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SINGULAR POINTS OF TUBULAR SURFACES IN MINKOWSKI 3-SPACE

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Abstract. In this paper, we examine singular points of tubular surfaces and its parallel surfaces, which is based on two-parameter spatial motion along a curve in Minkowski 3-space. Related results are presented also.

1. Introduction

Let $IR^3 = \{(r_1, r_2, r_3) \mid r_1, r_2, r_3 \in IR\}$ be a 3-dimensional vector space, $r = (r_1, r_2, r_3)$ and $s = (s_1, s_2, s_3)$ be two vectors in IR^3. The Lorentz scalar product of the vectors r and s is defined by

$$\langle r, s\rangle_L = -r_1 s_1 + r_2 s_2 + r_3 s_3.$$

The space $IR_1^3 = \left(IR^3, \langle , \rangle_L\right)$ is called a 3-dimensional Lorentz space, or a Minkowski 3-space. The Lorentz vector product of the vectors r and s is defined by

$$r \wedge_L s = \left(r_2 s_3 - r_3 s_2, r_1 s_3 - r_3 s_1, r_2 s_1 - r_1 s_2\right).$$

This yields

$$e_1 \wedge_L e_2 = -e_3, e_2 \wedge_L e_3 = e_1, e_3 \wedge_L e_1 = -e_2$$

where e_1, e_2, e_3 are the base of the space IR_1^3. The vector r in IR_1^3 is called a spacelike vector, a lightlike(null) vector or a timelike vector if $\langle r, r\rangle_L > 0$, $\langle r, r\rangle_L = 0$ or $\langle r, r\rangle_L < 0$ respectively. The norm of the vector r is defined by $\|r\|_L = \sqrt{|\langle r, r\rangle_L|}$, and r is called a unit vector if $\|r\|_L = 1$. Semi-orthogonal matrices provide a rotation by the angle (hyperbolic) t around the vector $\overrightarrow{c}$. The shape of the matrix depends on the type of the vector $\overrightarrow{c}$ as seen in [4].

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i) If $\overrightarrow{c(s)}$ is a spacelike vector,then

$$A_1(s,t) = I + C\sinh t + C^2(-1+\cosh t). \tag{1.1}$$

ii) If $\overrightarrow{c(s)}$ is a timelike vector,then

$$A_2(s,t) = I + C\sin t + C^2(1-\cos t). \tag{1.2}$$

If C is a semi-skew symetric matrix, then

$$C(3,1) = \left\{ \begin{array}{ll} C \in IR^3_3|, \quad C^T = -\varepsilon C \varepsilon, & C = \begin{bmatrix} 0 & c_3 & -c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix} \\ c_i \in IR, \quad \varepsilon = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \end{array} \right\}.$$

Let $\overrightarrow{p}$ denote the ground vector and P denote the column matrix form of the point. The equations

$$C.P = \overrightarrow{c} \wedge_L \overrightarrow{p} \tag{1.3}$$

and

$$\overrightarrow{c} \wedge_L (\overrightarrow{c} \wedge_L \overrightarrow{p}) = -\langle \overrightarrow{c}, \overrightarrow{p} \rangle_L \overrightarrow{c} + \langle \overrightarrow{c}, \overrightarrow{c} \rangle_L \overrightarrow{p} \tag{1.4}$$

are valid. Therefore, from equation (1.1) if $\overrightarrow{c(s)}$ is a spacelike vector, then

$$A_1(s,t)P = \left[I + C\sinh t + C^2(-1+\cosh t) \right] P.$$

From the equation (1.2) and if $\overrightarrow{c(s)}$ is a timelike vector, then

$$A_2(s,t)P = \left[I + C\sin t + C^2(1-\cot t) \right] P.$$

Using the equations (1.3) and (1.4), we get

$$A_1(s,t)P = \overrightarrow{p}\cosh t + \langle \overrightarrow{c}, \overrightarrow{p} \rangle_L \overrightarrow{c}(1-\cosh t) + (\overrightarrow{c} \wedge_L \overrightarrow{p})\sinh t \tag{1.5}$$

and

$$A_2(s,t)P = \overrightarrow{p}\cos t - \langle \overrightarrow{c}, \overrightarrow{p} \rangle_L \overrightarrow{c}(1-\cos t) + (\overrightarrow{c} \wedge_L \overrightarrow{p})\sin t. \tag{1.6}$$

Let α be a space curve given by

$$\alpha : I \to IR^3_1, \;\; s \to \alpha(s)$$

be differentiable for $s \in I \subset IR$. In addition, let a vector field $c(s)$ defined along the curve $\alpha(s)$ be given by

$$c : \alpha(I) \to \bigcup_{s \in I} T_{IR^3_1}$$

$$s \to c(s) = \left(\alpha(s), \overrightarrow{c(s)} \right) = \overrightarrow{c(s)} \mid_{\alpha(s)} .$$

Let $C(s)$ be a semi-skew symetric matrix defined by the vector $\overrightarrow{c}$ for all $s \in I$. The matrices $A_1(s,t)$ and $A_2(s,t)$ are semi-orthogonal matrices defined by $C(s)$. The moving Frenet frame defined along the curve $\alpha(I)$ is $\left\{\alpha(s), \overrightarrow{T(s)}, \overrightarrow{N(s)}, \overrightarrow{B(s)}\right\}$ and p is a fixed point in this frame. With these notations and assumptions we can give the following definition:

Definition 1.1. *The motion $\varphi(s,t)(P) = A_{1,2}(s,t)P + \alpha(s)$ is called the two parameter motion defined along the curve in Minkowski $3-$space* [2].

Here, $\varphi(s,t)(P)$ indicates a trajectory level. We now give some properties of $\varphi(s,t)(P)$. We will always use the frame $\left\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}\right\}$ instead of the Frenet frame

$$\left\{\overrightarrow{T(s)}, \overrightarrow{N(s)}, \overrightarrow{B(s)}\right\}$$

in the remainder of this paper. We will also choose the tangent vector field $\overrightarrow{T}$ instead of the vector field $\overrightarrow{c}$. A trajectory of the point P indicates a surface under the two parameter motion. The equation of this surface is

i) If $\overrightarrow{c}$ is a spacelike vector, then from equations (1.1) and (1.5), we have

$$\varphi_1(s,t)(P) = \overrightarrow{p}\cosh t + \langle\overrightarrow{T}, \overrightarrow{p}\rangle_L (1-\cosh t)\overrightarrow{T} + \sinh t.(\overrightarrow{T} \wedge_L \overrightarrow{p}) + \alpha(s). \tag{1.7}$$

ii) If $\overrightarrow{c}$ is a timelike vector, then from equations (1.2) and (1.6), we have

$$\varphi_2(s,t)(P) = \overrightarrow{p}\cos t - \langle\overrightarrow{T}, \overrightarrow{p}\rangle_L (1-\cos t)\overrightarrow{T} + \sin t.(\overrightarrow{T} \wedge_L \overrightarrow{p}) + \alpha(s). \tag{1.8}$$

2. Characterizations of tubular surfaces

i) If $\alpha(s)$ is a spacelike curve, then tangent $\overrightarrow{T}$ is a spacelike and we have the following cases:

a) $\overrightarrow{T}$ spacelike, $\overrightarrow{N}$ timelike and $\overrightarrow{B}$ spacelike. In this case and from equation (1.7) we have [2]

$$M_1(s,t) = \alpha(s) + \lambda\left(\overrightarrow{N}\cosh t + \overrightarrow{B}\sinh t\right) \tag{2.1}$$

where $\overrightarrow{p} = \lambda\overrightarrow{N}$, $\lambda \in IR$, $\lambda > 0$.

Thus, equation (2.1) is the parametric equation of a tubular surface defined along the curve $\alpha(s)$ in Minkowski 3-space.

b) $\overrightarrow{T}$ spacelike, $\overrightarrow{N}$ spacelike and $\overrightarrow{B}$ timelike. In this case and from equation (1.7) we have

$$M_2(s,t) = \alpha(s) + \lambda\left(\overrightarrow{N}\cosh t - \overrightarrow{B}\sinh t\right) \tag{2.2}$$

where $\overrightarrow{p} = \lambda\overrightarrow{N}$, $\lambda \in IR$, $\lambda > 0$.

Thus, equation (2.2) is the parametric equation of a tubular surface defined along the curve $\alpha(s)$ in Minkowski 3-space.

ii) If $\alpha(s)$ is a timelike curve, then the tangent $\overrightarrow{T}$ is a timelike and we have the following:

The tangent $\overrightarrow{T}$ timelike, $\overrightarrow{N}$ spacelike and $\overrightarrow{B}$ spacelike. In this case and from equation (1.8), we have [2]

$$M_3(s,t) = \alpha(s) + \lambda\left(\overrightarrow{N}\cos t + \overrightarrow{B}\sin t\right) \tag{2.3}$$

where $\overrightarrow{p} = \lambda\overrightarrow{N}$, $\lambda \in IR$, $\lambda > 0$.

Thus, equation (2.3) is the parametric equation of a tubular surface defined along the curve $\alpha(s)$ in Minkowski 3-space.

3. Singular points of the tubular surface $M_1(s,t)$

From equation (2.1) we have

$$M_{1s} \wedge_L M_{1t} = \lambda(1 + \lambda\kappa\cosh t)\left(\overrightarrow{N}\cosh t + \overrightarrow{B}\sinh t\right)$$

where

$$\frac{\partial M_1}{\partial s} = M_{1s} = (1 + \lambda\kappa\cosh t)\,\overrightarrow{T} + (\lambda\tau\sinh t)\,\overrightarrow{N} + (\lambda\tau\cosh t)\,\overrightarrow{B}$$

$$\frac{\partial M_1}{\partial t} = M_{1t} = (\lambda\sinh t)\,\overrightarrow{N} + (\lambda\cosh t)\,\overrightarrow{B}$$

and the Frenet formula (see [5])

$$\overrightarrow{T}' = \kappa\overrightarrow{N}, \overrightarrow{N}' = \kappa\overrightarrow{T} + \tau\overrightarrow{B}, \overrightarrow{B}' = \tau\overrightarrow{N}.$$

Here κ and τ are the curvature and torsion of curve $\alpha(s)$, respectively. The singular points are the points on the surface $M_1(s,t)$ such that

$$M_{1s} \wedge_L M_{1t} = 0$$

or equivalently

$$(1 + \lambda\kappa\cosh t) = 0.$$

Thus, we have the following theorem:

Theorem 3.1. *The point $M_1(s_0,t_0)$ of a surface $M_1(s,t)$ is a singular point if and only if*

$$(1 + \lambda\kappa(s_0)\cosh(t_0)) = 0.$$

Corollary 3.2. *If the vector M_{1s} is on the normal plane which is constructed using $\overrightarrow{N}$ and $\overrightarrow{B}$, then all points on the surface are singular.*

Corollary 3.3. *If $\kappa = 0$ then $(1 + \lambda\kappa\cosh t) \neq 0$.*

Theorem 3.4. *If M_1 is a cylindrical tubular surface then it has no singular points on M_1.*

Proof. $\kappa = 0$ on the cylindrical tubular surface. For $\kappa = 0$, $(1 + \lambda\kappa \cosh t) = 1 \neq 0$. □

Corollary 3.5. *Singular points on the tubular surface are independent of τ, the torsion of the curve $\alpha(s)$, and only depend on the curvature κ of the curve $\alpha(s)$.*

Theorem 3.6. *Let $\alpha(s)$ be a Lorentzian circle. In this case, $\kappa =$ const. and the singular points satisfy*

$$\cosh t = -\frac{1}{\lambda\kappa} = a$$

or equivalently, the singular points (s, t) on M_1 satisfy

$$t = \ln\left(a + \sqrt{a^2 - 1}\right), \quad 1 \leq a.$$

We next find which points on which surfaces are singular points of M_1.

Theorem 3.7. *If $\lambda = -\frac{1}{\kappa} =$ const. then all points on the curve $(t = 0)$, which is the locus of centre of curvature of the curve $\alpha(s)$, are singular points of surface M_1.*

Proof. Since $(1 + \lambda\kappa \cosh t) = 0$ for $t = 0$, $(1 + \lambda\kappa) = 0$, i.e., $\lambda = -\frac{1}{\kappa} =$ const. and the parametric curve of the surface is $M_1(s, 0) = \alpha(s) - \frac{1}{\kappa}\overrightarrow{N}$. This curve is the locus of centre of curvature of the curve $\alpha(s)$. Since for $t = 0$ and $\lambda = -\frac{1}{\kappa}$, $(1 + \lambda\kappa \cosh t) = 1 - 1 = 0$ at the points on this curve, these points are singular. □

Theorem 3.8. *Let $\kappa =$ const. for all s. With the conditions of $t =$ const. and $\lambda = -\frac{1}{a\kappa}$ $(\cosh t = a)$ on the surface M_1 all points on the curve $\alpha(s)$ (Lorentzian circle, helix) are singular.*

Proof. Since $(1 + \lambda\kappa \cosh t) = 1 - \frac{1}{a\kappa}\kappa \cosh t = 0$, these points are singular. □

Theorem 3.9. *Let $\alpha(s)$ be the Bertrand curve. On the conditions of $t = 0$ and $\lambda = -\frac{1}{\kappa} =$ const. the points on $\beta(s)$ curve, which is the couple of the $\alpha(s)$ curve, are singular.*

Proof. Let $\beta(s) = \alpha(s) + \eta\overrightarrow{N}$, $\eta = \lambda = -\frac{1}{\kappa}$ for the $\alpha(s)$ Bertrand curve. Then we have

$$(1 + \lambda\kappa \cosh t) = 1 - \lambda\frac{1}{\lambda} = 0.$$

□

4. Singular points of the parallel tubular surface $M_1^*(s,t)$

Definition 4.1. *The parallel surface of the surface $M_1(s,t)$ defined by*

$$M_1^*(s,t) = M_1(s,t) + \mu U_1(s,t),$$

where

$$U_1 = \frac{M_{1s} \wedge_L M_{1t}}{\|M_{1s} \wedge_L M_{1t}\|_L}$$

is the unit normal of the surface M_1.

Theorem 4.2. *The parallel surface $M_1^*(s,t)$ of the tubular surface $M_1(s,t)$ is still tubular surface* [3], *where*

$$M_1^*(s,t) = \alpha(s) + (\lambda+\mu)\left(\overrightarrow{N}\cosh t + \overrightarrow{B}\sinh t\right).$$

Now, let's investigate whether or not the points on surface $M_1(s,t)$ and its parallel surface $M_1^*(s,t)$ are the same. The normal of the parallel surface $M_1^*(s,t)$ is

$$M_{1s}^* \wedge_L M_{1t}^* = (\lambda+\mu)\left(1+(\lambda+\mu)\kappa\cosh t\right)\left((\cosh t)\overrightarrow{N} + (\sinh t)\overrightarrow{B}\right),$$

where

$$\frac{\partial M_1^*}{\partial s} = M_{1s}^* = \left(1+(\lambda+\mu)\kappa\cosh t\right)\overrightarrow{T} + \left((\lambda+\mu)\tau\sinh t\right)\overrightarrow{N} + \left((\lambda+\mu)\tau\cosh t\right)\overrightarrow{B}$$

$$\frac{\partial M_1^*}{\partial t} = M_{1t}^* = \left((\lambda+\mu)\sinh t\right)\overrightarrow{N} + \left((\lambda+\mu)\cosh t\right)\overrightarrow{B}.$$

Theorem 4.3. *The singular points (s,t) on the surface $M_1(s,t)$ are not singular on the parallel surface $M_1^*(s,t)$.*

Proof. The points (s,t) on the parallel surface $M_1^*(s,t)$ are singular points if and only if

$$\left(1+(\lambda+\mu)\kappa\cosh t\right) = 0.$$

But

$$\left(1+(\lambda+\mu)\kappa\cosh t\right) = 1+\lambda\kappa\cosh t+\mu\kappa\cosh t = 0,$$

which is impossible if $\mu = 0$. Therefore, the points (s,t) on $M_1^*(s,t)$ are not singular. □

Theorem 4.4. *The point $M_1^*(s_0,t_0)$ of the parallel surface $M_1^*(s,t)$ is singular point if and only if $\left(1+(\lambda+\mu)\kappa(s_0)\cosh t_0\right) = 0$.*

Proof. The point (s_0,t_0) on the surface M_1^* is singular point if and only if

$$M_{1s}^*(s_0,t_0) \wedge_L M_{1t}^*(s_0,t_0) = 0 \Leftrightarrow \left(1+(\lambda+\mu)\kappa(s_0)\cosh t_0\right) = 0.$$

□

Corollary 4.5. *If the vector M_{1s}^* is on the normal plane, constructed by $\overrightarrow{N}$ and $\overrightarrow{B}$, then all the points on the surface are singular.*

Corollary 4.6. *If $\kappa = 0$, then $(1 + (\lambda + \mu)\, \kappa \cosh t) \neq 0$.*

Theorem 4.7. *If M_1^* is a cylindrical tubular surface there are not any singular points on M_1^*.*

Proof. $\kappa = 0$ on the cylindrical tubular surface. For $\kappa = 0$, $(1 + (\lambda + \mu)\, \kappa \cosh t) = 1 \neq 0$. □

Corollary 4.8. *Singular points on the tubular surface M_1^* are independent of τ, which is torsion of the curve $\alpha(s)$, and only depend on the κ curvature of the curve $\alpha(s)$.*

Theorem 4.9. *Let $\alpha(s)$ be a Lorentzian circle. Then, since $\kappa =$ const., if*

$$\cosh t = -\frac{1}{(\lambda + \mu)\, \kappa} = a$$

and

$$t = \ln\left(a + \sqrt{a^2 - 1}\right), \ \ 1 \leq a$$

the points (s, t) on the surface M_1^* are singular.

We now examine which points on which curves are singular for $t = 0$ in parametric curves.

Theorem 4.10. *If $\lambda + \mu = -\frac{1}{\kappa} =$ const. all points on the curve $(t = 0)$, which is the locus of the centre of curvature of the curve $\alpha(s)$, are singular points of surface M_1^*.*

Proof. Since $(1 + (\lambda + \mu)\, \kappa \cosh t) = 0$ for $t = 0$, $(1 + (\lambda + \mu)\, \kappa) = 0$, i.e $\lambda + \mu = -\frac{1}{\kappa} =$ const. which implies

$$M_1^*(s, 0) = \alpha(s) \ - \frac{1}{\kappa}\overrightarrow{N}.$$

This curve is the locus of the centre of curvature of the curve $\alpha(s)$. Since for $t = 0$ and $\lambda + \mu = -\frac{1}{\kappa}$, $(1 + (\lambda + \mu)\kappa \cosh t) = 1 - 1 = 0$ at the points on this curve, these points are singular. □

Theorem 4.11. *Let $\kappa =$ const. for all s. With the conditions of all $t =$ const. and $\lambda + \mu = -\frac{1}{a\kappa}$ $(\cosh t = a)$ on the surface M_1^* all points on the curve $\alpha(s)$(Lorentzian circle, helix) are singular.*

Proof. Since $(1 + (\lambda + \mu)\, \kappa \cosh t) = 1 - \frac{1}{a\kappa}\kappa \cosh t = 0$, these points are singular. □

Theorem 4.12. *Let $\alpha(s)$ be the Bertrand curve. On the conditions of $t = 0$ and $\lambda + \mu = -\frac{1}{\kappa} =$ const. the points on the curve $\beta(s)$, the couple of the curve $\alpha(s)$, are singular.*

Proof. Let $\beta(s) = \alpha(s) + \delta\overrightarrow{N}$, $\delta = \lambda + \mu = -\frac{1}{\kappa}$ for the Bertrand curve $\alpha(s)$. For the points on this curve

$$(1 + (\lambda + \mu)\,\kappa \cosh t) = 1 - (\lambda + \mu)\,\frac{1}{(\lambda + \mu)} = 0$$

is obtained. □

5. Singular points of the tubular surface $M_2(s,t)$

From equation (2.2) we have

$$M_{2s} \wedge_L M_{2t} = \lambda(1 - \lambda\kappa \cosh t)\left(\overrightarrow{N} \cosh t - \overrightarrow{B} \sinh t\right)$$

where

$$M_{2s} = (1 - \lambda\kappa \cosh t)\,\overrightarrow{T} - (\lambda\tau \sinh t)\,\overrightarrow{N} + (\lambda\tau \cosh t)\,\overrightarrow{B}$$
$$M_{2t} = (\lambda \sinh t)\,\overrightarrow{N} - (\lambda \cosh t)\,\overrightarrow{B}$$

and the Frenet formula (see [5]),

$$\overrightarrow{T}' = \kappa\overrightarrow{N}, \overrightarrow{N}' = -\kappa\overrightarrow{T} + \tau\overrightarrow{B}, \overrightarrow{B}' = \tau\overrightarrow{N}.$$

Here κ and τ are curvature and torsion of the curve $\alpha(s)$, respectively. $M_{2s} \wedge_L M_{2t} = 0$ if and only if $(1 - \lambda\kappa \cosh t) = 0$. The study of the geometric properties the singular points can be done using the same methods as in Section 3.

6. Singular points of the parallel tubular surface $M_2^*(s,t)$

Theorem 6.1. *The parallel surface $M_2^*(s,t)$ of the tubular surface $M_2(s,t)$ is still tubular surface.*

$$M_2^*(s,t) = \alpha(s) + (\lambda + \mu)\left(\overrightarrow{N} \cosh t - \overrightarrow{B} \sinh t\right).$$

Now, let's investigate whether or not the points on surface $M_2(s,t)$ and its parallel surface $M_2^*(s,t)$ are the same. The normal of the parallel surface $M_2^*(s,t)$ is

$$M_{2s}^* \wedge_L M_{2t}^* = (\lambda + \mu)\,(1 - (\lambda + \mu)\,\kappa \cosh t)\left((\cosh t)\,\overrightarrow{N} - (\sinh t)\,\overrightarrow{B}\right),$$

where

$$M_{2s}^* = (1 - (\lambda + \mu)\,\kappa \cosh t)\,\overrightarrow{T} - ((\lambda + \mu)\,\tau \sinh t)\,\overrightarrow{N} + ((\lambda + \mu)\,\tau \cosh t)\,\overrightarrow{B}$$
$$M_{2t}^* = ((\lambda + \mu) \sinh t)\,\overrightarrow{N} - ((\lambda + \mu) \cosh t)\,\overrightarrow{B}.$$

The study can be carried out using the same technique as in Section 4.

7. Singular points of the tubular surface $M_3(s,t)$

From equation (2.3), we have

$$M_{3s} \wedge_L M_{3t} = \lambda(1+\lambda\kappa\cos t)\left(-\overrightarrow{N}\cos t - \overrightarrow{B}\sin t\right)$$

where

$$M_{3s} = (1+\lambda\kappa\cos t)\,\overrightarrow{T} - (\lambda\tau\sin t)\,\overrightarrow{N} + (\lambda\tau\cos t)\,\overrightarrow{B}$$
$$M_{3t} = -\,(\lambda\sin t)\,\overrightarrow{N} + (\lambda\cos t)\,\overrightarrow{B}$$

and the Frenet formula (see [5])

$$\overrightarrow{T}' = \kappa\overrightarrow{N}, \overrightarrow{N}' = \kappa\overrightarrow{T} + \tau\overrightarrow{B}, \overrightarrow{B}' = -\tau\overrightarrow{N}$$

where κ and τ are curvature and torsion of $\alpha(s)$ curve, respectively. $M_{3s} \wedge M_{3t} = 0$ if and only if $(1+\lambda\kappa\cos t) = 0$. The study is similar to Section 5.

8. Singular points of the parallel tubular surface $M_3^*(s,t)$

Theorem 8.1. *The $M_3^*(s,t)$ parallel surface of $M_3(s,t)$, a tubular surface, is still a tubular surface.*

$$M_3^*(s,t) = \alpha(s) + (\lambda+\mu)\left(\overrightarrow{N}\cos t + \overrightarrow{B}\sin t\right).$$

Now, let's investigate whether or not the points on surface $M_3(s,t)$ and its parallel surface $M_3^*(s,t)$ are the same. The normal to the parallel surface $M_3^*(s,t)$ is

$$M_{3s}^* \wedge_L M_{3t}^* = (\lambda+\mu)\,(1+(\lambda+\mu)\,\kappa\cos t)\left(-\,(\cos t)\,\overrightarrow{N} - (\sin t)\,\overrightarrow{B}\right),$$

where

$$M_{3s}^* = (1+(\lambda+\mu)\,\kappa\cos t)\,\overrightarrow{T} - ((\lambda+\mu)\,\tau\sin t)\,\overrightarrow{N} + ((\lambda+\mu)\,\tau\sin t)\,\overrightarrow{B}$$
$$M_{3t}^* = -\,((\lambda+\mu)\sin t)\,\overrightarrow{N} + ((\lambda+\mu)\cos t)\,\overrightarrow{B}.$$

The study is carried out using the same steps as in Section 6.

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