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ON THE DUAL BASIS OF PROJECTIVE SEMIMODULES AND ITS APPLICATIONS

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ABSTRACT. The dual basis lemma for projective semimodule over a semiring is proved. We show under which conditions the two categories *cs mod* $-$ *R* and $S -$ *cs mod* of cancellative semimodules are equivalent and how these equivalences are realized.

1. INTRODUCTION

Projective semimodules over semirings are characterized in [2]. Here we generalize one of the classical tools from the theory of modules over rings called the dual basis lemma, for projective semimodule over a semiring. We define generator and progenerator semimodules over semirings and show *under which conditions the two categories* $cs\,mod - R$ *and* $S - cs\,mod$ *of cancellative semimodules are equivalent and how such equivalences are realized.*

2. RESULTS

Dual Basis Lemma. *Let M be an $R-$semimodule. Then M is projective if and only if there exists $\{m_i\}_{i\in I} \subset M$ and $\{f_i\}_{i\in I} \subset \mathrm{Hom}_R(M,R)$ (I some index set) such that*

a) *for every $m \in M$, $f_i(m) = 0$ for all but finitely many $i \in I$ and*
b) *for every $m \in M$, $\sum\limits_{i\in I} f_i(m)\, m_i = m$.*

The collection $\{m_i, f_i\}$ is called a dual basis for M.

Proof. Let $R^{(I)}$ be a free $R-$semimodule and θ be a surjective $R-$homomorphism from $R^{(I)}$ to M where $R^{(I)}$ is the set of all functions from I to R with finite support.

Since M is a projective semimodule, there exists an $R-$ homomorphism $\psi : M \to R^{(I)}$ such that $\theta\psi = Id_M$. Let $\pi_i : R^{(I)} \to R$ be given by

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$\pi_i(f) = f(i)$ for all $f \in R^{(I)}$, then for any f in $R^{(I)}$ we have $\sum_{i\in I} \pi_i(f)e_i = f$, since $\big[\sum_{i\in I} \pi_i(f)e_i\big](j) = \pi_j(f) = f(j)$ where $e_i \in R^{(I)}$ defined by

$$e_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Now set $m_i = \theta(e_i)$ and $\pi_i\psi = f_i$. For $m \in M$ clearly $f_i(m) = 0$, for all but finitely many i.

Now,

$$\begin{aligned} \sum_{i\in I} f_i(m)m_i &= \sum_{i\in I} (\pi_i\psi)(m)m_i \\ &= \sum_{i\in I} \pi_i\big(\psi(m)\big)\theta(e_i) \\ &= \theta\Big(\sum_{i\in I} \pi_i\big(\psi(m)\big)(e_i)\Big) \\ &= \theta\big(\psi(m)\big) \\ &= (\theta\psi)(m) \\ &= m, \text{ for all } m \in M. \end{aligned}$$

Thus $\{m_i, f_i\}$ forms a dual basis for M.

Conversely, suppose that $\{m_i, f_i\}$ is a dual basis for $R-$semimodule M. Define $\psi : M \to R^{(I)}$ by $\psi(m)(i) = f_i(m)$ for all $m \in M$ and $\theta : R^{(I)} \to M$ by $\theta(f) = \sum_{i\in I} f(i)m_i$ for $m \in M$ and $f \in R^I$. Then θ and ψ are $R-$homomorphisms of left $R-$semimodules and

$$\begin{aligned} (\theta\psi)(m) &= \theta\big(\psi(m)\big) \\ &= \theta\big(f_i(m)\big) \\ &= \sum_{i\in I} f_i(m)m_i \\ &= m, \text{ for all } m \in M. \end{aligned}$$

Let $\phi : L \to K$ be a surjective $R-$homomorphism of left $R-$semimodules and $\alpha : M \to K$ be an $R-$homomorphism. Since $R^{(I)}$ is projective, then there exists an $R-$homomorphism $\beta : R^{(I)} \to L$ such that $\phi\beta = \alpha\theta \Rightarrow \phi\beta\psi = \alpha\theta\psi = \alpha$ and $\beta\psi : M \to L$ is a map having the property that we seek in order to prove the first condition of projectiveness. Now let $\phi : L \to K$ be a steady $R-$homomorphism of left $R-$semimodules and $\alpha, \alpha' : M \to L$ be $R-$homomorphisms satisfying $\phi\alpha = \phi\alpha'$ which implies that $\phi\alpha\theta = \phi\alpha'\theta$.

Since $R^{(I)}$ is projective, there exist $R-$homomorphisms $\beta\,,\,\beta' : R^{(I)} \to L$ satisfying $\phi\beta = \phi\beta'$ and $\alpha\theta + \beta \quad = \alpha'\theta + \beta'$. This implies $\phi(\beta\psi) = \phi(\beta'\psi)$ and $\alpha + \beta\psi = \alpha\theta\psi + \beta\psi = (\alpha\theta + \beta)\psi = (\alpha'\theta + \beta')\psi = \alpha' + \beta'\psi$.

Hence the second condition of projectiveness. □

Tensor product is as defined in [2]. Note that if M is a cancellative left $R-$semimodule then $R \otimes M \cong M$.

Proposition 1. *Let R be a cancellative semiring and M be a cancellative $R-$semimodule. Then* $\mathrm{Hom}_R(R, M) \cong M$.

Proposition 2. *Let R be a commutative semiring and let A and B be R-semialgebras. Let M be a finitely generated and projective $A-$semimodule and let N be a finitely generated and projective $B-$semimodule. Then*

$$\mathrm{Hom}_A(M, M) \otimes \mathrm{Hom}_B(N, N) \cong \mathrm{Hom}_{A\otimes B}(M \otimes N, M \otimes N)$$

where $\otimes = \otimes_R$.

Proof. Let $\{x_j, f_j\}$, $\{y_i, g_i\}$ be the dual bases for M and N respectively. Then for any m in M and n in N, $\sum_j f_j(m)\, x_j = m$ and $\sum_i g_i(n)\, y_i = n$.

Define,

$$\theta_j : M \otimes \mathrm{Hom}_B(N, N) \to \mathrm{Hom}_A(M, M) \otimes \mathrm{Hom}_B(N, N)$$

by

$$\theta_j\,(a \otimes h) = f_j(\ \ )\, a \otimes h$$

and

$$\pi_i : M \otimes N \to M \otimes \mathrm{Hom}_B(N, N)$$

given by

$$\pi_i\ (b_1 \otimes b_2) = b_1 \otimes g_i(\ \ )\, b_2.$$

Now define

$$\psi : \mathrm{Hom}_{A\otimes B}(M \otimes N, M \otimes N) \to \mathrm{Hom}_A(M, M) \otimes \mathrm{Hom}_B(N, N)$$

by

$$\psi(f) = \sum_{i,j} \theta_j(\pi_i[f(x_j \otimes y_i)])$$

and

$$\psi' : \mathrm{Hom}_A(M, M) \otimes \mathrm{Hom}_B(N, N) \to \mathrm{Hom}_{A\otimes B}(M \otimes N, M \otimes N)$$

by

$$\psi'(h_1 \otimes h_2) = h_1 \otimes h_2.$$

Consider,

$$
\begin{aligned}
\psi\psi'(h_1 \otimes h_2)(m \otimes n) &= \psi(h_1 \otimes h_2)(m \otimes n) \\
&= \sum_{i,j} \theta_j(\pi_i[h_1 \otimes h_2(x_j \otimes y_i)])(m \otimes n) \\
&= \sum_{i,j} \theta_j(\pi_i[(h_1(x_j) \otimes h_2(y_i)])(m \otimes n)) \\
&= \sum_{i,j} \theta_j(h_1(x_j) \otimes g_i(\ \ )h_2(y_i))(m \otimes n) \\
&= (h_1 \otimes h_2)\Big(\sum_{i,j} f_j(m)x_j \otimes g_i(n)y_i\Big) \\
&= (h_1 \otimes h_2)(m \otimes n) \\
&\Rightarrow \psi\psi'(h_1 \otimes h_2)(m \otimes n) = (h_1 \otimes h_2)(m \otimes n).
\end{aligned}
$$

Clearly, $\psi'\psi(f) = f$. Hence ψ is a one–one onto homomorphism. □

For any $R-$semimodule M, consider the subset $I_R(M)$ of R consisting of the element of the form $\sum_{i=1}^{n} f_i(m_i)$ where the $f_i \in \mathrm{Hom}_R(M, R)$ and the $m_i \in M$. The $I_R(M)$ is two–sided ideal in R so $I_R(M)$ is an ideal in R and is called the **trace ideal** of M. An $R-$semimodule M is an $R-$**generator** if $I_R(M) = R$. Thus M is an $R-$generator if and only if there exist $f_1, f_2, \ldots, f_n \in \mathrm{Hom}_R(M, R)$ and $m_1, m_2, \ldots, m_n \in M$ with $\sum_{i=1}^{n} f_i(m_i) = 1$.

An $R-$semimodule M is an $R-$**progenerator** if M is a finitely generated, projective and generator over R.

Proposition 3. *Let R be a commutative semiring and let M and N be $R-$semimodules. Then*

i) *$M \otimes_R N$ is finitely generated over R if both M and N are.*
ii) *$M \otimes_R N$ is $R-$projective if both $M$and N are.*
iii) *$M \otimes_R N$ is $R-$generator if both M and N are.*

Henceforth we show that ***csmod***-R and S-***csmod*** are equivalent categories where S is chosen as the cancellative semiring of endomorphisms of some cancellative $R-$**progenerator**.

Let R be any cancellative semiring and let M be any cancellative $R-$semimodule. Define $M^* = \mathrm{Hom}_R(M, R)$ and $S = \mathrm{Hom}_R(M, M)$. Note that M^*, S are cancellative. Since R is a cancellative $(R-R)$ bisemimodule, M^* is a cancellative right $R-$semimodule under the operation $(f.r)m = f(m)r$.

Moreover M is a cancellative left S-semimodule with $s.m = s(m)$. Under this operation M is a cancellative left $R-$left S bisemimodule. Hence M^* becomes a cancellative right $S-$semimodule under the operation $(f.s)(m) =$

$f(s(m))$. M^* is a cancellative right $R-$right $S-$bisemimodule. We can form $M^* \otimes_R M$ and $M^* \otimes_S M$. Moreover $M^* \otimes_R M$ is a cancellative left $S-$right $S-$bisemimodule by virtue of M being a cancellative left $R-$left $R-$bisemimodule and M^* being a cancellative right $R-$right S bisemimodule. Similarly $M^* \otimes_S M$ is a cancellative left $R-$right R bisemimodule.

Let θ_R denote the map from $M^* \otimes_R M$ to $S = \mathrm{Hom}_R(M, M)$ given by $[\theta_R \sum_i (f_i \otimes m_i)](m) = \sum_i f_i(m) m_i$. θ_R is both a left and a right $S-$semimodule homomorphism. Let θ_S denote the map from $M^* \otimes_S M$ to R given by $\theta_S(\sum_i f_i \otimes m_i) = \sum_i f_i(m_i)$. θ_S is a right and left $R-$semimodule homomorphism, whose image is the trace ideal $I_R(M)$.

Lemma 1. *Let R be any cancellative semiring and M be any cancellative $R-$semimodule. θ_R is onto iff M is finitely generated and projective. Moreover if θ_R is onto then it is one–one.*

Proof. Suppose that M is finitely generated and projective. Therefore there exists a dual basis $f_1, f_2, \ldots, f_n \in M^*$ and $m_1, m_2, \ldots, m_n \in M$, such that $\theta_R[\sum_{i=1}^n (f_i g) \otimes m_i] = g$ for any g in $S = \mathrm{Hom}_R(M, M)$. Hence θ_R is onto.

Conversely, assume that θ_R is onto. Then there exist $\sum_{i=1}^n f_i \otimes m_i \in M^* \otimes_R M$ such that $\theta_R(\sum_{i=1}^n f_i \otimes m_i)$ is the identity map from M to M, that is, $\sum_{i=1}^n f_i(m) m_i = m$ for all $m \in M$.

Thus the set $f_1, f_2, \ldots, f_n$, and $m_1, m_2, \ldots, m_n$ forms a finite dual basis for M. Therefore by the dual basis lemma, M is finitely generated and projective.

Now given that θ_R is onto, we know that M possesses a dual basis $f_1, f_2, \ldots, f_n \in M^*$ and $m_1, m_2, \ldots, m_n \in M$.

We claim that θ_R is one–one. Let $\sum_j g_j \otimes n_j$, $\sum_k h_k \otimes p_k \in M^* \otimes_R M$ satisfy

$$\theta_R\Big(\sum_j g_j \otimes n_j\Big)(m) = \theta_R\Big(\sum_k h_k \otimes p_k\Big)(m), \ \forall m \in M.$$

Then

$$\sum_j g_j(m) n_j = \sum_k h_k(m) p_k.$$

Now

$$\sum_j g_j \otimes n_j = \sum_j g_j \otimes \Big(\sum_i f_i(n_j)\Big) m_i = \sum_{i,j} g_j f_i(n_j) \otimes m_i.$$

But

$$\sum_j \big(g_j f_i(n_j)\big)(m) = \sum_j \big(g_j(f_i(n_j)(m)\big)$$

$$
\begin{aligned}
&= \sum_j (g_j(m)f_i(n_j)) \\
&= f_i\Big(\sum_j g_j(m)(n_j)\Big) \\
&= f_i\Big(\sum_k h_k(m)(p_k)\Big) \\
&= \sum_k h_k(m)f_i(p_k) \\
&= \sum_k \big(h_k f_i(p_k)\big)(m).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\Big[\sum_j g_j f_i(n_j)\Big](m) = \Big[\sum_k h_k f_i(p_k)\Big](m), \ \forall m \in M \\
&\Rightarrow \sum_j g_j f_i(n_j) = \sum_k h_k f_i(p_k) \\
&\Rightarrow \sum_{i,j} g_j f_i(n_j) \otimes m_i = \sum_{i,k} h_k f_i(p_k) \otimes m_i \\
&\Rightarrow \sum_j g_j \otimes n_j = \sum_k h_k \otimes p_k.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\theta_R\Big(\sum_j g_j \otimes n_j\Big) = \theta_R\Big(\sum_k h_k \otimes p_k\Big) \\
&\Rightarrow \sum_j g_j \otimes n_j = \sum_k h_k \otimes p_k.
\end{aligned}
$$

Hence θ_R is one-one. □

Lemma 2. *Let R be any cancellative semiring, M be any cancellative $R-$semimodule and $S = \mathrm{Hom}_R(M, M)$ be a cancellative semiring. θ_S is onto if and only if M is a generator. Moreover if θ_S is onto then it is one–one.*

Proof. Since the image of θ_S is equal to $I_R(M)$, θ_S is onto if and only if $I_R(M) = R$, that is M is a generator over R.

Suppose θ_S is onto. We claim that θ_S is one-one. Let $\sum_j h_j \otimes n_j$, $\sum_k g_k \otimes p_k \in M^* \otimes_S M$ satisfy

$$
\theta_S\Big(\sum_j h_j \otimes n_j\Big) = \theta_S\Big(\sum_k g_k \otimes p_k\Big).
$$

Then
$$\sum_j h_j(n_j) = \sum_k g_k(p_k).$$
Since θ_S is onto, there exist $f_1, f_2 \dots, f_n \in M^*$ and $m_1, m_2, \dots, m_n \in M$ with
$$\sum_i f_i(m_i) = 1.$$
Now
$$\begin{aligned}\sum_j h_j \otimes n_j &= \sum_j h_j \otimes \Big(\sum_i f_i(m_i)\Big) n_j \\ &= \sum_{i,j} h_j \otimes \theta_R(f_i \otimes n_j)(m_i) \\ &= \sum_i \Big(\sum_j h_j \theta_R(f_i \otimes n_j)\Big) \otimes m_i.\end{aligned}$$
Note that for every i and every $m \in M$,
$$\begin{aligned}\Big[\sum_j h_j \theta_R(f_i \otimes n_j)\Big](m) &= \sum_j h_j\big(f_i(m) n_j\big) \\ &= f_i(m)\Big(\sum_j h_j(n_j)\Big) \\ &= f_i(m)\Big(\sum_k g_k(p_k)\Big) \\ &= \sum_k g_k\big(f_i(m) p_k\big) \\ &= \Big[\sum_k g_k \theta_R(f_i \otimes p_k)\Big](m).\end{aligned}$$
Therefore
$$\Big[\sum_j h_j \theta_R(f_i \otimes n_j)\Big](m) = \Big[\sum_k g_k \theta_R(f_i \otimes p_k)\Big](m), \ \ \forall m \in M.$$
So,
$$\begin{aligned}&\Big[\sum_j h_j \theta_R(f_i \otimes n_j)\Big] = \Big[\sum_k g_k \theta_R(f_i \otimes p_k)\Big] \\ &\Rightarrow \Big[\sum_{i,j} h_j \theta_R(f_i \otimes n_j)\Big] \otimes m_i = \Big[\sum_{i,k} g_k \theta_R(f_i \otimes p_k)\Big] \otimes m_i \\ &\Rightarrow \sum_j h_j \otimes n_j = \sum_k g_k \otimes p_k.\end{aligned}$$

Thus

$$\theta_S\Big(\sum_j h_j \otimes n_j\Big) = \theta_S\Big(\sum_k g_k \otimes p_k\Big)$$
$$\Rightarrow \sum_j h_j \otimes n_j = \sum_k g_k \otimes p_k.$$

Hence θ_S is one–one. □

For any cancellative left $R-$semimodule M, we have seen that M is a left $R-$left S cancellative bisemimodule and $M^* = \mathrm{Hom}_R(M, R)$ is a right $R-$right S cancellative bisemimodule where $S = \mathrm{Hom}_R(M, M)$ is a cancellative semiring. Therefore for any cancellative right $R-$semimodule L, $L \otimes_R M$ has the structure of a left cancellative $S-$semimodule, while for any cancellative left $S-$semimodule N, $M^* \otimes_S N$ has the structure of a cancellative right $R-$semimodule.

Then

$$(\ ) \otimes_R M : cs \bmod -R \to S - cs \bmod$$

and

$$M^* \otimes_S (\ ) : S - cs \bmod \to cs \bmod -R$$

are functors.

Theorem 4. *Let R be any cancellative semiring, M be any cancellative left $R-$semimodule and left R progenerator. Consider the cancellative semiring $S = \mathrm{Hom}_R(M, M)$ and the cancellative semimodule $M^* = \mathrm{Hom}_R(M, R)$. Then the functors*

$$(\ ) \otimes_R M : cs \bmod -R \to S - cs \bmod,$$
$$M^* \otimes_S (\ ) : S - cs \bmod \to cs \bmod -R$$

are inverse equivalences.

Proof. Let $L \in cs \bmod -R$. Then we have

$$\begin{aligned} M^* \otimes_S (L \otimes_R M) &\cong M^* \otimes_S (M \otimes_{R^0} L) \\ &\cong (M^* \otimes_S M) \otimes_{R^0} L \\ &\cong (R \otimes_{R^0} L) \\ &\cong L \otimes_R R \cong L. \end{aligned}$$

Similarly for any cancellative left $S-$semimodule N,

$$\begin{aligned} (M^* \otimes_S N) \otimes_R M &\cong (N \otimes_{S^0} M^*) \otimes_R M \\ &\cong N \otimes_{S^0} (M^* \otimes_R M) \\ &\cong N \otimes_{S^0} S \\ &\cong S \otimes_S N \cong N. \end{aligned}$$

Hence the functors are inverse equivalences. □

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References

[1] F. De Mayer and E. Ingrahm, *Separable Algebras over Commutative Rings*, V-181, Springer - Verlag, New York, 1971.

[2] J. S. Golan, *The Theory of Semirings with Applications in Mathematics and Theoretical Computer Sciences*, (Pitman Monographs and Surveys in Pure and Applied Mathematics) 54, Harlow: Longman Scientific and Technical (1992).

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