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A NOTE ON A PAPER OF I. GOLEŢ

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Abstract. In a recent paper in this journal [2] (previously named Radovi Matematički), a theorem generalizing a well-known result of Radu [6] was proved. The aim of this short note is to show that, in turn, this theorem can be obtained from the fixed point theorem of Radu.

1. Preliminaries

In this section we briefly recall some definitions and results from [2]. For more details concerning the theory of probabilistic metric spaces and the fixed point theory in probabilistic metric spaces we refer the reader to the books [1], [3] and [7].

The following class of probabilistic contractions (now known as Hicks C -contractions or C-contractions) has been introduced by T. L. Hicks.

Definition 1.1. [4]. *Let F be a probabilistic distance on S that is, F is a function from $S \times S$ to Δ_+ ($F(p,q)$ is denoted by F_{pq}). A mapping $f : S \to S$ is called a probabilistic C-contraction or simply a C-contraction if, for some $k \in (0,1)$, the following implication holds for every $p, q \in S$ and $t > 0$:*

$$F_{pq}(t) > 1 - t \Rightarrow F_{f(p)f(q)}(kt) > 1 - kt.$$

Radu [6] showed that a C-contraction has a unique fixed point under a very weak condition on the t-norm. Namely, he has proved the following:

Theorem 1.1. [6]. *Let (S, F, T) be a complete Menger space satisfying that $\mathrm{Range}(F) \subset D_+$ and $\sup_{a<1} T(a,a) = 1$. Then every C-contraction f on S has a unique fixed point which is the limit of the sequence $(f^n(p))_{n\in\mathbf{N}}$ for every $p \in S$.*

It is worth noting that Radu's theorem holds even in the absence of the condition $\mathrm{Range}(F) \subset D_+$ (see e.g. [5]).

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In a recent paper in this journal [2], Goleţ defined a Hicks-type probabilistic contraction in the following way:

Definition 1.2. [2, Definition 3.] *Let (S, F, T) be a Menger space and f, g be two self mappings of S. f is said to be a probabilistic g-contraction if, for some $k \in (0, 1)$, the following implication holds for every $p, q \in S$ and $t > 0$:*

$$F_{g(p)g(q)}(t) > 1 - t \Rightarrow F_{f(p)f(q)}(kt) > 1 - kt.$$

The main result of [2] is Theorem 1.2 below:

Theorem 1.2. [2, Theorem 3.] *Let f be a probabilistic g-contraction on a complete Menger space (S, F, T) satisfying that* $\mathrm{Range}(F) \subset D_+$ *and* $\sup_{a<1} T(a, a) = 1$. *If g is bijective, then there is a unique point $p \in S$ such that $f(p) = g(p)$.*

When $g = 1_S$ one obtains Theorem 1.1.

2. The result

We will show that, in turn, Theorem 1.2 can be obtained from the theorem of Radu. As a matter of fact, we will obtain from Theorem 1.1 of Radu a more general result on the existence of the common fixed point.

Theorem 2.1. *Let (S, F, T) be a complete Menger space satisfying that* $\sup_{a<1} T(a, a) = 1$. *If $g : X \to X$ is a given surjection and $f : S \to S$ is a probabilistic g-contraction, then there is $u \in S$ such that $f(u) = g(u)$.*

Proof. It is well known that a mapping $s : A \to B$ is surjective if and only if it has a right inverse that is, there is a mapping $j : B \to A$ such that

$$s \circ j = 1_B.$$

Therefore, for some mapping $h : S \to S$, we have

$$g \circ h = 1_S.$$

Now, let $r, s \in S$ and $t > 0$ be such that $F_{rs}(t) > 1-t$. As this inequality can be rewritten as $F_{\ g\circ h(r) g\circ h(s)}(t) > 1 - t$ and f is a g-contraction, it follows that $F_{f\circ h(r) f\circ h(s)}(kt) > 1 - kt$. Therefore, the implication

$$F_{rs}(t) > 1 - t \Rightarrow F_{f\circ h(r) f\circ h(s)}(kt) > 1 - kt$$

holds for all $x, y \in S$ and $t > 0$. This says that $f \circ h$ is a C-contraction on the complete Menger space (S, F, T) and hence, from Theorem 1.1, there is $w \in S$ such that $f(h(w)) = w$. Let $u = h(w)$. As $w = g(h(w))$, one obtains $f(u) = g(u)\ (= w)$, as desired.

If in addition g is injective (that is, it is a bijection) then $h = g^{-1}$, hence $f \circ g^{-1}$ is a C-contraction, and $f(u) = g(u)$ implies $f \circ g^{-1}(w) = w$ where $w = g(u)$, that is $w = g(u)$ is a fixed point for $f \circ g^{-1}$. If v satisfies

$f(v) = g(v)$, from the uniqueness of the fixed point of a C-contraction it follows that $g(u) = g(v)$ and then, from the injectivity of g, $u = v$.

Note that generally, if f is a probabilistic g-contraction and u, v are common fixed points of f and g, then $g(u) = g(v)$. Indeed, since $f(u) = g(u)$ and $f(v) = g(v)$, if $F_{g(u)g(v)}(t) > 1 - t$ for some $t > 0$, then $F_{g(u)g(v)}(kt) > 1 - kt$ and, by induction, $F_{g(u)g(v)}(k^n t) > 1 - k^n t$, $\forall t > 0$. Taking the limit as $n \to \infty$ one obtains $F_{g(u)g(v)}(0+) = 1$, which is possible only if $g(u) = g(v)$. As the inequality $F_{g(u)g(v)}(t) > 1 - t$ holds for $t > 1$, it follows that $g(u) = g(v)$. $\square$

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