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ON RICCI CURVATURE OF CERTAIN SUBMANIFOLDS IN A COSYMPLECTIC SPACE FORM

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Abstract. In this paper, we obtain some sharp inequalities between the Ricci curvature and the squared mean curvature for slant, bi-slant and semi-slant submanifolds in cosymplectic space forms. Estimates of the scalar curvature and the k-Ricci curvature, in terms of the squared mean curvature, are also proved respectively.

1. Introduction

According to B.Y. Chen, one of the basic problems in submanifold theory is to find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. Scalar curvature and Ricci curvature are among the main intrinsic invariants, while the squared mean curvature is the main extrinsic invariant. In [5], B.Y. Chen establishes a relationship between sectional curvature function K and the shape operator for submanifolds in real space forms. In [6], he also gives a relationship between Ricci curvature and squared mean curvature.

A contact version of B.Y. Chen's inequality and its applications to slant immersions in a Sasakian space form $\tilde{M}\,(c)$ are given in [4]. There is another interesting class of almost contact metric manifolds, namely cosymplectic manifolds [8].

In the present paper, we will study the Ricci curvature of certain submanifolds, i.e., slant, bi-slant and semi-slant submanifolds in a cosymplectic space form, and get some very interesting results. The rest of this paper is organized as follows. Necessary details about cosymplectic manifolds and the submanifolds are reviewed in Section 2. In Section 3, some inequalities between Ricci curvature and squared mean curvature function for bi-slant, semi-slant and slant submanifolds in cosymplectic space forms. We also discuss the equality cases. In the last section, we establish some relationship

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between the k-Ricci curvature and the squared mean curvature for bi-slant, semi-slant and slant submanifolds in cosymplectic space forms. In particular, we give similar results for invariant, anti-invariant and contact CR submanifolds.

2. Preliminaries

Let $\tilde{M}$ be a $(2m+1)$-dimensional almost contact manifold ([1]) endowed with an almost contact structure (φ, ξ, η), that is, φ is a $(1,1)$ tensor field, ξ is a vector field and η is a 1-form such that $\varphi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$. Then, $\varphi(\xi) = 0$ and $\eta \circ \varphi = 0$. The almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $\tilde{M} \times R$ defined by $J(X, \lambda d/dt) = (\varphi X - \lambda\xi, \eta(X)\, d/dt)$ is integrable, where X is tangent to $\tilde{M}$, t the coordinate of R and λ a smooth function on $\tilde{M} \times R$.

Let g be a compatible Riemannian metric with (φ, ξ, η), that is, $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ or equivalently, $\Phi(X, Y) = g(X, \varphi Y) = -g(\varphi X, Y)$ and $g(X, \xi) = \eta(X)$ for all $X, Y \in T\tilde{M}$. Then, $\tilde{M}$ becomes an almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g).

If the fundamental 2-form Φ and 1-form η are closed, then $\tilde{M}$ is said to be almost cosymplectic manifold ([1]). A normal almost cosymplectic manifold is cosymplectic. An almost contact metric structure is cosymplectic if and only if $\tilde{\nabla}_X \varphi = 0$, where $\tilde{\nabla}$ is the Levi-Civita connection of the Riemannian metric g. From the formula $\tilde{\nabla}_X \varphi = 0$ it follows that $\tilde{\nabla}_X \xi = 0$.

A plane section σ in $T_p\tilde{M}$ of an almost contact metric manifold $\tilde{M}$ is called a φ-section if $\sigma \perp \xi$ and $\varphi(\sigma) = \sigma$. $\tilde{M}$ is of constant φ-sectional curvature if at each point $p \in \tilde{M}$, the sectional curvature $\tilde{K}(\sigma)$ does not depend on the choice of the φ-section σ of $T_p\tilde{M}$, and in this case for $p \in \tilde{M}$ and for any φ-section σ of $T_p\tilde{M}$, the function c defined by $c(p) = \tilde{K}(\sigma)$ is called the φ-sectional curvature of $\tilde{M}$. A cosymplectic manifold $\tilde{M}$ with constant φ-sectional curvature c is said to be a cosymplectic space form and is denoted by $\tilde{M}(c)$.

The curvature tensor $\tilde{R}$ of a cosymplectic space form $\tilde{M}(c)$ is given by [8].

$$\begin{aligned}
4\tilde{R}(X, Y, Z, W) &= c[g(X, W)\, g(Y, Z) - g(X, Z)\, g(Y, W) \\
&+ g(X, \varphi W)\, g(Y, \varphi Z) - g(X, \varphi Z)\, g(Y, \varphi W) - 2g(X, \varphi Y)\, g(Z, \varphi W) \\
&- g(X, W)\, \eta(Y)\, \eta(Z) + g(X, Z)\, \eta(Y)\, \eta(W) \\
&- g(Y, Z)\, \eta(X)\, \eta(W) + g(Y, W)\, \eta(X)\, \eta(Z)] \qquad (1)
\end{aligned}$$

for all $X, Y, Z, W \in T\tilde{M}$.

Let M be an n-dimensional Riemannian manifold. The scalar curvature τ at p is given by $\tau = \sum_{i<j} K_{ij}$, where K_{ij} is the sectional curvature of M associated with a plane section spanned by e_i and e_j at $p \in M$ for any orthonormal basis $\{e_1, dots, e_n\}$ for T_pM. Now let M be a submanifold of an m-dimensional manifold $\tilde{M}$ equipped with a Riemannian metric g. The Gauss and Weingarten formulae are given respectively by $\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$ and $\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N$ for all $X, Y \in TM$ and $N \in T^\perp M$, where $\tilde{\nabla}$, ∇ and $\nabla^\perp$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\tilde{M}$, M and the normal bundle $T^\perp M$ of M respectively, and σ is the second fundamental form related to the shape operator A by $g(h(X, Y), N) = g(A_N X, Y)$. Then the equation of Gauss is given by

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)) \tag{2}$$

for any vectors X, Y, Z, W tangent to M, where $\tilde{R}$ and R are the curvature tensors of $\tilde{M}$ and M respectively.

The relative null space of M at a point $p \in M$ is defined by

$$\mathcal{N}_p = \{X \in T_pM \,|\, \sigma(X,Y) = 0, \text{ for all } Y \in T_pM\}.$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM. The mean curvature vector H at $p \in M$ is

$$H = \frac{1}{n}\,\text{trace}\,(\sigma) = \frac{1}{n}\sum_{i=1}^{n} \sigma(e_i, e_i). \tag{3}$$

The submanifold M is totally geodesic in $\tilde{M}$ if $\sigma = 0$, and minimal if $H = 0$. If $\sigma(X,Y) = g(X,Y)H$ for all $X, Y \in TM$, then M is totally umbilical. We put

$$\sigma_{ij}^r = g(\sigma(e_i,e_j),e_r) \quad \|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i,e_j),\sigma(e_i,e_j)),$$

where e_r belongs to an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the normal space $T_p^\perp M$.

Suppose L is a k-plane section of T_pM and X a unit vector in L. We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of L such that $e_1 = X$.

Define the Ricci curvature Ric_L of L at X by

$$\mathrm{Ric}_L(X) = K_{12} + K_{13} + \cdots + K_{1k}, \tag{4}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j. We simply call such a curvature a k-Ricci curvature.

The scalar curvature τ of the k-plane section L is given by

$$\tau(L) = \sum_{1 \leq i < j \leq k} K_{ij}. \tag{5}$$

For each integer k, $2 \leq k \leq n$, the Riemannian invariant θ_k on an n-dimensional Riemannian manifold M is defined by

$$\Theta_k(p) = \frac{1}{k-1} \inf_{L,X} \mathrm{Ric}_L(X), \ \ p \in M, \tag{6}$$

where L runs over all k-plane sections in T_pM and X runs over all unit vectors in L.

Now let M be an n-dimensional submanifold in an almost contact metric manifold. For a vector field X in M, we put

$$\varphi X = PX + FX, \ \ PX \in TM, \ \ FX \in T^{\perp}M.$$

Thus, P is an endomorphism of the tangent bundle of M and satisfies $g(X, PY) = -g(PX, Y)$ for all $X, Y \in TM$. The squared norm of P is given by

$$\|P\|^2 = \sum_{i,j=1}^{n+1} g(e_i, Pe_j)^2$$

for any local orthonormal basis $\{e_1, e_2, \ldots, e_{n+1}\}$ for T_pM.

A submanifold M of an almost contact metric manifold with $\xi \in TM$ is called a semi-invariant submanifold or a contact CR submanifold ([8]) if there exists two differentiable distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ on M such that **(i)** $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \mathcal{E}$, **(ii)** the distribution $\mathcal{D}$ is invariant by φ, i.e., $\varphi(\mathcal{D}) = \mathcal{D}$, and **(iii)** the distribution $\mathcal{D}^{\perp}$ is anti-invariant by φ, i.e., $\varphi(\mathcal{D}^{\perp}) \subseteq T^{\perp}M$.

The submanifold M tangent to ξ is said to be invariant or anti-invariant ([9]) according as $F = 0$ or $P = 0$. Thus, a CR-submanifold is invariant or anti-invariant according as $\mathcal{D}^{\perp} = \{0\}$ or $\mathcal{D} = \{0\}$. A proper CR-submanifold is neither invariant nor anti-invariant.

For each non zero vector $X \in T_pM$, such that X is not proportional to ξ_p, we denote the angle between φX and T_pM by $\theta(X)$. Then M is said to be slant ([2], [7]) if the angle $\theta(X)$ is constant, that is, it is independent of the choice of $p \in M$ and $X \in T_pM - \{\xi\}$. The angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \pi/2$ respectively. A proper slant immersion is neither invariant nor anti-invariant.

We say that a submanifold M tangent to ξ is a bi-slant submanifold of $\tilde{M}$ ([3]) if there exists two orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on M such that

(i) $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$.
(ii) For any $i = 1, 2$, the distribution $\mathcal{D}_i$ is slant distribution with angle θ_i.

Let $2d_1 = \dim\, \mathcal{D}_1$ and $2d_2 = \dim\, \mathcal{D}_2$.

Remark. If either d_1 or d_2 vanishes, the bi-slant submanifold is a slant submanifold. Thus slant submanifolds are particular cases of bi-slant submanifolds.

We say that M tangent to ξ is a semi-slant submanifold of $\tilde{M}$ ([3]) if there exists two orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on M such that

(i) $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$.
(ii) The distribution $\mathcal{D}_1$ is invariant by φ, i.e., $\varphi(\mathcal{D}_1) = \mathcal{D}_1$.
(iii) The distribution $\mathcal{D}_2$ is slant with angle $\theta \neq 0$.

Let $2d_1 = \dim\, \mathcal{D}_1$ and $2d_2 = \dim\, \mathcal{D}_2$.

The invariant distribution of a semi-slant submanifold is a slant distribution with zero angle. Moreover, it is clear that, if $\theta = \frac{\pi}{2}$, then the semi-slant submanifold is a semi-invariant submanifold.

(a) If $d_2 = 0$, then M is an invariant submanifold.
(b) If $d_1 = 0$ and $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold.
(c) If $d_1 = 0$ and $\theta \neq \frac{\pi}{2}$, then M is a proper slant submanifold, with slant angle θ.

We say that a semi-slant submanifold is proper if $d_1 d_2 \neq 0$ and $\theta \neq \frac{\pi}{2}$.

3. Ricci curvature and squared mean curvature

Chen established a sharp relation between the Ricci curvature and squared mean curvature for submanifold in real space forms ([6]).

In this section we want to prove some similar inequalities for bi-slant, semi-slant and slant submanifolds in a cosymplectic space form.

Theorem 3.1. *Let M be an $(n = 2d_1 + 2d_2 + 1)$-dimensional bi-slant submanifold in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$ tangential to the structure vector field ξ. Then,*

(i) *For each unit vector $X \in T_pM$ orthogonal to ξ and if $X \in \mathcal{D}_1$ we have*

$$4Ric\,(X) \leq n^2\|H\|^2 + (n-1)c + \frac{1}{2}\left(3cos^2\theta_1 - 2\right)c, \tag{7}$$

and if $X \in \mathcal{D}_2$ we have

$$4Ric\,(X) \leq n^2\|H\|^2 + (n-1)c + \frac{1}{2}\left(3cos^2\theta_2 - 2\right)c. \tag{8}$$

(ii) *If $H(p) = 0$, a unit vector $X \in T_pM$ orthogonal to ξ satisfies the equality case of* (7) *or* (8) *if and only if X belongs to the relative null space $\mathcal{N}_p$.*

(iii) *The equality case of* (7) *or* (8) *holds for all unit vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

Proof. We choose an orthonormal basis $\{e_1, \dots, e_n = \xi,\ e_{n+1}, \dots, e_{2m+1}\}$ such that $e_1, \dots, e_n$ are tangent to M at p, with $e_1 = X$.

From the equation of Gauss, we have

$$n^2\|H\|^2 = 2\|\sigma\|^2 + 2\tau - \frac{n(n-1)}{4}c - [6(d_1cos^2\theta_1 + d_2cos^2\theta_2) - 2n + 2]\frac{c}{4}. \quad (9)$$

From above equation, we have

$$\begin{aligned} n^2\|H\|^2 = 2\tau &+ \sum_{r=n+1}^{2m+1}\Big[(\sigma_{11}^r)^2 + (\sigma_{22}^r + \cdots + \sigma_{nn}^r)^2 + 2\sum_{i<j}(\sigma_{ij}^r)^2\Big] \\ - 2\sum_{r=n+1}^{2m+1}&\sum_{2\le i<j\le n}\sigma_{ii}^r\sigma_{jj}^r - \frac{n(n-1)}{4}c - [6(d_1cos^2\theta_1 + d_2cos^2\theta_2) - 2n + 2]\frac{c}{4} \\ &= 2\tau + \frac{1}{2}\sum_{r=n+1}^{2m+1}[(\sigma_{11}^r + \cdots + \sigma_{nn}^r)^2 + (\sigma_{11}^r - \sigma_{22}^r - \cdots - \sigma_{nn}^r)^2] \\ &+ 2\sum_{r=n+1}^{2m+1}\sum_{i<j}(\sigma_{ij}^r)^2 - 2\sum_{r=n+1}^{2m+1}2\sum_{2\le i<j\le n}\sigma_{ii}^r\sigma_{jj}^r - \frac{n(n-1)}{4}c \\ &- [6(d_1cos^2\theta_1 + d_2cos^2\theta_2) - 2n + 2]\frac{c}{4}. \end{aligned} \quad (10)$$

From the equation of Gauss we can get

a) If $X \in \mathcal{D}_1$

$$K_{ij} = \sum_{r=n+1}^{2m+1}[\sigma_{11}^r + \sigma_{22}^r - (\sigma_{12}^r)^2] + \frac{c}{4} + 3cos^2\theta_1\frac{c}{4}, \quad (11)$$

and consequently

$$\begin{aligned} \sum_{2\le i<j\le n}K_{ij} = \sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n}[\sigma_{ii}^r\sigma_{jj}^r - (\sigma_{ij}^r)^2] + \frac{(n-1)(n-2)}{2}\frac{c}{4} \\ + [6(d_1cos^2\theta_1 + d_2cos^2\theta_2) - 3cos^2\theta_1 - 2n + 4]\frac{c}{8}. \end{aligned} \quad (12)$$

Substituting (12) in (10), we have

$$\frac{1}{2}n^2\|H\|^2 \ge 2\mathrm{Ric}\,(X) - 2(n-1)\frac{c}{4} - \frac{c}{4}(3cos^2\theta_1 - 2)$$

which is equivalent to (7).

b) If $X \in \mathcal{D}_2$

$$K_{ij} = \sum_{r=n+1}^{2m+1} [\sigma_{11}^r + \sigma_{22}^r - (\sigma_{12}^r)^2] + \frac{c}{4} + 3cos^2\theta_2 \frac{c}{4}, \tag{13}$$

and consequently

$$\sum_{2 \le i < j \le n} K_{ij} = \sum_{r=n+1}^{2m+1} \sum_{2 \le i < j \le n} [\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2] + \frac{(n-1)(n-2)}{2} \frac{c}{4}$$
$$+ [6(d_1 cos^2\theta_1 + d_2 cos^2\theta_2) - 3cos^2\theta_2 - 2n + 4]\frac{c}{8}. \tag{14}$$

Substituting (14) in (10), we have

$$\frac{1}{2} n^2 \|H\|^2 \ge 2\mathrm{Ric}\,(X) - 2(n-1)\frac{c}{4} - \frac{c}{4}(3cos^2\theta_2 - 2)$$

which is equivalent to (8).

The equality case of (7) or (8) is satisfied if and only if

$$\sigma_{11}^r = \sigma_{22}^r + \cdots + \sigma_{nn}^r,$$
$$\sigma_{12}^r = \cdots = \sigma_{1n}^r = 0, \; r \in \{n+1, \ldots, 2m+1\}. \tag{15}$$

If $H(p) = 0$, (15) implies that $e_1 = X$ belongs to the relative null space N_p at p. Conversely, if $e_1 = X$ lies in the relative null space, then (15) holds because $H(p) = 0$ is assumed. This proves statement (ii).

Now, we prove (iii). Assume that the equality case of (7) or (8) for all unit tangent vectors orthogonal to ξ at $p \in M$ is true. Then for each $r \in \{n+1, \ldots, 2m+1\}$, we have

$$2\sigma_{ii}^r = \sigma_{11}^r + \cdots + \sigma_{nn}^r, \; i \in \{1, \ldots, n\},$$
$$\sigma_{ij}^r = 0, \; i \ne j. \tag{16}$$

Thus, we have two cases, namely either $n = 1$ or $n \ne 1$. In the first case p is a totally umbilical point, while in the second case p is a totally geodesic point. Since $\xi \in TM$, therefore each totally umbilical point is totally geodesic. Thus in both cases, p is a totally geodesic point. The proof of converse part is straightforward. □

Similarly we can prove the following theorems:

Theorem 3.2. *Let M be an $(n = 2d_1 + 2d_2 + 1)$-dimensional semi-slant submanifold in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$ tangential to the structure vector field ξ. Then,*

(i) *For each unit vector* $X \in T_pM$ *orthogonal to* ξ *and if* $X \in \mathcal{D}_1$ *we have*

$$4\,Ric\,(X) \leq n^2\|H\|^2 + nc - \frac{1}{2}c, \tag{17}$$

and if $X \in \mathcal{D}_2$ *we have*

$$4\,Ric\,(X) \leq n^2\|H\|^2 + (n-1)c + \frac{1}{2}\left(3cos^2\theta - 2\right)c. \tag{18}$$

(ii) *If* $H(p) = 0$*, a unit vector* $X \in T_pM$ *orthogonal to* ξ *satisfies the equality case of* (17) *or* (18) *if and only if* X *belongs to the relative null space* $\mathcal{N}_p$*.*

(iii) *The equality case of* (17) *or* (18) *holds for all unit vectors orthogonal to* ξ *at* p *if and only if* p *is a totally geodesic point.*

Theorem 3.3. *Let* M *be an* n*-dimensional* θ*-slant submanifold in a* $(2m+1)$*-dimensional cosymplectic space form* $\tilde{M}(c)$ *tangential to the structure vector field* ξ*. Then,*

(i) *For each unit vector* $X \in T_pM$ *orthogonal to* ξ*, we have*

$$4\,Ric\,(X) \leq n^2\|H\|^2 + (n-1)c + \frac{1}{2}\left(3cos^2\theta - 2\right)c. \tag{19}$$

(ii) *If* $H(p) = 0$*, a unit vector* $X \in T_pM$ *orthogonal to* ξ *satisfies the equality case of* (19) *if and only if* X *belongs to the relative null space* $\mathcal{N}_p$*.*

(iii) *The equality case of* (19) *holds for all unit vectors orthogonal to* ξ *at* p *if and only if* p *is a totally geodesic point.*

For invariant, anti-invariant and contact CR submanifolds in a cosymplectic space form, we have the following results.

Corollary 3.1. *Let* M *be an* n*-dimensional invariant submanifold in a* $(2m+1)$*-dimensional cosymplectic space form* $\tilde{M}(c)$ *tangential to the structure vector field* ξ*. Then,*

(i) *For each unit vector* $X \in T_pM$ *orthogonal to* ξ*, we have*

$$4\,Ric\,(X) \leq nc - \frac{c}{2}. \tag{20}$$

(ii) *If* $H(p) = 0$*, a unit vector* $X \in T_pM$ *orthogonal to* ξ *satisfies the equality case of* (20) *if and only if* X *belongs to the relative null space* $\mathcal{N}_p$*.*

(iii) *The equality case of* (20) *holds for all unit vectors orthogonal to* ξ *at* p *if and only if* p *is a totally geodesic point.*

Corollary 3.2. *Let* M *be an* n*-dimensional anti-invariant submanifold in a* $(2m+1)$*-dimensional cosymplectic space form* $\tilde{M}(c)$ *tangential to the structure vector field* ξ*. Then,*

(i) *For each unit vector $X \in T_pM$ orthogonal to ξ, we have*

$$4\,Ric\,(X) \leq n^2\|H\|^2 + (n-2)c. \tag{21}$$

(ii) *If $H(p) = 0$, a unit vector $X \in T_pM$ orthogonal to ξ satisfies the equality case of* (21) *if and only if X belongs to the relative null space $\mathcal{N}_p$.*

(iii) *The equality case of* (21) *holds for all unit vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

Corollary 3.3. *Let M be an n-dimensional contact CR submanifold in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$ tangential to the structure vector field ξ. Then,*

(i) *For each unit vector $X \in T_pM$ orthogonal to ξ and if $X \in \mathcal{D}$ we have*

$$4\,Ric\,(X) \leq n^2\|H\|^2 + nc - \frac{1}{2}c. \tag{22}$$

and if $X \in \mathcal{D}^\perp$ we have

$$4\,Ric\,(X) \leq n^2\|H\|^2 + (n-2)c. \tag{23}$$

(ii) *If $H(p) = 0$, a unit vector $X \in T_pM$ orthogonal to ξ satisfies the equality case of* (22) *or* (23) *if and only if X belongs to the relative null space $\mathcal{N}_p$.*

(iii) *The equality case of* (22) *or* (23) *holds for all unit vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

4. k-Ricci curvature

In this section, we study the relationship between the k-Ricci curvature and the squared mean curvature for bi-slant, slant and semi-slant submanifolds in a cosymplectic space form.

First, we state a relationship between the scalar curvature and the squared mean curvature for bi-slant submanifolds in a cosymplectic space form.

Theorem 4.1. *Let M be an $(n = 2d_1 + 2d_2 + 1)$-dimensional bi-slant submanifold in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$ tangential to the structure vector field ξ. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c}{4} - \frac{[3(d_1 cos^2\theta_1 + d_2 cos^2\theta_2) - n + 1]c}{2n(n-1)}. \tag{24}$$

Proof. We choose an orthonormal basis $\{e_1, \ldots, e_n = \xi, e_{n+1}, \ldots, e_{2m+1}\}$ at $p \in M$ such that $e_1, \ldots, e_n$ are tangent to M at p such that e_{n+1} is parallel to the mean curvature vector $H(p)$ and $e_1, \ldots, e_n$ diagonalize the shape operator A_{n+1}. Then the shape operators take the forms

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ . & . & \ldots & . \\ . & . & \ldots & . \\ 0 & 0 & \ldots & a_n. \end{pmatrix} \tag{25}$$

$$A_r = \left(h_{ij}^r\right), \quad i, j = 1, \ldots, n; \;\; r = n+2, \ldots, 2m; \;\; \mathrm{trace} A_r = \sum_{i=1}^{n} h_{ii}^r = 0. \tag{26}$$

From (9), we get

$$n^2\|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - \frac{n(n-1)}{4}c - [6(d_1 cos^2\theta_1 + d_2 cos^2\theta_2) - 2n + 2]\frac{c}{4}. \tag{27}$$

On the other hand, since

$$0 \leq \sum_{i<j} (a_i - a_j)^2 = (n-1)\sum_i a_i^2 - 2\sum_{i<j} a_i a_j,$$

we obtain

$$n^2 \|H\|^2 = (\sum_{i=1}^{n} a_i)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \leq n \sum_{i=1}^{n} a_i^2,$$

which implies

$$\sum_{i=1}^{n} a_i^2 \geq n \|H\|^2 .$$

So we have

$$n^2\|H\|^2 = 2\tau + n\|H\|^2 - \frac{n(n-1)}{4}c - [3(d_1 cos^2\theta_1 + d_2 cos^2\theta_2) - 2n + 2]\frac{c}{4}, \tag{28}$$

which is equivalent to (24). $\square$

Using Theorem 4.1, we obtain the following.

Theorem 4.2. *Let M be an $(n = 2d_1 + 2d_2 + 1)$-dimensional bi-slant submanifold in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$ tangential to the structure vector field ξ. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2 (p) \geq \Theta_k(p) - \frac{c}{4} - \frac{[3(d_1 cos^2\theta_1 + d_2 cos^2\theta_2) - n + 1]c}{2n(n-1)}. \tag{29}$$

Proof. Let $\{e_1, \ldots, e_n = \xi\}$ be an orthonormal basis of T_pM. Denote by $L_{i_1 \ldots i_k}$ the k-plane section spanned by $e_{i_1}, \ldots, e_{i_k}$. It follows from (4) and (5) that

$$\tau(L_{i_1 \ldots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \ldots, i_k\}} Ric_{L_{i_1 \ldots i_k}}(e_i), \tag{30}$$

$$\tau(p) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \tau(L_{i_1 \ldots i_k}). \tag{31}$$

Combining (6), (30) and (31), we find

$$\tau(p) \geq \frac{n(n-1)}{2} \Theta_k(p). \tag{32}$$

From (24) and (32), we obtain (29). □

Similarly we can prove the following theorems:

Theorem 4.3. *Let M be an $(n = 2d_1 + 2d_2 + 1)$-dimensional semi-slant submanifold in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$ tangential to the structure vector field ξ. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c}{4} - \frac{[3(d_1 + d_2 cos^2\theta) - n + 1]c}{2n(n-1)}. \tag{33}$$

Theorem 4.4. *Let M be an $(n = 2d_1 + 2d_2 + 1)$-dimensional semi-slant submanifold in a $(2m + 1)$-dimensional cosymplectic space form $\tilde{M}(c)$ tangential to the structure vector field ξ. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c}{4} - \frac{[3(d_1 + d_2 cos^2\theta) - n + 1]c}{2n(n-1)}. \tag{34}$$

Theorem 4.5. *Let M be an n-dimensional θ-slant submanifold in a $(2m + 1)$-dimensional cosymplectic space form $\tilde{M}(c)$ tangential to the structure vector field ξ. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c}{4} - \frac{[3(n-1)cos^2\theta - 2n + 2]c}{4n(n-1)}. \tag{35}$$

Theorem 4.6. *Let M be an n-dimensional θ-slant submanifold in a $(2m + 1)$-dimensional cosymplectic space form $\tilde{M}(c)$ tangential to the structure vector field ξ. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c}{4} - \frac{[3(n-1)cos^2\theta - 2n + 2]c}{4n(n-1)}. \tag{36}$$

Finally for invariant, anti-invariant contact CR submanifolds in a cosymplectic space form, we have:

Corollary 4.1. *Let M be an n-dimensional invariant submanifold in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$ tangential to the structure vector field ξ. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\Theta_k(p) \leq \frac{c}{4} + \frac{c}{4n}. \tag{37}$$

and

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c}{4} + \frac{c}{2n}. \tag{38}$$

Corollary 4.2. *Let M be an $(n = 2d_1 + 2d_2 + 1)$-dimensional contact CR submanifold ($\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$) in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$ tangential to the structure vector field ξ. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c}{4} - \frac{(3d_1 - n + 1)c}{2n(n-1)}. \tag{39}$$

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