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UNIVERSALITY RESULTS FOR WELL-FOUNDED POSETS

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Abstract. In this paper it is shown that the univerality spectrum of well-founded posets is exactly the same as the spectrum of the class of well-orders. A universality result for a restricted class of well-founded posets under rank and order preserving embeddings is also proved. This is done using a club guessing method generalised by Kojman which demonstrates a surjective homomorphism with subsets of the reals of bounded size ordered by inclusion.

1. Introduction

Well-founded partial orders arise naturally in many different areas of mathematics. In the context of universality, they are interesting structures because they provide a simple example of a theory which is not first order.

To clarify our use of universality, we will present the basic definitions. *Embeddings* for ordered sets are normally injective order-preserving maps, but other types of embeddings will be considered. This will be the definition of the embedding unless otherwise specified. Given a set of structures, $\mathcal{A}_\lambda$ each of size λ, a *universal model* for $\mathcal{A}_\lambda$ is one which embeds all other structures in $\mathcal{A}_\lambda$. If there does not exist a universal model for $\mathcal{A}_\lambda$, then we consider its *complexity*, or the smallest size of a family of structures of $\mathcal{A}_\lambda$ which embeds the rest. Such a family of structures is called a *universal family.* The complexity of a class is also referred to as the cofinality by some authors. The *universal spectrum* for a class of structures $\mathcal{A}$ is the family of cardinals for which $\mathcal{A}$ has a universal model (given cardinal arithmetic assumptions).

In Section 2 we will give an account of the universality spectrum of well-founded partial orders under the usual embeddings.

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Recall that a *well-founded partial order* is a partial order which does not have infinite decreasing sequences. We sometimes abbreviate well-founded partial order as WFPO. Kojman and Shelah in [3] prove in ZFC that if any complete first order theory with the strict order property has a universal model in some power, then so does the theory of linear orders. If we restrict ourselves to well-founded partial orders, then we cannot use the results in [3] since the strict order property only applies to complete first order theories. In [9] it is proved that every partial order has an extension to a linear order. We also cannot immediately use this extension to deny the existence of universal WFPOs as the extending linear orders are generally not well-founded (see [6]).

However, in [1] one can find a proof that every WFPO has a linear augmentation that is well ordered. So any universal well order is also a universal WFPO. Moreover, we shall show that WFPOs have universal models in exactly the cardinalities that ordinals do, namely that for every $\lambda \geq \aleph_0$ there are λ^+ many well orders which are jointly universal for all well-founded partial orders of size λ. Thus, the universality spectrum for both ordinals and WFPOs is only the set of finite cardinals.

In Section 3, we shall consider a different type of universality. We will restrict both the set of structures and the embedding to get a different universality result for a certain kind of well-founded partial orders. This is done via a club guessing method first used by Shelah and Kojman in [3] and then generalized by Kojman in [2]. Kojman's method involves finding a surjective homomorphism between certain subsets of $\mathcal{P}(\omega)$ ordered by the subset relation and the structures in question ordered by embeddability.

The embeddings considered in that section preserve rank as well as order. This type of embedding has been mostly considered in the context of trees (see e.g. [10]).

The main result then is as follows. Let $\mathcal{A}_\lambda$ be the set of isomorphism types of well-founded posets A of size λ regular for $\aleph_1 < \lambda < 2^{\aleph_0}$ such that for all $x \in A$ and $\alpha < \mathrm{rk}(x)$ we have that $|\{y \leq x : \mathrm{rk}(y) = \alpha\}| < \lambda$ and $|\{z \in A : x \leq z\}| < \lambda$. Then the complexity of $\mathcal{A}_\lambda$ under rank and order preserving embeddings is at least $2^{\aleph_0}$.

We do not know if our result can be obtained without the use of the club guessing method, or if it is consistent for our class to have complexity $\aleph_1 < 2^{\aleph_0}$ at $\aleph_1$. A consistency proof of such a complexity is provided by Shelah in [7], where there is a universal linear order of size $\aleph_1$. It is not clear if this complex method could be applied to our class of structures. There is also the method for proving positive consistency results about universality due to Džamonja and Shelah in [5]. It applies to abstract elementary classes (with certain additional properties) but the class of well-founded posets is not an abstract elementary class because it is not closed under increasing unions.

2. Well-founded posets and well-orders

We will first examine the universal spectrum of well-orders under ZFC with the usual embeddings and show how this is related to the universal spectrum of well-founded partial orders. The results regarding these spectra do not seem to appear in the literature even though the proofs are easy corollaries of known results.

Fraisse in [1, Section 2.9.2] showed that any well-founded partial can be augmented to a well-order. This can be done by well-ordering the antichain of elements at each rank of the poset. These augmentations do not affect embeddings since the original order is preserved. Thus, the existence of a universal well-order in a given cardinal is equivalent to the existence of a universal well-founded partial order in that cardinal. Moreover, the two classes must have the same complexity in every cardinality.

Fact 2.1. For any infinite cardinal λ, the complexity of the set of well-orders of size $< \lambda$ is $\mathrm{cf}(\lambda)$.

In particular, the complexity of well-orders of size λ is λ^+. Fewer than λ^+ well-orders do not suffice since by the regularity of λ^+ there would be an ordinal with order type larger than all of them.

So we may say that for an infinite cardinal λ, the complexity of the set of WFPOs of size λ is λ^+. This fact is non-trivial if $\lambda^+ < 2^\lambda$. That is, λ^+ would form a "small" universal family, i.e. a family whose size is less than the number of isomorphism types of WFPOs of size λ.

3. A universality result using the preservation/construction method

In this section we will use a stronger embedding and get a result about the complexity of a restricted class of well-founded partial orders of a certain size under these embeddings. The embedding will preserve not only order, but rank as well, a notion introduced in the last section. First, we will give an intuitive idea of the type of argument used in the proof.

Kojman in [2] generalized a method used in [3] to show that graphs omitting a certain type of subgraph have no universal model at certain $\lambda \in (\aleph_1, 2^{\aleph_0})$, in fact they have complexity at least $\max\{\lambda^+, 2^{\aleph_0}\}$. We will call this the preservation/construction method. We shall use this method in a certain class of well-founded partial orders.

Now we will fix a regular cardinal $\lambda \in (\aleph_1, 2^{\aleph_0})$, so in particular we assume CH fails.

In the next result, we will use the preservation/construction method for certain well-founded partial orders. Given a WFPO with universe λ, one assigns an index to club many elements of λ (that is, the set of elements of λ to which we assign an index forms a club of λ). Modulo a club guessing sequence, club many of these indices are PRESERVED under embeddings. Then we show that for every infinite subset of ω we can CONSTRUCT a well-founded partial order with universe λ which has that set as the index of club many elements. The contradiction comes as the universal must have an element corresponding to all $2^{\aleph_0}$ many infinite subsets of ω. The definition of the index can be quite complicated and in this case, it is only fixed up to a filtration. Since any two filtrations agree on a club, the index is fixed modulo the club filter on λ and thus the preservation is obtained when we use a club guessing sequence.

To prove the PRESERVATION property, we construct a surjective homomorphism from a set of structures of size λ, to subsets of (a certain set of the size of) the continuum. A homomorphism in this context is a map which preserves a specified ordering. We will define the homomorphism for our context.

Notation 3.1. If $A_1, A_2 \in \mathcal{A}_\lambda$ are structures of size λ, then $A_1 \preceq A_2$ means that there exists an embedding $f : A_1 \to A_2$.

The homomorphism Φ that we will study is from $\langle \mathcal{A}_\lambda, \preceq \rangle$, that is, the set of structures $\mathcal{A}_\lambda$ together with the embedding order, to a certain structure $\mathcal{P}$ ordered by the subset relation, denoted by $\langle \mathcal{P}, \subseteq \rangle$. We shall define $\mathcal{P}$ below. To say that the homomorphism preserves these relations means that if $A_1, A_2 \in \mathcal{A}_\lambda$ with $A_1 \preceq A_2$ then $\Phi(A_1) \subseteq \Phi(A_2)$.

Let $\bar{\mathcal{P}}(\omega) = \mathcal{P}(\omega)/\mathrm{Fin}$, where Fin is the ideal of all finite subsets of ω. Let

$$\mathcal{P} = \{B \in [\bar{\mathcal{P}}(\omega)]^{\leq \lambda} : \emptyset \in B\}.$$

One can see that when the surjective homomorphism from $\mathcal{A}_\lambda$ to $\mathcal{P}$ exists at $\lambda \in (\aleph_1, 2^{\aleph_0})$, the complexity of the set $\mathcal{A}_\lambda$ is at least $2^{\aleph_0}$. In fact, if Φ is a homomorphism from one quasi-ordered set with cofinal range in another, then the complexity of the first set is at least that of the second. The theorem specifically for graphs forbidding a certain subgraph is due to Kojman in [2].

In particular, no universal model exists. This will be the CONSTRUCTION lemma as defined above. That is, given that the homomorphism preserves the ordering, we can construct an element that cannot embed into any set of $< 2^{\aleph_0}$ structures.

Now our aim is to show that this homomorphism exists for certain families of well-founded partial orders. However, we must use a different type of embedding than the usual one for ordered sets. If we used the usual order preserving embedding then every member of the class would embed into an ordinal of the same size. Therefore, the complexity would be as discussed in Section 2. We will give here the definition of the embedding that we use.

Definition 3.2. *Let $(A, \leq_A)$, $(B, \leq_B)$ be well-founded partial orders of size λ. A* rank preserving embedding *is a (one-to-one) map $f : A \to B$ where for all $x, y \in A$ we have $x \leq_A y$ implies that $f(x) \leq_B f(y)$ and $\mathrm{rk}_A(x) = \mathrm{rk}_B(f(x))$.*

So the embedding preserves order and rank. Note that this does not imply that incomparability is preserved. The result given will hold for injective and non-injective embeddings. Embeddings like these (without one-to-one) have been considered extensively in the literature, see e.g. Todorčević and Väänänen's paper [10]. Now that we have defined the embedding, we will concentrate on the homomorphism.

Using part of [2, Th. 1.10] we can reduce the homomorphism we are looking for to one modulo a normal ideal.

First we will explain the notation $(\mathcal{P}, \subseteq)^\lambda / I$ used below. Let $s : \lambda \to \mathcal{P}$ and $t : \lambda \to \mathcal{P}$ so for all $\alpha < \lambda$ we have $s(\alpha) \in \mathcal{P}$ and similarly for t. Then $s \subseteq_I t$ if and only if $\{\alpha : s(\alpha) \not\subseteq t(\alpha)\} \in I$.

Below is the precise statement of the reduction from [2] just mentioned.

Lemma 3.3. *Let λ be a regular infinite cardinal. For every proper ideal I over λ, if there exists $\phi : \langle \mathcal{A}_\lambda, \preceq \rangle \to (\mathcal{P}, \subseteq)^\lambda / I$ a surjective homomorphism then there exists $\Phi : \langle \mathcal{A}_\lambda, \preceq \rangle \to (\mathcal{P}, \subseteq)$ a surjective homomorphism.*

Proof. We will construct $\psi : (\mathcal{P}, \subseteq)^\lambda / I \to (\mathcal{P}, \subseteq)$ such that $\psi \circ \phi = \Phi$. That is, we will show that $(\mathcal{P}, \subseteq)$ is a homomorphic image of $(\mathcal{P}, \subseteq)^\lambda / I$ for every ideal I. Suppose that $\bar{A} = \langle A(\delta) : \delta < \lambda \rangle$ is a representative of an equivalence class of $(\mathcal{P}, \subseteq)^\lambda / I$. Define

$$\psi([\bar{A}]) := \{x \in \bar{\mathcal{P}}(\omega) : \{\delta < \lambda : x \in \bar{A}(\delta)\} \notin I\}.$$

In words, $\psi([\bar{A}])$ is the set of all elements of $\mathcal{P}$ which appear in a positive set of coordinates.

One can check that the definition of $\psi([\bar{A}])$ does not depend on the choice of representative. So suppose $\langle A(\delta) : \delta < \lambda \rangle =_I \langle B(\delta) : \delta < \lambda \rangle$. This means that $\{\delta < \lambda : A(\delta) \neq B(\delta)\} \in I$. Hence for any $x \in \bar{\mathcal{P}}(\omega)$ we have $\{\delta < \lambda : x \in \bar{A}(\delta)\} \notin I$ if and only if $\{\delta < \lambda : x \in \bar{B}(\delta)\} \notin I$.

To see that ψ is a homomorphism, suppose $\langle A(\delta) : \delta < \lambda \rangle \subseteq_I \langle B(\delta) : \delta < \lambda \rangle$. Hence for every $x \in \bar{\mathcal{P}}(\omega)$ we have $\{\delta < \lambda : x \in A(\delta)$ does not imply $x \in B(\delta)\} \in I$.

Finally, we show that ψ is surjective. Given $A \in \mathcal{P}$ let $\bar{A}$ be the equivalence class of $\langle A(\delta) : \delta < \lambda \rangle$ where each $A(\delta) = A$. Then $\psi([\bar{A}]) = \{x \in$

$\bar{\mathcal{P}}(\omega) : \{\delta < \lambda : x \in A(\delta)\} \notin I\} = \{x \in \bar{\mathcal{P}}(\omega) : x \in A\} = A$. Here we use that I is proper. $\square$

Now we will restrict ourselves to finding this homomorphism for certain WFPOs. Let $\mathcal{A}_\lambda$ be the set of well-founded partial orders A with size λ and such that the following two conditions hold:

1. For all $x \in A$ and $\alpha < \mathrm{rk}(x)$ we have that $|\{y \leq x : \mathrm{rk}(y) = \alpha\}| < \lambda$,
2. For all $x \in A$ we have $|\{z \in A : x \leq z\}| < \lambda$.

Examples of such structures include any disjoint union λ copies of an ordinal $< \lambda$. Let $\langle \mathcal{A}_\lambda, \preceq \rangle$ be the set defined above with the rank preserving embedding.

Theorem 3.4. *Suppose that $\lambda > \aleph_1$ is regular. Then there is a proper, normal ideal I over λ such that there exists a surjective homomorphism $\phi : \langle \mathcal{A}_\lambda, \leq \rangle \to (\mathcal{P}, \subseteq)^\lambda / I$.*

Proof. We will first define the ideal I and show that it is normal and λ-complete. Then we define the homomorphism ϕ and show that this is indeed a surjective homomorphism.

Fix a club guessing sequence $\bar{C} = \langle c_\delta : \delta \in S \rangle$ such that $S \subseteq \lambda$ is stationary and $\mathrm{type}(c_\delta) = \omega$ for all $c_\delta \in \bar{C}$. This sequence is known to exist, see Shelah's introduction to club guessing in [8] for more information. For each $\delta \in S$, let $\langle \alpha_n^\delta : n < \omega \rangle$ be the increasing enumeration of c_δ. The notation used below, $A \subseteq^* B$, means that all but finitely many members of A are also members of B.

Let

$$I := \{X \subseteq \lambda : (\exists E \subseteq \lambda \text{ a club})\ (\forall \delta \in X \cap S)\ c_\delta \not\subseteq^* E\}.$$

One can easily check that this is an ideal. Since $\bar{C}$ is a club guessing sequence, I is proper, that is, $\lambda \notin I$.

The ideal is normal and λ-complete as we now show. $\square$

Lemma 3.5. *The ideal I, defined above, is normal, i.e. it is closed under diagonal unions of length λ.*

Proof. Given a sequence $\langle N_\alpha : \alpha < \lambda \rangle$ such that each N_α is in I, let $N = \bigtriangledown\{N_\alpha : \alpha < \lambda\}$.

For each $\alpha < \lambda$ let E_α witness the fact that N_α is in I. The diagonal intersection of the witnesses

$$E = \triangle_{\alpha < \lambda} E_\alpha = \{\beta : (\forall \alpha < \beta)\ \beta \in E_\alpha\},$$

is itself a club, as it is the diagonal intersection of λ club subsets of λ (the proof of this statement can be found in [4, Ch. II, Lemma 6.14]).

Now we will show that E witnesses that $N \in I$. If E does not witness that $N \in I$ then there exists a $\delta \in N$ such that $c_\delta \subseteq^* E$. We know that there is an $\alpha < \delta$ such that $c_\delta \not\subseteq^* E_\alpha$ by the definition of N. Let $\beta \in c_\delta \setminus (\alpha+1)$ be arbitrary such that $\beta \in E$. Such a β exists as it is assumed that $c_\delta \subseteq^* E$ and c_δ is unbounded in δ. By definition of E we have $\beta \in E_\alpha$ since $\beta > \alpha$. Hence we are left with

$$c_\delta \subseteq^* \{\beta \in c_\delta : \beta \in E \setminus (\alpha+1)\} \subseteq \{\beta \in c_\delta : \beta \in E_\alpha\} \subseteq E_\alpha,$$

which contradicts our assumptions on c_δ. □

Lemma 3.6. *Let λ be as above. Then the ideal I given in the proof of Theorem* 3.4 *is λ-complete, i.e. it is closed under unions of size $< \lambda$.*

Proof. Given an ordinal $i^* < \lambda$ and a sequence $\langle X_i : i < i^* \rangle$ of members of I, we will show that $X = \bigcup_{i<i^*} X_i$ is a member of I.

For each $i < i^*$, fix a club E_i such that E_i witnesses the fact that X_i is in I. Let $E = \bigcap_{i<i^*} E_i$. We know that E is a club since it is the intersection of fewer than λ clubs of λ (again, this fact and proof can be found in [4, Ch. 2, Lemma 6.8]).

Now we will show that E witnesses that X is in I. For each $\delta \in X$, there exists an $\alpha < i^*$ such that $\delta \in X_\alpha$, thus $c_\delta \not\subseteq^* E_\alpha$. Suppose, contrary to the statement of the lemma, that $c_\delta \subseteq^* E$ for some $\delta \in X$. However by the definition of E, we get $c_\delta \subseteq^* E \subseteq E_\alpha$, which is a contradiction. □

Now by Lemma 3.3 it suffices to define the homomorphism modulo the proper ideal I. Since in the homomorphism we are only concerned with preservation of relations modulo I, we can work with representatives of each equivalence class of I.

Since each $A \in \mathcal{A}_\lambda$ has an isomorphic copy whose universe is $\lambda \times \lambda \times 2$ we shall only work with such orders. This is because such a coding will allow us to construct the appropriate elements of $\mathcal{A}_\lambda$ in the surjection argument in Lemma 3.9.

In order to define the homomorphism ϕ, we must introduce a couple of auxiliary functions that might be different for each $A \in \mathcal{A}_\lambda$, but will be fixed now for the rest of the proof. Let A be such a WFPO and let $A^\alpha := \lambda \times \alpha \times 2$ for $\alpha < \lambda$. Also, if $(A, \leq_A)$ is a well-founded partial order and $x \in A$ then define $A[x] := \{y \in A : y \leq_A x\}$.

Now, fix $A \in \mathcal{A}_\lambda$ (with universe $\lambda \times \lambda \times 2$). Recall that α_n^δ is the n-th term in the enumeration of c_δ. Also recall that c_δ has order type ω and is an element of the club guessing sequence we fixed at the beginning of this proof. For $x \in A$ and $\delta < \lambda$ let

$$\phi(A, x, \delta) = \begin{cases} [\{n < \omega : A[x] \cap A^{\alpha_n^\delta} \neq A[x] \cap A^{\alpha_{n+1}^\delta}\}]_{\mathrm{Fin}} & \text{if } \delta \in S \\ \emptyset \text{ otherwise} \end{cases}$$

$$\phi(A,\delta) = \{\phi(A,x,\delta) : x \in A\} \cup \{\emptyset\}$$
$$\phi(A) = [\langle \phi(A,\delta) : \delta < \lambda \rangle]_I.$$

One can see that $\phi(A)$ is a member of $\mathcal{P}^\lambda/I$ as $\phi(A,x,\delta) \in \bar{\mathcal{P}}(\omega)$ and $\phi(A,\delta) \subseteq [\bar{\mathcal{P}}(\omega)]^{\leq\lambda}$ while $\emptyset \in \phi(A,\delta)$. One can see that $\phi(A,x,\delta)$ is a subset of ω mod Fin. Since $|A| = \lambda$ then $|\phi(A,\delta)| \leq \lambda$.

We need to show that ϕ is a homomorphism as required. This next lemma corresponds to the PRESERVATION lemma in [3] where the invariance preserved here is the subset relation on the reals.

Lemma 3.7. *The map ϕ as defined above is a homomorphism. In particular, if $A_1, A_2 \in \mathcal{A}_\lambda$, then $A_1 \preceq A_2$ implies that $\phi(A_1) \subseteq \phi(A_2)$ modulo I.*

Proof. Let A_1, A_2 be as in the statement of the lemma. We need to show that
$$S \setminus \{\delta \in S : \phi(A_1,\delta) \subseteq \phi(A_2,\delta)\} \in I.$$
Since we are assuming that $A_1 \preceq A_2$, there is a rank preserving embedding (in the sense of Definition 3.2) $f : A_1 \to A_2$. In this proof we use the notation $(\alpha,\gamma,l) < \delta$ for $\alpha,\gamma < \lambda$ and $l < 2$ to mean $\alpha,\gamma < \delta$. Also, for $X \subseteq \lambda \times \lambda \times 2$ we define $\sup(X)$ to be $\sup(\{\alpha : (\exists\beta,l)(\alpha,\beta,l) \in X$ or $(\beta,\alpha,l) \in X\})$. Let

$$\begin{aligned} C = \{\delta < \lambda :& (\forall\alpha,\gamma < \delta)\,(\forall l < 2) f(\alpha,\gamma,l) < \delta, \text{ and} \\ & (\forall\alpha,\gamma < \delta \text{ and } \forall l < 2 \text{ such that } f^{-1}(\alpha,\gamma,l) \text{ is well-defined}) \\ & f^{-1}(\alpha,\gamma,l) < \delta\} \end{aligned}$$

and let D be the set of all $\delta < \lambda$ such that for all $\alpha,\gamma < \delta$ for all $l < 2$ and for all β,ε,m, if there exists $z \geq_{A_i} (\alpha,\gamma,l), (\beta,\varepsilon,m)$ then we have that $\mathrm{rk}_{A_i}((\alpha,\gamma,l)) = \mathrm{rk}_{A_i}((\beta,\varepsilon,m))$ implies $\beta,\varepsilon < \delta$ for $i = 1,2$. □

Claim 3.8. If $A_1 \preceq A_2$ and f is as above then $E = C \cap D$ is a club subset of λ.

Proof. This proof will be given in two parts; both C and D will be shown to form a club. Then E, as the intersection of these two clubs is a club.

We will first show that C is unbounded in λ, that is, given $\beta < \lambda$ we can find $\alpha \in (\beta,\lambda)$ such that $\alpha \in C$. To this end, we will define a sequence $\langle \beta_n : n < \omega \rangle$ with $\beta_0 = \beta$ and $\beta_\omega = \sup_{n<\omega} \beta_n = \alpha$. Let

$$\begin{aligned} \beta_{n+1} = \sup\big(&\{f(\delta,\varepsilon,m) : \delta,\varepsilon < \beta_n \text{ and } m < 2\} \\ & \cup \{f^{-1}(\alpha,\gamma,l) : \alpha,\gamma < \beta_n,\, l < 2 \text{ and } f^{-1}(\alpha,\gamma,l) \text{ is well-defined}\}\big) \\ & \cup (\beta_n + 1). \end{aligned}$$

Note that each $\beta_n < \lambda$.

To show that $\alpha \in C$, note that for all $\delta, \varepsilon < \alpha$ and $m < 2$ there exists an $n < \omega$ such that $\delta, \varepsilon < \beta_n$. Therefore, there also exists an $n < \omega$ such that $f(\delta, \varepsilon, m) < \beta_{n+1}$ and so we have $f(\delta, \varepsilon, m) < \alpha$.

On the other hand, if we have $f(\delta, \varepsilon, m) < \alpha$ then there exists an $n < \omega$ with $f(\delta, \varepsilon, m) < \beta_n$ and so there exists $n < \omega$ with $(\delta, \varepsilon, m) < \beta_n$ so $f^{-1}(\delta, \varepsilon, m) < \beta_{n+1} < \alpha$.

To show that C is closed, suppose $\sup(C \cap \delta) = \delta$ and thus, δ is a limit ordinal. Let $\langle \beta_\xi : \xi < \xi^* \rangle$ be a sequence increasing to δ such that $\beta_\xi \in C$ for all $\xi < \xi^*$. So for all $\alpha, \gamma < \delta$ and $l < 2$, there exists $\xi < \xi^*$ such that $\alpha, \gamma < \beta_\xi$. Thus, $f(\alpha, \gamma, l) < \beta_\xi < \delta$ as $\beta_\xi \in C$. If $\alpha, \gamma < \delta$ and $l < 2$ are such that $f^{-1}(\alpha, \gamma, l)$ is defined, then similarly, we can find a $\beta_\xi < \delta$ such that $\alpha, \gamma < \beta_\xi$. Again, $f^{-1}(\alpha, \gamma, l) < \beta_\xi < \delta$ as $\beta_\xi \in C$.

Now we will show that D is unbounded in λ. So given $\alpha_0 < \lambda$ we define $\alpha_{n+1} > \alpha_0$ by induction such that $\alpha_\omega = \sup\{\alpha_n : n < \omega\} \in D$. To that end, assume α_n is defined and let $\alpha_{n+1} = \sup\{\beta : (\exists i \in \{1,2\})(\exists \alpha, \gamma < \alpha_n)(\exists l < 2)(\exists \varepsilon, m)(\exists z \geq_{A_i} (\alpha, \gamma, l), (\beta, \varepsilon, m))\mathrm{rk}_{A_i}(\alpha, \gamma, l) = \mathrm{rk}_{A_i}(\beta, \varepsilon, m)\} + 1$. By our assumptions on $\mathcal{A}_\lambda$, we know that $\alpha_{n+1} < \lambda$. Namely, for each α the set $B_\alpha = \{\beta : (\exists i \in \{1,2\})(\exists \varepsilon, m, \gamma, l)\ \mathrm{rk}_{A_i}(\beta, \varepsilon, m) = \mathrm{rk}_{A_i}(\alpha, \gamma, l)$ and $\exists z \geq_{A_i} (\alpha, \gamma, l), (\beta, \varepsilon, m)\}$ has size $< \lambda$, so $\alpha_{n+1} = \sup \bigcup_{\alpha < \alpha_n} B_\alpha$ is $< \lambda$.

Let $i \in \{1,2\}$ be given. For α_ω as defined above, let $\alpha, \gamma < \alpha_\omega$ and β, ε, m, l be such that $\mathrm{rk}_{A_i}(\alpha, \gamma, l) = \mathrm{rk}_{A_i}(\beta, \varepsilon, m)$ and there exists $z \geq_{A_i} (\alpha, \gamma, l)$, (β, ε, m). Then there exists $n < \omega$ such that $\alpha, \gamma < \alpha_n$ and hence $\beta, \varepsilon < \alpha_{n+1} < \alpha_\omega$.

To show that D is closed, let $\delta = \sup(D \cap \delta)$ and we will show that $\delta \in D$. Given $\alpha, \gamma < \delta$ and β, ε, m, l such that there exists $z \geq_{A_i} (\alpha, \gamma, l), (\beta, \varepsilon, m)$ and $\mathrm{rk}_{A_i}(\alpha, \gamma, l) = \mathrm{rk}_{A_i}(\beta, \varepsilon, m)$, we will show that $\beta, \varepsilon < \delta$. There exists $\eta < \delta$ such that $\eta \in D$ and $\alpha, \gamma < \eta$ as δ is a limit. Since $\eta \in D$ we can conclude that $\beta, \varepsilon < \eta < \delta$. □

If we define $N(E)$ to be $\{\delta \in S : c_\delta \subseteq^* E\}$ then we have that $S \setminus N(E) \in I$ by definition of I. This is because these are exactly the $\delta \in S$ for which the club E witnesses that $c_\delta \not\subseteq^* E$.

Now we need to show that for all $\delta \in N(E)$, we have $\phi(A_1, \delta) \subseteq \phi(A_2, \delta)$. This implies that $\phi(A_1) \subseteq \phi(A_2)$ modulo I as required.

Let $\delta \in N(E)$ and we will show that for each $x \in A_1$ we have $\phi(A_1, x, \delta) \in \phi(A_2, \delta)$. In fact, we will demonstrate that $\phi(A_1, x, \delta) = \phi(A_2, f(x), \delta)$ modulo Fin and this will suffice as clearly $\phi(A_2, f(x), \delta) \in \phi(A_2, \delta)$.

So for each $x \in A_1$, we will first consider $n \in \phi(A_1, x, \delta)$ and show that $n \in \phi(A_2, f(x), \delta)$. We may assume that $\alpha^\delta_{n+1}, \alpha^\delta_n \in E$ as this is the case for all but finitely many n. By the definition of ϕ, we have that there exists $y \leq_{A_1} x$ such that $y \in A_1^{\alpha^\delta_{n+1}} \setminus A_1^{\alpha^\delta_n}$. By the definition of E, we have that $f(y)$ must also be in $A_2^{\alpha^\delta_{n+1}} \setminus A_2^{\alpha^\delta_n}$. Because of the order preservation in the embedding, it is the case that $f(y) \leq_{A_2} f(x)$. This shows that $n \in \phi(A_2, f(x), \delta)$.

In the other direction, we will now consider $n \in \phi(A_2, f(x), \delta)$ for $x \in A_1$. Again we may assume that $\alpha^\delta_{n+1}, \alpha^\delta_n \in E$. In this case there exists $w \leq_{A_2} f(x)$ such that $w \in A_2^{\alpha^\delta_{n+1}} \setminus A_2^{\alpha^\delta_n}$. Since $w \leq_{A_2} f(x)$ we have that the $\mathrm{rk}_{A_2}(w) \leq \mathrm{rk}_{A_2}(f(x)) = \mathrm{rk}_{A_1}(x)$ by the rank preservation of the embedding. Hence, there exists $z \leq_{A_1} x$ such that $\mathrm{rk}_{A_1}(z) = \mathrm{rk}_{A_2}(w)$. This means that $f(z) \leq_{A_2} f(x)$ by the order preservation of the embedding and therefore, $\mathrm{rk}_{A_2}(f(z)) = \mathrm{rk}_{A_1}(z) = \mathrm{rk}_{A_2}(w)$. This shows that since $w \in A_2^{\alpha^\delta_{n+1}}$, also $f(z) \in A_2^{\alpha^\delta_{n+1}}$ by the definition of D.

Now suppose that $f(z) \in A_2^{\alpha^\delta_n}$ and we will arrive at a contradiction. There exists $a \geq_{A_2} w, f(z)$ namely $a = f(x)$, and we know that $\mathrm{rk}_{A_2}(w) = \mathrm{rk}_{A_1}(z)$. Now by the definition of D we have $w \in A_2^{\alpha^\delta_n}$, which contradicts our assumptions on w.

So we can conclude that $f(z) \in A_2^{\alpha^\delta_{n+1}} \setminus A_2^{\alpha^\delta_n}$ and thus, $z \in A_1^{\alpha^\delta_{n+1}} \setminus A_1^{\alpha^\delta_n}$. This z witnesses that $n \in \phi(A_1, x, \delta)$. □

This concludes the proof that ϕ is a homomorphism. We will now see that it is surjective as well.

Lemma 3.9. *Given any $B \in (\mathcal{P})^\lambda$, there exists an $A \in \mathcal{A}_\lambda$ such that $\phi(A) = [B]_I$.*

Proof. We begin by decomposing the elements of B into the corresponding subsets of ω and then we will construct A to match.

List the members of B by $\langle B_\alpha : \alpha < \lambda \rangle$ so that $B_\alpha \in \mathcal{P}$. For all $\delta \in S$ let $B_\delta = \langle X_{\delta,\alpha} : \alpha < \alpha^*(\delta) \rangle$ for some $\alpha^*(\delta) \leq \lambda$, that is, $X_{\delta,\alpha} \in \bar{\mathcal{P}}(\omega)$ and $\emptyset \in X_{\delta,\alpha}$. Then we shall list the members of $X_{\delta,\alpha}$ (that is, the representatives of the Fin equivalence classes) increasingly as $\langle \beta^{\delta,\alpha}_n : n < \omega \rangle$ so that each $\beta^{\delta,\alpha}_n \subseteq \omega$.

For the WFPO $(A, \leq_A)$, let the universe $A = L \cup R$ where $L = \lambda \times \lambda \times \{0\}$ and $R = \lambda \times \lambda \times \{1\}$. Let h be a bijection from $\lambda \times \lambda$ into λ.

We shall define the relations $\leq_A$ on A in the following way. For every $\delta \in S$ and $\gamma < \alpha^*(\delta)$, find a sequence $\langle e^{\delta,\gamma}_n : n < \omega \rangle$ in R such that for each $n < \omega$ we have $e^{\delta,\gamma}_n = (h(\delta, \gamma), d^{\delta,\gamma}_n, 1)$ and $d^{\delta,\gamma}_n = \alpha^\delta_{\beta^{\delta,\gamma}_n}$. In particular note that the $\min\{\alpha < \delta : e^{\delta,\gamma}_n \in A^\alpha\} = \alpha^\delta_{\beta^{\delta,\gamma}_n} + 1$. Connect each $e^{\delta,\gamma}_n$ to $x = (\delta, \gamma, 0) \in L$, by making $e^{\delta,\gamma}_n \leq_A x$ for all $n < \omega$. These will be the only relations on A.

This is clearly transitive, so A is a poset. One can show that $A \in \mathcal{A}_\lambda$ as $|A| = \lambda$ and there are no chains of size > 2 in A, so it is well-founded. The other conditions on $A \in \mathcal{A}_\lambda$ follow as each point has a maximum of $\aleph_0$ points less than it and a maximum of one point greater than it. Also, the universe of A is $\lambda \times \lambda \times 2$. So we can now explore what $\phi(A)$ is. We will show that $\phi(A) = [B]_I$ for A defined as above.

There exists a club subset $E \subseteq \lambda$ such that for all $\alpha \in E$ it holds that for all δ, γ we have $\delta, \gamma < \alpha$ iff $h(\delta, \gamma) < \alpha$. Namely, $E = \{\alpha < \lambda : h$ is a bijection from $\alpha \times \alpha$ into $\alpha\}$. This can be shown to be a club by a proof very similar to that of Claim 3.8.

Suppose that $\delta \in N(E)$ (recall the definition of $N(E)$ given in the proof of Lemma 3.7) and let $\gamma < \alpha^*_\delta$ be given. For $x = (\delta, \gamma, 0) \in L$, we will show that $\phi(A, x, \delta) = X_{\delta,\gamma}$:

$$\begin{aligned}\phi(A, x, \delta) &= [\{n < \omega : A[x] \cap A^{\alpha^\delta_n} \neq A[x] \cap A^{\alpha^\delta_{n+1}}\}]_{\mathrm{Fin}}\\ &= [\{n < \omega : (\exists m < \omega)\ e^{\delta,\gamma}_m \leq_A x \text{ and } e^{\delta,\gamma}_m \in A^{\alpha^\delta_{n+1}} \setminus A^{\alpha^\delta_n}\}]_{\mathrm{Fin}}\\ &= [\{\beta^{\delta,\gamma}_n : n < \omega\}]_{\mathrm{Fin}}\\ &= X_{\delta,\gamma}.\end{aligned}$$

If $\gamma \in [\alpha^*_\delta, \delta)$ then for $x = (\delta, \gamma, 0)$ we have $A[x] = \emptyset$ so $\phi(A, x, \delta) = \emptyset$. For all $x \in R$ the point x has at most one element comparable to it (and that would be greater than x) so we have $A[x] = \emptyset$ for all $x \in R$ so $\phi(A, x, \delta) = \emptyset$.

Then for all $\delta \in N(E)$ we have

$$\phi(A, \delta) = \{\phi(A, x, \delta) : x \in A\} = \{X_{\delta,\gamma} : \gamma < \lambda\} \cup \{\emptyset\} = B_\delta$$

as $\emptyset \in B_\delta$.

If $\delta \notin N(E)$ then for each x we have $\phi(A, x, \delta) = \emptyset$. Hence $\phi(A, \delta) = \{\emptyset\}$. Hence, $\phi(A) = [\{\phi(A, \delta) : \delta < \lambda\}]_I = B$ because $S \setminus N(E) \in I$. □

This completes the proof of Lemma 3.9. We have shown that ϕ is a surjective homomorphism, also completing the proof of Theorem 3.4. □

The existence of such a homomorphism proves the following result by Lemma 3.3.

Theorem 3.10. *The complexity of $\mathcal{A}_\lambda$ for $\lambda \in (\aleph_1, 2^{\aleph_0})$ a regular cardinal under well-founded partial order embeddings is at least $2^{\aleph_0}$.*

Note that because the surjection argument only involves posets of rank 2, we have also shown that the complexity of rank 2 posets of size $\lambda \in (\aleph_1, 2^{\aleph_0})$ a regular cardinal is at least $2^{\aleph_0}$. However, if we restricted our argument only to posets of rank 2, then we would have showed that posets of rank 2 do not have a small universal family which consists of posets of rank 2. This does not suffice for our class of well-founded partial orders as there could be a more general poset that embeds all posets of rank 2.

The theorem above works for regular cardinals λ. Now we will extend the complexity result to some singular cardinals. We will keep the notation and definitions of I, r, ϕ, Φ as in the previous proof for regular cardinals. Now we will need a couple more facts and definitions. We will define a class of subsets of well-founded partial orders which have a certain continuity property in the homomorphism Φ. Here, λ will remain a regular cardinal.

Lemma 3.11. *Let Φ be as in Lemma 3.3. If $A \in \mathcal{A}_\lambda$ and $A = \bigcup_{\alpha<\alpha^*} A_\alpha$ is an increasing union where $\alpha^* < \lambda$ and each A_α is a sub-poset of A for $\alpha < \alpha^*$ then*

$$\Phi(A) \subseteq \bigcup_{\alpha<\alpha^*} \Phi(A_\alpha).$$

Proof. Suppose $B \in \Phi(A)$ and we will demonstrate the existence of an $\alpha < \alpha^*$ such that $B \in \Phi(A_\alpha)$.

For every $\alpha < \alpha^*$ there is a set $X_\alpha \in I$ such that for every $\delta \in \lambda \setminus X_\alpha$ and every $x \in A_\alpha$ we have $\phi(A_\alpha, x, \delta) = \phi(A, f(x), \delta)$ by Lemma 3.7. For all $x \in A_\alpha$ we have $f(x) = x$ so for δ as above,

$$\phi(A_\alpha, x, \delta) = \phi(A, x, \delta).$$

Let $X = \bigcup_{\alpha<\alpha^*} X_\alpha$. Because I is λ-complete by Lemma 3.6 and because $\alpha^* < \lambda$, we deduce that $X \in I$.

Since $B \in \Phi(A)$, the set $Y := \{\delta < \lambda : B \in \phi(A, \delta)\} \notin I$ and in particular, is non-empty. For every $\delta \in Y$, pick x_δ such that $\phi(A, x_\delta, \delta) = B$ and choose some $\alpha_\delta < \alpha^*$ such that $x_\delta \in A_{\alpha_\delta}$.

Since $Y \notin I$, this implies that Y is not the union of $< \lambda$ sets which are all in I since I is λ-complete. For each $\alpha < \alpha^*$ let $Y_\alpha = \{\delta \in Y : \alpha_\delta = \alpha\}$. We know that $Y = \bigcup_{\alpha<\alpha^*} Y_\alpha$ so there is an $\alpha < \alpha^*$ such that $Y_\alpha \notin I$. Fix this α. Now it follows that for every $\delta \in Y_\alpha \setminus X$ it holds that

$$\phi(A_\alpha, x_\delta, \delta) = \phi(A, x_\delta, \delta) = B$$

by the definition of X. Since $Y_\alpha \notin I$, we conclude that $B \in \Phi(A_\alpha)$. $\square$

The following covering lemma can be found in [3].

Lemma 3.12. *Suppose μ is a fixed point of first order (i.e. $\mu = \aleph_\mu$), but not of second order, (i.e. $|\{\gamma < \mu : \gamma$ is a fixed point of first order$\}| = \sigma < \mu$) and further suppose $\max\{\sigma, \mathrm{cf}(\mu)\} < \lambda < \mu$. Then the minimal size of a family $\mathcal{D} \subseteq [\mu]^{<\lambda^+}$, where $\mathcal{D}$ satisfies the property that for all $X \in [\mu]^{<\lambda^+}$ there are $< \lambda$ members of $\mathcal{D}$ whose union covers X, is μ.*

The theorem below is the singular cardinal version of the construction lemma proved for regular cardinals. In contrast to the regular cardinal version, this one will be proved specifically for our well-founded partial orders. The preservation in the homomorphism for regular cardinals will suffice to prove the complexity theorem for singular cardinals.

We will find a regular cardinal below our singular that has the surjective homomorphism defined above and then use a combinatorial argument with the covering lemma to show that this contradicts the existence of any small universal family. Note that the existence of a cardinal μ as in Theorem 3.13 strongly violates CH.

Theorem 3.13. *If $\mu < 2^{\aleph_0}$ satisfies the requirements in Lemma 3.12 and $\mathrm{cf}(\mu) \geq \aleph_1$, then the complexity of $(\mathcal{A}_\mu, \preceq)$ is $\geq 2^{\aleph_0}$.*

Proof. Let μ be as in the statement of the theorem and let $\lambda \in (\mathrm{cf}(\mu), \mu)$ be any regular cardinal. Suppose for contradiction that $\kappa < 2^{\aleph_0}$ and for every $\alpha < \kappa$ we are given $A_\alpha \in \mathcal{A}_\mu$ so that $\{A_\alpha : \alpha < \kappa\}$ forms a universal family for $\mathcal{A}_\mu$. We will demonstrate the existence of $A \in \mathcal{A}_\lambda$ which does not embed into any A_α for $\alpha < \kappa$.

By the covering Lemma 3.12 above, for every $\alpha < \kappa$, there is a family $\mathcal{F}_\alpha \subseteq [A_\alpha]^\lambda$ with $|\mathcal{F}_\alpha| = \mu$ with the property that for all $X \in [A_\alpha]^\lambda$ there is an $\mathcal{F} \in [\mathcal{F}_\alpha]^{<\lambda}$ so that $X \subseteq \bigcup \mathcal{F}$.

Fix a homomorphism Φ as in Lemma 3.3 for λ. Then if we set $D := \{\Phi(\mathcal{F}) : \mathcal{F} \in \bigcup_{\alpha<\kappa} \mathcal{F}_\alpha\}$ we know that D size $\kappa \times \mu < 2^{\aleph_0}$. Thus, for $\Phi(\mathcal{F}) \in D$ we have $|\bigcup D| < 2^{\aleph_0}$. So we can find a set $B \in \mathcal{P}$ such that $B \not\subseteq \bigcup D$. Using the surjectivity of Φ, we can fix a well-founded partial order $A \in \mathcal{A}_\lambda$ such that $\Phi(A) = B$.

Suppose to the contrary that A embeds into A_α for some $\alpha < \kappa$. Without loss of generality say $A \preceq A_0$. By the covering property of $\mathcal{F}_0$ stated in its definition we can find a subset $\mathcal{F} \in [\mathcal{F}_0]^{<\lambda}$ such that $A \preceq \bigcup \mathcal{F}$. Therefore, note that $\Phi(\bigcup \mathcal{F}) \subseteq \bigcup_{F \in \mathcal{F}} \Phi(F)$ by the continuity property proved in Lemma 3.11. Thus, we have

$$B = \Phi(A) \subseteq \Phi(\bigcup \mathcal{F}) \subseteq \bigcup_{F \in \mathcal{F}} \Phi(F) \subseteq \bigcup D,$$

contradicting the choice of B. ☐

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