and [5].

Recently, the second author and Zhu [6] investigated some free - derivative characterizations of the spaces Q_k and $Q_{K,0}$. In this paper, we continue to give some free - derivative characterizations of the spaces Q_k and $Q_{K,0}$.

2. Preliminaries

If the function K is only defined on $(0,1]$, then we extend it to $(0,\infty)$ by setting $K(t) = K(1)$ for $t > 1$. We define an auxiliary function (see [4] and [6]) as follows:

$$\varphi_K(s) = \sup_{0<t\le 1} \frac{K(st)}{K(t)}, 0 < s < \infty. \tag{3}$$

We assume that K is continuous and nondecreasing on $(0,1]$. This ensures that the above function is continuous and nondecreasing on $(0,\infty)$.

The following Lemma is very useful in the proof of the main theorem in [6].

Lemma 2.1. *Let K be any nonnegative and Lebesgue measurable function on $(0,\infty)$ and $f(z) = K(1-|z|^2)$. If*

$$\int_0^\infty \frac{\varphi_K(x)}{(1+x)^3} dx < \infty, \tag{4}$$

then there exists a positive constant C such that $Bf(z) \le Cf(z)$ for all $z \in \mathbb{D}$, where Bf is a Berezin transform of f.

Here and elsewhere constants are denoted by C which are positive and may be different from one occurrence to the next.

Let $\beta(z,w)$ denote the Bergman metric between two points z and w in $\mathbb{D}$. It is well known that

$$\beta(z,w) = \frac{1}{2} \log \frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}.$$

For $z \in \mathbb{D}$ and $R > 0$ we use

$$\mathbb{D}(z, R) = \{w \in \mathbb{D} : \beta(z, w) < R\}$$

to denote the Bergman metric ball at z with radius R. If R is fixed, then it can be checked that the area of $\mathbb{D}(z, R)$, denoted by $|\mathbb{D}(z, R)|$, is comparable to $(1 - |z|^2)^2$ as z approaches the unit circle (see [8]).

Fix a positive r and denote by

$$\widehat{f}_r(z) = \frac{1}{|\mathbb{D}(z, r)|} \int_{\mathbb{D}(z,r)} f(w) dA(w)$$

the average of f over the Bergman metric ball $\mathbb{D}(z, r)$. We define the mean oscillation of f at z in the Bergman metric to be (see [8])

$$MO_r(f)(z) = \left\{ \frac{1}{|\mathbb{D}(z, r)|} \int_{\mathbb{D}(z,r)} |f(w) - \hat{f}_r(z)|^2 dA(w) \right\}^{1/2}.$$

It is easy to verify that for any $z \in \mathbb{D}$,

$$\begin{aligned}[MO_r(f)(z)]^2 &= \widehat{|f|^2}_r(z) - |\widehat{f}_r(z)|^2 \\ &= \frac{1}{2|\mathbb{D}(z, r)|^2} \int_{\mathbb{D}(z,r)} \int_{\mathbb{D}(z,r)} |f(u) - f(v)|^2 dA(u) dA(v).\end{aligned}$$

Given a function $f \in L^2_a(\mathbb{D}, dA)$, define

$$MO(f)(z) = \left[\int_{\mathbb{D}} |f \circ \varphi_z(w) - f(z)|^2 dA(w) \right]^{1/2}.$$

We call $MO(f)(z)$ as the invariant mean oscillation of f in the Bergman metric at the point z, since we have

$$MO(f \circ \varphi)(z) = MO(f)(\varphi(z)),$$

where $\varphi \in$ Aut $(\mathbb{D})$, the group of Möbius maps of the unit disk $\mathbb{D}$.

The main results in [6] can be stated as follows:

Theorem 2.2. *Let K satisfy condition* (4) *and $r > 0$. Then the following statements are equivalent:*

(1) $f \in Q_K$;
(2) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} [MO(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) < \infty$;
(3) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} [MO_r(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) < \infty$;
(4) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z)-f(w)|^2}{|1-z\overline{w}|^4} K(1 - |\varphi_a(z)|^2) dA(z) dA(w) < \infty$.

Theorem 2.3. *Let K satisfy condition* (4) *and $r > 0$. Then the following statements are equivalent:*

(1) $f \in Q_{K,0}$;
(2) $\lim_{|a|\to 1} \int_{\mathbb{D}} [MO(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) = 0$;

(3) $\lim_{|a|\to 1}\int_{\mathbb{D}}[MO_r(f)(z)]^2 K(1-|\varphi_a(z)|^2)d\tau(z)=0;$

(4) $\lim_{|a|\to 1}\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{|f(z)-f(w)|^2}{|1-z\overline{w}|^4}K(1-|\varphi_a(z)|^2)dA(z)dA(w)=0.$

For a sub-arc I of $\partial\mathbb{D}$, $|I|$ is the length of I and

$$S(I)=\{r\zeta:\zeta\in I, 1-|I|<r<1\}$$

is the corresponding Carleson square.

A positive Borel measure μ on $\mathbb{D}$ is called $K-$Carleson measure if

$$\sup_I \int_{S(I)} K\Big(\frac{1-|z|}{|I|}\Big)d\mu(z)<\infty,$$

where the supremum is taken over all sub-arcs $I\subset\partial\mathbb{D}$.

A positive Borel measure μ on $\mathbb{D}$ is called a vanishing K-Carleson measure if

$$\lim_{|I|\to 0}\int_{S(I)} K\Big(\frac{1-|z|}{|I|}\Big)d\mu(z)=0.$$

See [4] for more results on the K-Carleson measure.

Lemma 2.4. *Suppose K satisfies the following two conditions:*

(a) *There exists a constant $C>0$ such that $K(2t)\le CK(t)$ for all $t>0$.*

(b) *The auxiliary function φ_K has the property that*

$$\int_0^1 \varphi_K(s)\frac{ds}{s}<\infty.$$

By [4] we know that a positive Borel measure μ on $\mathbb{D}$ is a K-Carleson measure if and only if

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} K(1-|\varphi_a(z)|^2)d\mu(z)<\infty.$$

Using Lemma 2.4, the second author and Zhu gave some characterizations of Q_K and $Q_{K,0}$ spaces in terms of K-Carleson and vanishing K-Carleson measures, respectively (see [6]).

3. The spaces Q_K

For a function f on $\mathbb{D}$, the function $w_r(f)(z)$ on $\mathbb{D}$ defined by

$$w_r(f)(z)=\sup_{w\in\mathbb{D}(z,r)}|f(z)-f(w)|$$

is called the oscillation of f at z in the Bergman metric (see [8]). Similarly, define another oscillation of f at z in the Bergman metric as follows:

$$\widehat{w}_r(f)(z)=\sup_{w\in\mathbb{D}(z,r)}|\hat{f}_r(z)-f(w)|.$$

For $1 < p < \infty$ write

$$O_1(f)(z) = \left\{ \frac{1}{|\mathbb{D}(z,r)|} \int_{\mathbb{D}(z,r)} |f(w) - f(z)|^p dA(w) \right\}^{1/p}$$

and

$$O_2(f)(z) = \left\{ \frac{1}{|\mathbb{D}(z,r)|} \int_{\mathbb{D}(z,r)} |f(w) - \widehat{f}_r(z)|^p dA(w) \right\}^{1/p}.$$

The main result in this note is the following.

Theorem 3.1. *Let K satisfy condition* (4) *and $r > 0$. Then the following statements are equivalent for all functions $f \in H(\mathbb{D})$.*

(1) $f \in Q_K$;
(2) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} [w_r(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) < \infty$;
(3) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} [\widehat{w}_r(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) < \infty$;
(4) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} [O_1(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) < \infty$;
(5) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} [O_2(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) < \infty$.

Proof. The proof will follow by the routes (2) $\Rightarrow$ (3) $\Rightarrow$ (1) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Since $(1 - |z|^2) \sim (1 - |w|^2)$ for $w \in \mathbb{D}(z, r)$, we have

$$\begin{aligned}
\sup_{w\in\mathbb{D}(z,r)} |\widehat{f}_r(z) - f(w)| &\leq C \sup_{w\in\mathbb{D}(z,r)} \frac{1}{|\mathbb{D}(z,r)|} \int_{\mathbb{D}(z,r)} |f(w) - f(u)| dA(u) \\
&\leq C \sup_{w\in\mathbb{D}(z,r)} \sup_{u\in\mathbb{D}(z,r)} |f(w) - f(u)| \\
&\leq C \sup_{w\in\mathbb{D}(z,r)} \sup_{u\in\mathbb{D}(z,r)} \big(|f(w) - f(z)| + |f(z) - f(u)|\big) \\
&\leq C\big(\sup_{w\in\mathbb{D}(z,r)} |f(w) - f(z)| + \sup_{u\in\mathbb{D}(z,r)} |f(z) - f(u)|\big) \\
&= C \sup_{w\in\mathbb{D}(z,r)} |f(w) - f(z)|.
\end{aligned}$$

Hence, there exists a constant C such that

$$\widehat{w}_r(f)(z) \leq C w_r(f)(z).$$

It follows that for each $a \in \mathbb{D}$ the integral

$$\int_{\mathbb{D}} [\widehat{w}_r(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z)$$

is less than or equal to C times the integral

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} [w_r(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z).$$

This shows that the condition (2) implies (3).

(3) $\Rightarrow$ (1). A simple calculation shows that

$$|g'(0)|^2 \le C \int_{\mathbb{D}(0,r)} |g(u)|^2 dA(u)$$

holds for all $g \in H(\mathbb{D})$, where C is a positive constant depending on r only. Replacing $g = f\circ\varphi_z - \widehat{f}_r(z)$ and using the fact that $(1-|z|^2)$ is comparable to $|1-\overline{z}w|$ and $|\mathbb{D}(z,r)|^{1/2}$ when $w \in \mathbb{D}(z,r)$, we get

$$\begin{aligned}(1-|z|^2)^2|f'(z)|^2 &\le C\int_{\mathbb{D}(0,r)} |f\circ\varphi_z(u) - \widehat{f}_r(z)|^2 dA(u)\\ &\le C\int_{\mathbb{D}(z,r)} |f(u)-\widehat{f}_r(z)|^2\frac{(1-|z|^2)^2}{|1-\overline{z}u|^4} dA(u)\\ &\le \frac{C}{|\mathbb{D}(z,r)|}\int_{\mathbb{D}(z,r)} |f(u)-\widehat{f}_r(z)|^2 dA(u)\\ &\le C\sup_{w\in\mathbb{D}(z,r)} |f(w)-\widehat{f}_r(z)|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f'(z)|^2K(1-|\varphi_a(z)|^2)dA(z)\\ &\qquad\qquad \le C\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} [\widehat{w}_r(f)(z)]^2K(1-|\varphi_a(z)|^2)d\tau(z).\end{aligned}$$

This completes that (3) implies (1).

(1) $\Rightarrow$ (4). For $w \in \mathbb{D}(z,r)$, we have (see [8])

$$\frac{1}{|\mathbb{D}(z,r)|} \sim \frac{(1-|z|^2)^2}{|1-z\overline{w}|^4} \sim \frac{1}{(1-|z|^2)^2}.$$

By making a change of variables, there exists a constant $C>0$ such that

$$\begin{aligned}&\left(\frac{1}{|\mathbb{D}(z,r)|}\int_{\mathbb{D}(z,r)} |f(w)-f(z)|^p dA(w)\right)^{\frac{1}{p}}\\ &\le C\left(\int_{\mathbb{D}(z,r)} |f(w)-f(z)|^p\frac{(1-|z|^2)^2}{|1-z\overline{w}|^4} dA(w)\right)^{\frac{1}{p}}\\ &= C\left(\int_{\mathbb{D}(0,r)} |f\circ\varphi_z(w)-f\circ\varphi_z(0)|^p dA(w)\right)^{\frac{1}{p}}\\ &\le C\left(\int_{\mathbb{D}(0,r)} (1-|w|^2)^p|(f\circ\varphi_z)'(w)|^p dA(w)\right)^{\frac{1}{p}}\end{aligned}$$

$$= C\left(\int_{\mathbb{D}(z,r)} (1-|\varphi_z(w)|^2)^p |f'\circ\varphi_z(w)|^p dA(w)\right)^{\frac{1}{p}}$$
$$= C\left(\int_{\mathbb{D}(z,r)} (1-|w|^2)^p |f'(w)|^p \frac{(1-|z|^2)^2}{|1-z\overline{w}|^4} dA(w)\right)^{\frac{1}{p}}$$
$$\leq C \sup_{w\in\mathbb{D}(z,r)} (1-|w|^2)|f'(w)| \left(\int_{\mathbb{D}(z,r)} \frac{(1-|z|^2)^2}{|1-z\overline{w}|^4} dA(w)\right)^{\frac{1}{p}}$$
$$\leq C(1-|w|^2)|f'(w)|.$$

Hence, (1) implies (4).

(4) $\Rightarrow$ (2). Since $|g(v)|^p$ is subharmonic for all functions $g \in H(\mathbb{D})$ for $0 < p < +\infty$,

$$|g(w)|^p \leq \frac{C}{|\mathbb{D}(w,r)|}\int_{\mathbb{D}(w,r)} |g(v)|^p dA(v).$$

Replacing g by $f - f(z)$, we get

$$|f(w)-f(z)|^p \leq \frac{C}{|\mathbb{D}(w,r)|}\int_{\mathbb{D}(w,r)} |f(u)-f(z)|^p dA(u).$$

For $w \in \mathbb{D}(z,r)$, we have $\mathbb{D}(w,r) \subset \mathbb{D}(z,2r)$ and there is a constant $N > 0$ such that

$$\frac{C}{|\mathbb{D}(w,r)|} \leq \frac{N}{|\mathbb{D}(z,2r)|}.$$

Therefore,

$$\sup_{w\in\mathbb{D}(z,r)} |f(w)-f(z)| \leq \left(\frac{C}{|\mathbb{D}(z,2r)|}\int_{\mathbb{D}(z,2r)} |f(w)-f(z)|^p dA(w)\right)^{\frac{1}{p}} \leq CO_1(f).$$

This gives (2) by (4). Carefully checking the above proof, we can see that (5) is equivalent to the other 4 conditions. □

Similarly to Theorems 10 and 12 in [6], we have the following K-Carleson measure characterizations of Q_K spaces.

Theorem 3.2. *Let K satisfy condition* (4) *and the conditions* (a) *and* (b) *in Lemma* 2.4 *and $r > 0$. Then the following statements are equivalent for all functions $f \in H(\mathbb{D})$.*

(1) $f \in Q_K$;
(2) $d\mu(z) = [w_r(f)(z)]^2 d\tau(z)$ *is a* $K-$*Carleson measure;*
(3) $d\mu(z) = [\widehat{w}_r(f)(z)]^2 d\tau(z)$ *is a* $K-$*Carleson measure;*
(4) $d\mu(z) = [O_1(f)(z)]^2 d\tau(z)$ *is a* $K-$*Carleson measure;*
(5) $d\mu(z) = [O_2(f)(z)]^2 d\tau(z)$ *is a* $K-$*Carleson measure.*

4. The spaces $Q_{K,0}$

Since our earlier estimates are pointwise estimates with respect to $a \in \mathbb{D}$, we have the corresponding "little o" version characterizations of $Q_{K,0}$ spaces. We omit the details of the proofs.

Theorem 4.1. *Let K satisfy the condition* (4) *and $r > 0$. Then the following statements are equivalent for all functions $f \in H(\mathbb{D})$.*

(1) $f \in Q_{K,0}$;
(2) $\lim_{|a|\to 1} \int_{\mathbb{D}} [w_r(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) = 0$;
(3) $\lim_{|a|\to 1} \int_{\mathbb{D}} [\widehat{w}_r(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) = 0$;
(4) $\lim_{|a|\to 1} \int_{\mathbb{D}} [O_1(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) = 0$;
(5) $\lim_{|a|\to 1} \int_{\mathbb{D}} [O_2(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) = 0$.

Theorem 4.2. *Let K satisfy the condition* (4) *and the conditions* (a) *and* (b) *in Lemma* 2.4 *and $r > 0$. Then the following statements are equivalent for all functions $f \in H(\mathbb{D})$.*

(1) $f \in Q_{K,0}$;
(2) $d\mu(z) = [w_r(f)(z)]^2 d\tau(z)$ *is a vanishing* $K-$*Carleson measure;*
(3) $d\mu(z) = [\widehat{w}_r(f)(z)]^2 d\tau(z)$ *is a vanishing* $K-$*Carleson measure;*
(4) $d\mu(z) = [O_1(f)(z)]^2 d\tau(z)$ *is a vanishing* $K-$*Carleson measure;*
(5) $d\mu(z) = [O_2(f)(z)]^2 d\tau(z)$ *is a vanishing* $K-$*Carleson measure.*

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