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# ASYMPTOTIC LOCATION OF THE ZEROS OF THE FABER POLYNOMIALS

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ABSTRACT. Let $E$ be a compact set of the complex plane containing more than one point whose complement in the extended complex plane is simply connected. Let $\omega = \phi(z)$ map conformally $\mathrm{Ext}(E)$ into $|\omega| > 1$ and with $\phi(\infty) = \infty$. The map $\phi(z)$ has the form

$$\phi(z) = \frac{z}{c} + a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots$$

The Faber polynomials for $E$, $\phi_n(z)$, consist of the polynomial part of $\phi(z)^n$.

For $\epsilon > 0$ given let

$$E_\epsilon := \bigcup_{z \in E} B(z, \epsilon),$$

where $B(z, \epsilon)$ denotes the disk of center $z$ and radius $\epsilon$, and let

$$B_\epsilon := \mathrm{Br}(E_\epsilon).$$

Let

$$\alpha := \inf_{z \in \mathrm{Br}(E)} \{\theta;\ \pi\theta \text{ is the exterior angle of } E \text{ at } z\}.$$

A typical result obtained in this work is the Theorem 3.2.

## 1. INTRODUCTION

In this section we state, mostly without proof, results of complex approximation theory which will be needed in the next section. We follow essentially three sources: [12], [18] and [19]. In [5] and [6] complex approximation is used in different contexts.

Let $E$ be a compact set of the complex plane containing more than one point whose complement in the extended complex plane is simply connected. Let $\omega = \phi(z)$ map conformally $\mathrm{Ext}(E)$ into $|\omega| > 1$ and with $\phi(\infty) = \infty$.

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The map $\phi(z)$ has the form

$$\phi(z) = \frac{z}{c} + a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots$$

The number $c > 0$ is the *capacity* $\mathrm{Cap}(E)$ of the set $E$. The Faber polynomials for $E$, $\phi_n(z)$, consist of the polynomial part of $\phi(z)^n$. The inverse map is denoted $z = \psi(\omega)$ and has the form

$$\psi(\omega) = c\,\omega + b_0 + \frac{b_{-1}}{\omega} + \frac{b_{-2}}{\omega^2} + \cdots$$

For $\rho > 1$ let $\Gamma_\rho$ be the *level curve*

$$\Gamma_\rho := \{z \in \mathbb{C}\ ;\ |\phi(z)| = \rho\}.$$

As an example consider $E = [-1, 1]$. Then

$$\omega = \phi(z) = z + \sqrt{z^2 - 1} = \frac{z}{\frac{1}{2}} - \frac{1}{2z} - \frac{1}{8z^3} + \cdots$$

and

$$z = \psi(\omega) = \frac{1}{2}\omega + \frac{1}{2\omega}.$$

Hence $\Gamma_\rho$ ($\rho > 1$) is the ellipse with foci $-1, 1$ and $\rho + \frac{1}{\rho}$, $\rho - \frac{1}{\rho}$ for lengths of its axes. This shows, in addition and as well known, that $\mathrm{Cap}\ [a, b] = \frac{b-a}{4}$. The Faber polynomials for $[-1, 1]$ are $\phi_n(z) = 2T_n(z)$ where $T_n(x) = \cos(n \arccos x)$ are the Chebyshev polynomials.

Notice that we deviate slightly from the usual definition of the Faber polynomials in the sense that our function $\omega = \phi(z)$ maps $\mathrm{Ext}(E)$ onto $|\omega| > 1$ instead of $|\omega| > c$, where $c = \mathrm{Cap}(E)$, as it is usually the case. It follows that the Faber polynomials as often described in the literature correspond to $c^n\phi_n(z)$ here. In particular the Faber polynomials for $[-1, 1]$ are usually $\frac{1}{2^{n-1}}T_n(z)$ instead of $2T_n(z)$ here. We found it more convenient to make this modification. Note that some authors, like Curtiss in [2], use the convention adopted here.

If the mapping function extends continuously and in an bijective manner to the boundary $\partial E$ of $E$, which is the case if $\partial E$ is a Jordan curve, we write $\Gamma = \mathrm{B}_r(E)$. In that case the mapping function and its inverse are still denoted by $\omega = \phi(z)$ and $z = \psi(\omega)$, respectively.

It is the purpose of this note to show that, asymptotically and in a sense made precise below in Section 3, the zeros of the Faber polynomials cluster near the boundary of $E$.

Recently we have witnessed a resurgence of research activity on the location of the zeros of the Faber polynomials. Back in 1972 Ulmann studied in [21] the zeros of the derivatives of the Faber polynomials. In an important paper [13] Kövari and Pommerenke showed that if $E$ is convex, $E$ not an interval, then the zeros of the associated Faber polynomials are included in Int$(E)$. In [1] Bartolomeo and He studied the zeros of the Faber polynomials associated with $m$-stars. See also [3, 4, 7, 8, 9, 10, 11, 15, 16, 17, 22]. In general, the zeros of the Faber polynomials need not be in $E$. In the remarkable paper cited above [13] Kövari and Pommerenke presents non-convex sets for which the zeros of the corresponding Faber polynomials are outside the convex hull of the sets. Unlike the sets that have been used for this purpose before, the sets have nonempty interiors and analytic boundaries. See also Theorem 3.1 below for *asymptotic* results in the case of analytic boundary. See also [3, 4, 7, 8, 9, 10, 11, 15, 16, 17, 22]

The purpose of this work consists of showing that the zeros of the Faber polynomials are "pushed" towards the boundary $\partial E$, as the degrees of the polynomials increase, in a sense made precise below.

The phenomenon of asymptotic zero-clustering occurs in many different contexts. We cite one of them where the clustering occurs near the point $z = \infty$. Let

$$f(z) = \sum_{k=0}^{\infty} a_k \, z^k$$

be an entire function with no zeros. Then, given $R > 0$, there exists $N$ such that $\sum_{k=0}^{n} a_k \, z^k$ has all its zeros in $\{|z| > R\}$ if $n \geq N$.

The verification of this (standard) fact is based on a simple compactness argument. $|f(z)| \geq \alpha > 0$ if $|z| \leq R$. On the other hand $\sup_{|z| \leq R} |\sum_{k=0}^{n} a_k z^k - f(z)| \leq \frac{\alpha}{2}$ if $n$ is big enough. Hence $\sum_{k=0}^{n} a_k z^k$ has no zeros in $\{|z| \leq R\}$.

This paper is organized as follows. In the next section we review the standard formulas relating the Faber polynomials with the conformal mapping. In Section 3 we state our main results and the next section is devoted to the proofs of these results. Section 5 states open problems.

## 2. The relationship between the Faber polynomials and the exterior conformal mapping

For the sake of completeness we first reproduce here, following essentially [18], the standard formulae relating the Faber polynomials $\phi_n(z)$ with the mapping function $\phi(z)$.

Let $\rho > 1$, $z \in \operatorname{int}\Gamma_\rho$, $z \notin E$. Then, because $\phi(\zeta)^n - \phi_n(\zeta)$ has a zero at $\infty$ of order at least one,

$$\zeta \mapsto \frac{\phi(\zeta)^n - \phi_n(\zeta)}{\zeta - z}$$

has a zero at $\infty$ of order at least two. Hence

$$\frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\phi(\zeta)^n - \phi_n(\zeta)}{\zeta - z} d\zeta = 0$$

so that

$$\phi_n(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\phi(\zeta)^n}{\zeta - z} d\zeta. \tag{2.1}$$

(See also [2].) Let now $1 \le \rho' < \rho$ be such that $z \in \mathrm{ext}\Gamma_{\rho'}$. Then

$$\phi(z)^n = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\phi(\zeta)^n}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\Gamma\rho'} \frac{\phi(\zeta)^n}{\zeta - z} d\zeta$$

so that, in view of (2.1),

$$\phi_n(z) = \phi(z)^n + \frac{1}{2\pi i} \int_{\Gamma_{\rho'}} \frac{\phi(\zeta)^n}{\zeta - z} d\zeta. \tag{2.2}$$

Recall now that $|\phi(\zeta)| = \rho'$ if $\zeta \in \Gamma_{\rho'}$. Hence

$$\left| \frac{1}{2\pi i} \int_{\Gamma_{\rho'}} \frac{\phi(\zeta)^n}{\zeta - z} d\zeta \right| \le M\rho'^n. \tag{2.3}$$

Recall now that $z \in \mathrm{ext}\left(\Gamma_{\rho'}\right)$, so that $|\phi(z)| > \rho'$. We then have, given $\epsilon > 0$,

$$(1-\epsilon)|\phi(z)^n| \le |\phi_n(z)| \le (1+\epsilon)|\phi(z)^n|$$

for $n$ big enough. That is to say

**Proposition 2.1.** *The asymptotic rate of growth of the Faber polynomials* $|\phi_n(z)|$ *at a point* $z \in \mathbb{C}$, $z \notin E$ *is the same as that of the mapping function* $|\phi(z)|^n$.

In the sequel we will assume that the boundary of $E$ is a Jordan curve so that the mapping function extends continuously up to the boundary, which will be denoted by $\Gamma$. We will assume that $\Gamma$ has enough regularity so that (2.1) and (2.2) hold with the curve $\Gamma_{\rho'}$ being replaced by the curve $\Gamma$. In that case (2.3) becomes

$$\left| \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(\zeta)^n}{\zeta - z} d\zeta \right| \le \frac{M}{\mathrm{Dist}(z, \Gamma)}. \tag{2.4}$$

2.1. **The Faber polynomials associated with the level curve $\boldsymbol{\Gamma_\rho}$.** Assume that the mapping function $z = \psi(\omega)$ extends conformally from $\{|\omega| > 1\}$ to $\{|\omega| > \delta\}$ for some $\delta < 1$, which is the case if the boundary $\partial E$ is analytic. For $\rho > \delta$ let $E_\rho$ be the compact set, with simply connected complement, whose boundary is $\Gamma_\rho$. (The case of interest in Theorem 3.1 will be $1 > \rho > \delta$, but what follows holds true for $\rho > \delta$. In the case $\rho > 1$, there is no need to assume the conformal extension.) The mapping function from $\mathrm{Ext}(E_\rho)$ to $\{|\omega| > 1\}$ is then

$$z = \psi(\rho\omega)$$

with inverse

$$\omega = \frac{1}{\rho}\phi(z)$$

where $\omega = \phi(z)$ is the inverse of $z = \psi(\omega)$. It follows that, if the Faber polynomials for $E$ are $\phi_n(z)$, then the Faber polynomials $\phi_{\rho,n}(z)$ for $E_\rho$ are

$$\phi_{\rho,n}(z) = \frac{1}{\rho^n}\phi_n(z). \tag{2.5}$$

As an example consider $E = [-1, 1]$ so that

$$z = \psi(\omega) = \frac{1}{2}\omega + \frac{1}{2\omega}.$$

Then $\Gamma_2$ is the ellipse with foci $-1, 1$ and $2 + \frac{1}{2}$, $2 - \frac{1}{2}$ for lengths of its axes. Hence in view of (2.5) the Faber polynomials for $\Gamma_2$ are $\phi_n(z) = \frac{1}{2^n}2T_n(z)$ where $T_n(x) = \cos(n \arccos x)$ are the Chebyshev polynomials. Note that $\frac{1}{2^n}2T_n(z) = z^n + \cdots$ which shows, in addition, that $\mathrm{Cap}(\Gamma_2) = 1$, as well known.

An example with $\rho = \frac{1}{2}$ is provided be reversing the steps above: The mapping

$$z = \omega + \frac{1}{4\omega}$$

transforms $|\omega| > 1$ into $\mathrm{Ext}(\Gamma_2)$. As seen above the Faber polynomials for $\Gamma_2$ are $\frac{1}{2^n}2T_n(z)$. The map extends conformally to $|\omega| > \frac{1}{2}$ and the image of $|\omega| = \frac{1}{2}$ is $[-1, 1]$. Hence in view of (2.5) the Faber polynomials for $[-1, 1]$ are $\phi_n(z) = 2^n\frac{1}{2^n}2T_n(z) = 2T_n(z)$, as already noticed above.

## 3. The zeros of the Faber polynomials

We introduce the following definition of the $\epsilon$-boundary $B_\epsilon$ of $E$.

**Definition.** Let $\epsilon > 0$ be given and let

$$E_\epsilon := \bigcup_{z\in E} B(z, \epsilon).$$

Here $B(z, \epsilon)$ denotes the disk of center $z$ and radius $\epsilon$. Define

$$B_\epsilon := \mathrm{Br}(E_\epsilon).$$

It can be seen that $B_\epsilon$ is the level curve $d = \epsilon$ of the distance function $d(z) = \mathrm{dist}(z, E)$. The following result is well known (see [22]). For the sake of completeness, and to illustrate the techniques used in the sequel, we present its proof.

**Theorem 3.1.** *Let $E$ and $\phi_n(z)$ be as above. Then given $\epsilon > 0$, there exists $N > 0$ such that $n \geq N$ implies*

$$\text{all the zeros of } \phi_n(z) \text{ are located in } \mathrm{Int}(B_\epsilon).$$

If we have additional information on the boundary $\partial E$ of $E$ then we can estimate the rate at which the zeros approach $\partial E$. Let

$$\alpha := \inf_{z \in \mathrm{Br}(E)} \{\theta;\ \pi\theta \text{ is the exterior angle of } E \text{ at } z\}. \tag{3.1}$$

If $\alpha > 0$, then $\mathrm{Br}(E)$ has no reentrant corner of vanishing angle. It is readily seen that if $E$ is convex or if $\mathrm{Br}(E)$ is a $C^1$ curve then $\alpha = 1$. We also note that $\alpha \leq 1$.

**Theorem 3.2.** *Let $\mathrm{B_r}(E)$ consists of piecewise smooth curves and assume that $\alpha$, as defined above in (3.1), is positive. Then there exists a constant $C$ depending only on $E$ with the following property: For every $\epsilon > 0$,*

$$n \geq \frac{-C \ \log \ \epsilon}{\epsilon^{\frac{1}{\alpha}}} \tag{3.2}$$

*implies*

$$\text{all the zeros of } \phi_n(z) \text{ are located in } \mathrm{Int}(B_\epsilon).$$

In Proposition 3.1 below we say that the boundary of $E$ is analytic if there exists an analytic parametization $\gamma(t)$ of $\mathrm{Br}(E)$ with $\gamma'(t) \neq 0$. The following result is essentially a consequence of the definition of the Faber polynomials.

**Proposition 3.1.** *Assume that the boundary $\partial E$ of $E$ is analytic. Then there exists $N > 0$ such that $n \geq N$ implies*

$$\text{all the zeros of } \phi_n(z) \text{ are located in } \mathrm{Int}(E).$$

## 4. Proofs of the main results

*Proof of Theorem* 3.1. Let $z \in \mathbb{C}$ be such that

$$\mathrm{Dist}(z, E) \geq \epsilon.$$

Observe that there exists $\rho > 1$ such that for the level curve $\Gamma_\rho$ we have

$$\Gamma_\rho \subset \mathrm{Int}(B_\epsilon).$$

It follows that with $z$ as above we have $|\phi(z)| > \rho$. We remark that if $\xi \in \mathrm{B_r}(E)$ then $|\phi(\xi)| = 1$. It follows, using

$$\phi_n(z) = \phi(z)^n + \frac{1}{2\pi i}\int_\Gamma \frac{\phi(\zeta)^n}{\zeta - z} d\zeta$$

and
$$\left|\frac{1}{2\pi i}\int_\Gamma \frac{\phi(\zeta)^n}{\zeta - z}d\zeta\right| \leq \frac{M}{\epsilon},$$
that
$$|\phi_n(z)| \geq \left|\rho^n - \frac{M}{\epsilon}\right|.$$
Hence, because $\rho > 1$,
$$|\phi_n(z)| > 0$$
if $n$ is big enough. That is to say, for these values of $n$, $\phi_n(z) \neq 0$ if $\mathrm{Dist}(z, E) \geq \epsilon$. □

In order to prove Theorem 3.2 we need the following classical result due to Szegö [20]. See also [19].

**Theorem 4.1.** *Let* $\mathrm{Br}(E)$ *consists of piecewise smooth curves and let* $z_0 \in \mathrm{Br}(E)$. *Let the exterior angle at* $z_0$ *be* $\pi\theta$, $\theta > 0$. *Then for the level curves* $\Gamma_{1+\frac{1}{n}}$ *we have*
$$\mathrm{Dist}\left(z_0, \Gamma_{1+\frac{1}{n}}\right) = \mathcal{O}\left(\frac{1}{n^\theta}\right).$$

**Lemma 4.1.** *Let* $E$ *be such that* $\alpha$, *as stated above, is positive. Then there exists a positive constant* $k$ *with the following property. For every* $\epsilon > 0$
$$\Gamma_{1+k\epsilon^{\frac{1}{\alpha}}} \subset \mathrm{Int}(B_\epsilon).$$

*Proof.* The proof follows from Szegö's theorem. Let $z \in \mathrm{Br}(E)$ and let $\pi\beta$ be the exterior angle at $z$. Then $\beta \geq \alpha$. We then have (with $\epsilon \leq 1$)
$$\mathrm{Dist}\left(z, \Gamma_{1+\epsilon^{\frac{1}{\alpha}}}\right) \leq K\epsilon^{\frac{\beta}{\alpha}} \leq K\epsilon.$$
Hence
$$\mathrm{Dist}\left(z, \Gamma_{1+\frac{1}{K^{\frac{1}{\beta}}}\epsilon^{\frac{1}{\alpha}}}\right) \leq \epsilon^{\frac{\beta}{\alpha}} \leq \epsilon.$$
Choose now $k < \frac{1}{K^{\frac{1}{\beta}}}$. Then
$$\mathrm{Dist}\left(z, \Gamma_{1+k\epsilon^{\frac{1}{\alpha}}}\right) < \epsilon.$$
The proof now follows from the compactness of the level curve $\Gamma_{1+k\epsilon^{\frac{1}{\alpha}}}$. □

*Proof of Theorem* 3.2. For $\epsilon > 0$, let $z \in \mathbb{C}$ be such that
$$\mathrm{Dist}(z, E) \geq \epsilon.$$
Because the level curve $\Gamma_{1+k\epsilon^{\frac{1}{\alpha}}}$, where $k$ is as in Lemma 4.1, is contained in $\mathrm{Int}(B_\epsilon)$,
$$|\phi(z)| \geq 1 + k\epsilon^{\frac{1}{\alpha}}.$$

We remark that if $\xi \in \mathrm{B_r}(E)$ then $|\phi(\xi)| = 1$. It follows, using again

$$\phi_n(z) = \phi(z)^n + \frac{1}{2\pi i}\int_\Gamma \frac{\phi(\zeta)^n}{\zeta - z}d\zeta$$

and

$$\left|\frac{1}{2\pi i}\int_\Gamma \frac{\phi(\zeta)^n}{\zeta - z}d\zeta\right| \leq \frac{M}{\epsilon},$$

that

$$|\phi_n(z)| \geq \left|\left(1 + k\epsilon^{\frac{1}{\alpha}}\right)^n - \frac{M}{\epsilon}\right|.$$

It follows that $\phi_n(z) \neq 0$ if

$$n > \frac{\log \frac{M}{\epsilon}}{\log\left(1 + k\epsilon^{\frac{1}{\alpha}}\right)}.$$

Using now $\log(1+x) > \frac{1}{2}x,\ x > 0,\ x$ small, and $\log M + \log\frac{1}{\epsilon} < 2\log\frac{1}{\epsilon}$ if $\epsilon$ is small enough, we obtain $\phi_n(z) \neq 0$ if

$$n > \frac{2\log\frac{1}{\epsilon}}{\frac{1}{2}k\epsilon^{\frac{1}{\alpha}}}.$$

Hence Theorem 3.2 is proved with $C = \frac{4}{k}$. □

The proof of Proposition 3.1 will depend on Theorem 3.1 and on the observation made in Subsection 2.1.

*Proof of Proposition* 3.1. Let the Boundary of $E$ be an analytic curve. Then the mapping function $z = \psi(\omega)$ extends in a conformal manner to $|\omega| > R$ for some $R < 1$. Consider now the Faber $\phi_{1,n}(z)$ polynomials associated with the level curve $\Gamma_{R+\frac{1-R}{3}}$. Remark now that

$$\operatorname{dist}\left(\Gamma_{R+\frac{2(1-R)}{3}},\ \Gamma_{R+\frac{1-R}{3}}\right) > 0.$$

Then, in view of Theorem 3.1, all the zeros of $\phi_{1,n}(z)$ will be contained in the level curve $\Gamma_{R+\frac{2(1-R)}{3}}$ if $n$ is big enough. But, in view of (2.5),

$$\phi_{1,n}(z) = \left(\frac{1}{R + \frac{1-R}{3}}\right)^n \phi_n(z)$$

where $\phi_n(z)$ are the Faber polynomials associated to $E$. Hence all the zeros of $\phi_n(z)$, $n$ big enough, are contained in $\operatorname{Int}\left(\Gamma_{R+\frac{2(1-R)}{3}}\right)$. Because $R+\frac{2(1-R)}{3} < 1$, this level curve is (strictly) contained in $E$. Hence all the zeros of $\phi_n(z)$ are contained in $\operatorname{Int}(E)$ for all $n$ big enough. □

## 5. Concluding remarks and an open problem

Examination of the proof of Theorem 3.1 shows that in fact we have slightly more. Let

$$R := \inf\{r\,; z = \psi(\omega) \text{ can be conformally extended into } |\omega| > r\}.$$

**Proposition 5.1.** *Assume that the boundary $\partial E$ of $E$ is analytic and let $R$ be as above. Then, for $\epsilon > 0$, there exists $N > 0$ such that $n \geq N$ implies*

$$\textit{all the zeros of } \phi_n(z) \textit{ are located in } \operatorname{Int}(\Gamma_{R+\epsilon}).$$

*Remark.* The condition that $\Gamma$ has enough regularity so that (2.1) and (2.2) hold with the curve $\Gamma_{\rho'}$ being replaced by the curve $\Gamma$ is not needed in the proofs of Theorem 3.1 but makes the proof slightly less technical. This result holds, more generally, with no conditions imposed on $\partial E$.

We do not know if the inequality for $n$ in (3.2) of Theorem 3.2 is sharp. Results of the location of the zeros of the Faber polynomials as stated in the papers quoted in the Introduction do not confirm nor invalidate the possible sharpness of this estimate. On the other hand we were not able to build Faber polynomials would zeros would confirm this estimate. It is our opinion that the estimate for $n$ in (3.2) is *not* sharp.

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