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A CLASS OF FUNCTIONAL EQUATIONS (ALMOST) CHARACTERIZING POLYNOMIALS ON INTEGRAL DOMAINS

BRUCE R. EBANKS AND PETER DE PLACE FRIIS

Abstract. Let R be an infinite integral domain not of characteristic 2. For a given $n \geq 2$, suppose functions $f : R \to R$ and $h : R \to R$ satisfy

$$[x_1, x_2, \dots, x_n; f] = h(x_1 + \cdots + x_n) \prod_{j>i} (x_j - x_i),$$

where the left side denotes the determinant of the $n \times n$ matrix with row i given by $(1, x_i, x_i^2, \dots, x_i^{n-2}, f(x_i))$. It is proved that Df is a polynomial of degree at most n over R, for some D in R. For $n = 2$ and $n = 3$ the conclusion can be strengthened to take $D = 1$, but surprisingly this is not possible for $n \geq 4$.

Several authors have used divided differences to characterize polynomials, first on $\mathbb{R}$ (the set of real numbers) and then on fields of characteristic different from 2. It is easy to see that quadratic polynomials on $\mathbb{R}$ enjoy the property that

$$\frac{f(x) - f(y)}{x - y} = f'\left(\frac{x + y}{2}\right).$$

A strong converse of this is proved in [1] and [5], where it is shown that $f : K \to K$ (where K is any field not of characteristic 2) satisfies

$$f(x) - f(y) = (x - y)h(x + y),$$

for some function $h : K \to K$, only if f is a quadratic polynomial function.

This result has been successively generalized to higher degrees in [3], [8], [7], [2], [4] by considering the equation

$$f[x_1, x_2, \dots, x_n] = g(x_1 + \cdots + x_n) \tag{1}$$

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for $n \geq 2$. Here $f[x_1, x_2, \ldots, x_n]$ denotes the n^{th} divided difference of f, defined as

$$f[x_1, \ldots, x_n] := \frac{[x_1, \ldots, x_n; f]}{[x_1, \ldots, x_n]}$$

where

$$[x_1, \ldots, x_n; f] = \begin{vmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-2} & f(x_1) \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-2} & f(x_2) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-2} & f(x_n) \end{vmatrix}$$

and

$$[x_1, \ldots, x_n] = \begin{vmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{vmatrix} = \prod_{j>i} (x_j - x_i).$$

In [7] and [2], equation (1) is solved for functions $f, g : \mathbb{R} \to \mathbb{R}$. It is shown that f is a polynomial of degree at most n and that g is linear.

The strongest results are [8] , [4], and [6] . In [8] and [4] it is supposed that $f, g : K \to K$ with K a field not of characteristic 2. There it is proved that f is a polynomial of degree at most n over K. In [6] the slightly more general equation

$$\frac{[x_1, \ldots, x_n; f_1, ..., f_n]}{[x_1, \ldots, x_n]} = g(x_1 + \cdots + x_n) \tag{2}$$

is solved for unknown functions $f_1, ..., f_n, g : K \to K$ where K is a field of characteristic different from 2 with enough distinct points, and where

$$[x_1, \ldots, x_n; f_1, ..., f_n] = \begin{vmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-2} & f_1(x_1) \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-2} & f_2(x_2) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-2} & f_n(x_n) \end{vmatrix}.$$

There it is shown that the f_i's are all equal to the same polynomial of degree at most n.

Our goal was to generalize these results to integral domains. Since every finite integral domain is a field, we suppose that R is an infinite integral domain throughout most of this article. Of course division is not defined in integral domains, so we consider the equation (1) in the form

$$[x_1, x_2, \ldots, x_n; f] = h(x_1 + \cdots + x_n)[x_1, \ldots, x_n], \;\; x_1, \ldots, x_n \in R. \tag{3}$$

In Section 1 we present our main result, showing that $f, h : R \to R$ satisfy (3) only if there exists an element $D \in R$ for which Df is a polynomial of degree at most n over R. For $n \geq 4$, we also present an example showing that the conclusion cannot be strengthened in general; that is, f itself need not be a polynomial over R.

We move to the more general structure of (commutative) rings in Section 2. There we show that for $n = 2$ one need not suppose that R is an integral domain to conclude that f is a polynomial over R.

We show in Section 3 that the conclusion can also be strengthened in the case $n = 3$. If R is an infinite integral domain, then f must be a polynomial of degree at most 3 over R.

Finally, we turn to the more general equation (2) in the last section, considering it in the form

$$[x_1, x_2, \ldots, x_n; f_1, ..., f_n] = h(x_1+\cdots+x_n)[x_1, \ldots, x_n], \;\; x_1, \ldots, x_n \in R. \quad (4)$$

1. The main result

First we prove a lemma that is a key to our main result. This is similar to the lemma in [7] which is valid when R is the field of real numbers. In fact, the proof given in [7] holds when R is any field of characteristic different from 2. But the proof given in [7] does not work when R is not divisible by 2, as for example when R is the ring of integers.

Lemma 1. *Let R be an infinite integral domain with characteristic different from 2. Let E be a finite subset of R. If $H : R \to R$ satisfies*

$$H(0) = 0, \quad (5)$$

and

$$tH(t) - sH(s) = (t - s)H(s + t), \;\; s, t \in R \setminus E, \quad (6)$$

then

$$H(u) = au, u \in R$$

for some constant $a \in R$.

Proof. First note that we can assume without loss of generality that E is symmetric about 0. If E is not symmetric we simply define $E_S = E \cup (-E)$ where as usual $-E = \{-e : e \in E\}$. Then E_S is finite, symmetric, and we have

$$tH(t) - sH(s) = (t - s)H(s + t), \;\; s, t \in R \setminus E_S, \;\; H(0) = 0.$$

So we will assume that E is symmetric. For $s \in R \setminus E$ put $t = -s$, since E is symmetric $-s \notin E$ so $t \in R \setminus E$. From (6) we get

$$-s[H(-s) + H(s)] = -2sH(0) = 0, \;\; s \in R \setminus E.$$

Thus (also for $s = 0$, since $H(0) = 0$ is given)

$$H(-s) = -H(s), \;\; s \in R \setminus E.$$

Replace s by $-s$ in (6) to get

$$tH(t) - sH(s) = tH(t) + sH(-s) = (t+s)H(t-s), \; s,t \in R \setminus E. \quad (7)$$

Next we put $u = t - s$, so $t = s + u$, in (7)

$$(s+u)H(s+u) - sH(s) = (2s+u)H(u), \; s, s+u \in R \setminus E.$$

Interchange s and u to get

$$(u+s)H(u+s) - uH(u) = (2u+s)H(s), \; u, u+s \in R \setminus E.$$

Comparison of these two last equations gives

$$2sH(u) = 2uH(s), \; u, s, u+s \in R \setminus E.$$

Since R is an integral domain and the characteristic is different from 2 this gives us

$$sH(u) = uH(s), \; u, s, s+u \in R \setminus E.$$

Note that if $b_1, \ldots, b_n \in R$ then the set of all $s \in R$ satisfying $s + b_1 \in R \setminus E$, $s + b_2 \in R \setminus E, \ldots, s + b_n \in R \setminus E$ is infinite since the n conditions are equivalent to $s \in R \setminus (\cup_{i=1}^{n}(E - b_i))$ where as usual $E - b = \{e - b : e \in E\}$. Since R is infinite and E is finite the set $R \setminus (\cup_{i=1}^{n}(E - b_i))$ is infinite.

Choose an element $\alpha \in R \setminus E$ such that $\alpha + 1 \in R \setminus E$ also. For this element α we have

$$\alpha H(u) = uH(\alpha), \; u, u+\alpha \in R \setminus E,$$

and

$$(\alpha+1)H(u) = uH(\alpha+1), \; u, u+\alpha+1 \in R \setminus E.$$

The difference gives

$$H(u) = [H(\alpha+1) - H(\alpha)]u, \; u, u+\alpha, u+\alpha+1 \in R \setminus E.$$

Defining $a = H(\alpha+1) - H(\alpha)$, we have

$$H(u) = au, \; u, u+\alpha, u+\alpha+1 \in R \setminus E.$$

Substituting this into (7) with $s, t, s+\alpha, t+\alpha, s+\alpha+1, t+\alpha+1 \in R \setminus E$ we have

$$(t+s)H(t-s) = at^2 - as^2 = a(t+s)(t-s).$$

If $t + s \neq 0$ it follows, since R is an integral domain, that

$$H(t-s) = a(t-s). \quad (8)$$

Given any $x \in R$ there are infinitely many choices of s which satisfy $s, s+x, s+\alpha, s+x+\alpha, s+\alpha+1, s+x+\alpha+1 \in R \setminus E$. There is at most one s which satisfies $x + 2s = 0$. So we can choose $s \in R$ such that $s, s+x, s+\alpha, s+x+\alpha, s+\alpha+1, s+x+\alpha+1 \in R \setminus E$ and $x + 2s \neq 0$. Put $t = x + s$

then $t-s=x$, $t+s\neq 0$ and $s,t,s+\alpha,t+\alpha,s+\alpha+1,t+\alpha+1\in R\setminus E$ so from (8) we get

$$H(x)=ax.$$

Since $x\in R$ was arbitrary this concludes the proof. □

Before proving the main result, we make the following observation.

Any polynomial $p(x)=\sum_{k=0}^{n-2}a_kx^k$ satisfies

$$[x_1,x_2,\dots,x_n;p]=\begin{vmatrix}1 & x_1 & \dots & x_1^{n-2} & p(x_1)\\ \dots & \dots & \dots & \dots & \dots\\ 1 & x_n & \dots & x_n^{n-2} & p(x_n)\end{vmatrix}=0,$$

so the function $\tilde{f}=f+p$ also satisfies (3) whenever f satisfies (3), by the linearity of (3).

Theorem 2. *Let R be an infinite integral domain, with characteristic different from 2, and let n be an integer greater than 1 . Then for $f,h:R\to R$ satisfying*

$$[x_1,x_2,\dots,x_n;f]=h(x_1+\dots+x_n)[x_1,\dots,x_n],\ x_1,\dots,x_n\in R, \tag{9}$$

there exist $a,b,d\in R$ such that

$$f(x)=ax^n+bx^{n-1}+q(x),\ \ h(x)=ax+b,\ \ x\in R,$$

for some map $q:R\to R$ for which dq is a polynomial of degree at most $n-2$ over R. (That is, q is a polynomial over K, the field of fractions of R.)

Remark. Before proving the main theorem we remark that the conclusion of Theorem 2 cannot be strengthened in general for $n\geq 4$, as the following example shows. Let $R=\mathbb{Z}$ (the integers), and define $q:\mathbb{Z}\to\mathbb{Z}$ by

$$q(x):=\frac{x(x+1)}{2},\ \ x\in\mathbb{Z}.$$

Then $2q$ is a polynomial over $\mathbb{Z}$, but q is not. The functions

$$f(x)=ax^n+bx^{n-1}+q(x),\ \ h(x)=ax+b$$

satisfy (9) for any $n\geq 4$.

On the other hand the conclusion of Theorem 2 can be strengthened in the cases $n=2$ and $n=3$, as we will show later.

Proof. We adapt the proof of the theorem in [7] to the present situation. Given an arbitrary set of $n-1$ distinct elements $y_1,\dots,y_{n-1}\in R$, we wish to choose coefficients $a_0,\dots,a_{n-2}$ so that the map $\tilde{f}$

$$\tilde{f}(x):=f(x)-\sum_{k=0}^{n-2}a_kx^k \tag{10}$$

has $y_1, \dots, y_{n-1}$ as zeros. In order to do this, we must allow for the possibility that $a_0, \dots, a_{n-2}$ are in in the field K of fractions of R. Notice, however, that we are not extending our functions to K. (In fact, there is no obvious way to do this.) We need to solve the system of equations

$$\begin{pmatrix} 1 & y_1 & y_1^2 & \dots & y_1^{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & y_n & y_n^2 & \dots & y_n^{n-2} \end{pmatrix} \begin{pmatrix} a_0 \\ \dots \\ a_{n-2} \end{pmatrix} = \begin{pmatrix} f(y_1) \\ \dots \\ f(y_{n-1}) \end{pmatrix}.$$

Since the y_i's are distinct, the determinant of the matrix is

$$D = \prod_{\substack{i,j=1 \\ j>i}}^{n-1} (y_j - y_i) \neq 0,$$

and Cramer's Rule gives the solution as

$$a_i = \frac{D_i}{D}, \;\; D_i = \begin{pmatrix} 1 & \dots & y_1^{i-1} & f(y_1) & y_1^{i+1} & \dots & y_1^{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & y_{n-1}^{i-1} & f(y_{n-1}) & y_{n-1}^{i+1} & \dots & y_{n-1}^{n-2} \end{pmatrix}, 0 \leq i \leq n-2. \tag{11}$$

Multiplying (10) by D, we have

$$D\tilde{f}(x) = Df(x) - \sum_{k=0}^{n-2} D_k x^k. \tag{12}$$

So the map $F : R \to R$ defined by

$$F(x) = D\tilde{f}(x), \;\; x \in R, \tag{13}$$

satisfies

$$F(y_j) = 0, \;\; 1 \leq j \leq n-1.$$

Furthermore, multiplying (9), with $\tilde{f}$ in place of f, by D , we have also

$$[x_1, \dots, x_n; F] = Dh(x_1 + \dots + x_n)[x_1, \dots, x_n], \;\; x_1, \dots, x_n \in R. \tag{14}$$

Substituting $x_1 = x \in R$, $x_{j+1} = y_j$ $(j = 1, \dots, n-1)$ in (14) we get

$$(-1)^{n-1} DF(x) = Dh\left(x + \sum_{j=1}^{n-1} y_j\right) \prod_{j=1}^{n-1} (y_j - x) \prod_{\substack{i,j=1 \\ j>i}}^{n-1} (y_j - y_i), \;\; x \in R,$$

which simplifies to

$$F(x) = Dh\left(x + \sum_{j=1}^{n-1} y_j\right) \prod_{j=1}^{n-1} (x - y_j), \;\; x \in R. \tag{15}$$

Next substitute $x_1 = x$, $x_2 = y$, $x_{j+2} = y_j$, $(1 \leq j \leq n-2)$ in (14) to get

$$(-1)^{n-1}F(x)\begin{vmatrix}1 & y & \dots & y^{n-2}\\ 1 & y_1 & \dots & y_1^{n-2}\\ \dots & \dots & \dots & \dots\\ 1 & y_{n-2} & \dots & y_{n-2}^{n-2}\end{vmatrix} - (-1)^{n-1}F(y)\begin{vmatrix}1 & x & \dots & x^{n-2}\\ 1 & y_1 & \dots & y_1^{n-2}\\ \dots & \dots & \dots & \dots\\ 1 & y_{n-2} & \dots & y_{n-2}^{n-2}\end{vmatrix}$$

$$= Dh\Big(x+y+\sum_{j=1}^{n-2} y_j\Big)(y-x)\prod_{j=1}^{n-2}\Big[(y_j-x)(y_j-y)\prod_{\substack{i=1\\ j>i}}^{n-3}(y_j-y_i)\Big],$$

or

$$(-1)^{n-1}\Big\{F(x)\prod_{j=1}^{n-2}\Big[(y_j-y)\prod_{\substack{i=1\\ j>i}}^{n-3}(y_j-y_i)\Big]-F(y)\prod_{j=1}^{n-2}\Big[(y_j-x)\prod_{\substack{i=1\\ j>i}}^{n-3}(y_j-y_i)\Big]\Big\}$$

$$= Dh\Big(x+y+\sum_{j=1}^{n-2} y_j\Big)(y-x)\prod_{j=1}^{n-2}\Big[(y_j-x)(y_j-y)\prod_{\substack{i=1\\ j>i}}^{n-3}(y_j-y_i)\Big].$$

Inserting (15) into this and simplifying, we get

$$h\Big(x+\sum_{j=1}^{n-1} y_j\Big)(y_{n-1}-x) - h\Big(y+\sum_{j=1}^{n-1} y_j\Big)(y_{n-1}-y)$$

$$=h\Big(x+y+\sum_{j=1}^{n-2} y_j\Big)(y-x),\ \ \forall x,y\in R\setminus\{y_1,\dots,y_{n-2}\}.$$

Now let $s=x-y_{n-1}$, $t=y-y_{n-1}$ so $t,s\in R\setminus\{y_1-y_{n-1},\dots,y_{n-2}-y_{n-1}\}$ and

$$-sh\Big(s+\sum_{j=1}^{n-2} y_j+2y_{n-1}\Big)+th\Big(t+\sum_{j=1}^{n-2} y_j+2y_{n-1}\Big)$$

$$=(t-s)h\Big(s+t+\sum_{j=1}^{n-2} y_j+2y_{n-1}\Big).$$

Defining the map $H: R\to R$ by

$$H(s):=h\Big(s+\sum_{j=1}^{n-2} y_j+2y_{n-1}\Big)-h\Big(\sum_{j=1}^{n-2} y_j+2y_{n-1}\Big),\ \ s\in R, \tag{16}$$

we have

$$H(0)=0,$$

and

$$tH(t)-sH(s)=(t-s)H(s+t),\ \ s,t\in R\setminus E,$$

where $E = \{y_1 - y_{n-1}, \ldots, y_{n-2} - y_{n-1}\}$. By the lemma $H(u) = au$, for all $u \in R$. We get from (16) that

$$h(x) = H\left(x - \sum_{j=1}^{n-2} y_j - 2y_{n-1}\right) + h\left(\sum_{j=1}^{n-2} y_j + 2y_{n-1}\right) = ax + b,$$

for some constant $b \in R$. So from (15) we get

$$F(x) = D\left(ax + a\sum_{j=1}^{n-1} y_j + b\right)\prod_{j=1}^{n-1}(x - y_j), \quad x \in R.$$

From (13) we get

$$D\tilde{f}(x) = D(ax^n + bx^{n-1} + r(x)), \quad x \in R,$$

where r is a polynomial of degree at most $n-2$ over R. Since $D \neq 0$ and R is an integral domain we get

$$\tilde{f}(x) = ax^n + bx^{n-1} + r(x), \quad x \in R.$$

So by (12) we have

$$f(x) = ax^n + bx^{n-1} + q(x), \quad x \in R,$$

where

$$q(x) = r(x) + \sum_{k=0}^{n-2} a_k x^k.$$

Finally by (11), $x \mapsto Dq(x)$ is a polynomial of degree at most $n-2$ over R. □

2. The case n=2 revisited

In the case $n = 2$ we show that stronger results hold, even on the more general structure of a commutative ring with unit. We begin with a lemma similar to Lemma 1.

Lemma 3. *Let R be any commutative ring with unit, and suppose 2 is not a zero-divisor. Then every map $h : R \to R$ fulfilling*

$$h(x+y)(x-y) = h(x)x - h(y)y, \quad x, y \in R, \tag{17}$$

is of the form

$$h(x) = ax + b$$

for some constants $a, b \in R$.

Proof. First, define $H : R \to R$ by

$$H(x) := h(x) - h(0), \;\; x \in R.$$

Then $H(0) = 0$ and H also satisfies (17). With $x = u$ and $y = v + w$, then $x = u + v$ and $y = w$ in (17), we have

$$\begin{aligned} H(u+v+w)(u-v-w) &= H(u)u - H(v+w)(v+w) \\ &= H(u)u - H(v+w)[(v-w)+2w] \\ &= H(u)u - H(v)v + H(w)w - 2H(v+w)w, \end{aligned}$$

respectively

$$\begin{aligned} H(u+v+w)(u+v-w) &= H(u+v)(u+v) - H(w)w \\ &= H(u+v)[(u-v)+2v] - H(w)w \\ &= H(u)u - H(v)v + 2H(u+v)v - H(w)w. \end{aligned}$$

Subtracting the first equation from the second, we get

$$2H(u+v+w)v = 2[H(u+v)v - H(w)w + H(v+w)w].$$

Since 2 is not a zero-divisor, this means

$$[H(u+v+w) - H(u+v)]v = [H(v+w) - H(w)]w, \;\; u, v, w \in R. \tag{18}$$

With $u = t - 1, v = 1$ we deduce that

$$H(t+w) - H(t) = H(1+w) - H(w)]w, \;\; w, t \in R. \tag{19}$$

Now define $J : R \to R$ by

$$J(w) := [H(1+w) - H(w)]w, \;\; w \in R.$$

Then reducing (18) by (19), we have

$$J(w)v = J(v)w, \;\; w, v \in R.$$

With $v = 1$, this yields (with $a := J(1)$)

$$J(w) = aw, \;\; w \in R.$$

Inserting this into (19) and putting $t = 0$, we get

$$H(w) = aw, \;\; w \in R.$$

By definition of H, this gives the asserted form of h. □

The following example shows the necessity of the condition that 2 is not a zero divisor.

Example 4. When $R = \mathbb{Z}/4\mathbb{Z}$ (the ring of integers modulo 4) then $h : R \to R$ given by $h(0) = h(2) = 0$, $h(1) = 1$, $h(3) = 3$ is a solution of (17) which is *not* of the form $h(x) = ax + b$ for any $a, b \in R$.

Theorem 5. *Let R be a commutative ring with unit, in which 2 is not a zero-divisor. Then $f, h : R \to R$ satisfy*

$$f(x) - f(y) = h(x+y)(x-y), \quad x, y \in R, \tag{20}$$

if and only if there exist $a, b, c \in R$ such that

$$f(x) = ax^2 + bx + c, \quad h(x) = ax + b, \quad x \in R.$$

Proof. Putting $y = 0$ in (20) yields

$$f(x) = h(x)x + f(0), \quad x \in R,$$

and with this (20) reduces to (17). Hence we have $h(x) = ax + b$ by Lemma 3. Then

$$f(x) = (ax+b)x + f(0).$$

With $c := f(0)$, we have the asserted form of f. The converse is easily checked. □

3. The case n=3 revisited

In the case $n = 3$ we can also strengthen Theorem 2. As in the case $n = 2$, the function q must be a polynomial over R.

Theorem 6. *Let R be an infinite integral domain, with characteristic different from 2. The general solution $f, h : R \to R$ of*

$$\begin{aligned} f(x)(y-z) - f(y)(x-z) + f(z)(x-y) \\ = h(x+y+z)(x-y)(y-z)(x-z), \quad x, y, z \in R, \end{aligned} \tag{21}$$

is given by

$$f(x) = ax^3 + bx^2 + cx + d, \quad h(x) = ax + b, \quad x \in R,$$

for arbitrary constants $a, b, c, d \in R$.

Proof. Define $F : R \to R$ by

$$F(x) = f(x) - [f(1) - f(0)]x - f(0), \quad x \in R. \tag{22}$$

Then F satisfies (21) with h and $F(0) = F(1) = 0$. Putting $y = 1, z = 0$ in (21) with F in place of f, we get

$$F(x) = h(x+1)(x-1)x, \quad x \in R. \tag{23}$$

With this (21) yields (taking $z = 0$)

$$h(x+1)(x-1)xy - h(y+1)(y-1)yx = h(x+y)(x-y)xy, \quad x, y \in R,$$

and from this, since R is an integral domain, we get

$$h(x+1)(x-1) - h(y+1)(y-1) = h(x+y)(x-y), \quad x, y \in R \setminus \{0\}.$$

Defining $H : R \to R$ by

$$H(s) := h(s+2) - h(2), \quad s \in R, \tag{24}$$

we have $H(0) = 0$ and

$$H(x-1)(x-1) - H(y-1)(y-1) = H(x+y-2)(x-y), \quad x, y \in R \setminus \{0\}.$$

With $s = x - 1$, $t = y - 1$, this last equation becomes

$$H(s)s - H(t)t = H(s+t)(s-t), \quad s, t \in R \setminus \{-1\},$$

which is equation (6) with $E = \{-1\}$. Hence by Lemma 1 we have $H(u) = au$ for all $u \in R$, for some constant $a \in R$. Therefore, by (24) we see that

$$h(x) = H(x-2) + h(2) = ax + b, \quad x \in R,$$

for some constant $b \in R$. Then (23) yields

$$F(x) = [a(x+1) + b](x-1)x = ax^3 + bx^2 - (a+b)x, \quad x \in R,$$

and from this we get the asserted form of f by application of (22). □

Our next example shows that Theorem 6 cannot be extended to rings in general, even to those rings in which 2 is not a zero-divisor.

Example 7. Let $R = \mathbb{Z}/9\mathbb{Z}$ be the ring of integers modulo 9. Then the pair $f, h : R \to R$ given by $h(2) = h(7) = 0$, $h(x) = 3$, for $x \in R \setminus \{2, 7\}$ and $f(x) = (x-1)xh(x+1)$ is a solution of (21) but h is *not* of the form $h(x) = ax + b$ for any $a, b \in R$.

4. The case of $n + 1$ unknown functions

We conclude with the following generalization of Theorem 2.

Theorem 8. *Let R be an infinite integral domain, with characteristic different from 2, and let n be an integer greater than 1. Then for $f_1, \ldots, f_n, h : R \to R$ satisfying*

$$[x_1, x_2, \ldots, x_n; f_1, \ldots, f_n] = h(x_1 + \cdots + x_n)[x_1, \ldots, x_n], \ x_1, \ldots, x_n \in R, \tag{25}$$

there exist $a, b, d \in R$ such that

$$f_1(x) = \cdots = f_n(x) = ax^n + bx^{n-1} + q(x), \quad h(x) = ax + b, \quad x \in R,$$

for some map $q : R \to R$ such that dq is a polynomial of degree at most $n-2$ over R.

Proof. By Theorem 2, we need only show that $f_1, \ldots, f_n$ are equal. To this end, let i be any index from $\{1, \ldots, n-1\}$, and put $x_n = x_i$ in (25). Then subtract row i from row n in the determinant on the left hand side of the resulting equation. Expanding that determinant about row n we obtain

$$[x_1, \ldots, x_{n-1}]\,(f_n(x_i) - f_i(x_i)) = 0.$$

Thus $f_n = f_i$. Since i was arbitrary, all f_i's are equal and the result follows from Theorem 2. □

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Bruce R. Ebanks
Department of Mathematics and Statistics
P.O. Box MA
Mississippi State University
Mississippi State, MS 39762, USA
E–mail: ebanks@math.msstate.edu

Peter de Place Friis
Millsaps College
P.O. Box 150695
Jackson, MS 39210, USA
E–mail: friisp@millsaps.edu