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TAUBERIAN CONDITIONS FOR DOUBLE SEQUENCES THAT ARE STATISTICALLY SUMMABLE BY WEIGHTED MEANS

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Abstract. The concept of statistical convergence of ordinary (single) sequences was introduced by Fast in 1951. Basic properties of statistical convergence were proved by Schönberg and Fridy. Móricz extended the concept of statistical convergence from single to multiple sequences and proved some basic results. Móricz and Orhan have recently proved necessary and sufficient Tauberian conditions under which statistical convergence follows from statistical summability by weighted means. We extend this result from single to double sequences.

1. Introduction

In 1951 Fast [1] was the first mathematician who introduced an extension of the usual concept of sequential limits which he called statistical convergence. Schönberg [9] gave some properties of statistical convergence and also studied the concept as a summability method. Both of these authors noted that if a bounded sequence is statistically convergent to L then it is Cesàro summable to L. Basic properties of statistical convergence were proved by Fridy [2] too. Móricz extended in [5] the concept of statistical convergence from single to multiple sequences and proved some basic results. Móricz and Orhan [6] have recently proved necessary and sufficient Tauberian conditions under which statistical convergence follows from statistical summability $(\overline{N}, p)$ by weighted means. We extend this result from single to double sequences.

A double sequence $(x_{jk} : j,\ k = 0, 1, 2, \dots)$ of (real or complex) numbers is said to be *statistically convergent* to some number L, in symbol

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st–lim $x_{jk} = L$, if for each $\epsilon > 0$,

$$\lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} |\{j \le m \quad \text{and} \quad k \le n : |x_{jk} - L| \ge \epsilon\}| = 0. \quad (1.1)$$

Let $p := \{p_j\}_{j=0}^{\infty}$, $q := \{q_k\}_{k=0}^{\infty}$ be two sequences of nonnegative numbers $(p_0, q_0 > 0)$ with the property that

$$P_m := \sum_{j=0}^{m} p_j \to \infty \text{ as } m \to \infty \quad \text{and} \quad Q_n := \sum_{k=0}^{n} q_k \to \infty \text{ as } n \to \infty. \quad (1.2)$$

The weighted means of a given double sequence (x_{jk}) are the $(\overline{N}, p, q)$ means t_{mn}, which are defined by

$$t_{mn} = \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k x_{jk}, \quad m, n = 0, 1, 2 \ldots . \quad (1.3)$$

We say that the sequence x_{jk} is *statistically summable* $(\overline{N}, p, q)$ to L if st–lim $t_{mn} = L$, that is,

$$\lim_{M,N\to\infty} \frac{1}{(M+1)(N+1)} |\{m \le M \quad \text{and} \quad n \le N : |t_{mn} - L| \ge \epsilon\}| = 0. \quad (1.4)$$

Our goal is to find conditions under which st–lim $t_{mn} = L$ implies that st–lim $x_{jk} = L$.

The concepts of statistical limit inferior and limit superior of a sequence of real numbers was introduced by Fridy and Orhan [3]. We recall that if $\alpha :=$ st–lim $\inf_{k\to\infty} p_k$ is finite, then for every $\alpha_1 < \alpha$,

$$\lim_{n\to\infty} \frac{1}{n+1} |\{k \le n : p_k < \alpha_1\}| = 0 \quad (1.5)$$

and for every $\alpha_2 > \alpha$,

$$\lim_{n\to\infty} \frac{1}{n+1} |\{k \le n : p_k < \alpha_2\}| \ne 0. \quad (1.6)$$

The latter statement means that either the limit does not exist, or the limit exists and is greater than 0. If (1.5) is satisfied for every real number α_1, then we set st–$\lim_{k\to\infty} p_k =$ st–lim $\inf_{k\to\infty} p_k := +\infty$. If (1.6) is satisfied for every real number α_2, then we set st–lim $\inf_{k\to\infty} p_k := -\infty$.

The dual statement for $\beta :=$ st–lim $\sup_{k\to\infty} p_k$ can be formulated analogously.

2. New results

First, we consider sequences (x_{jk}) of real numbers and give *one–sided Tauberian conditions.*

Theorem 1. *Let $p := \{p_j\}_{j=0}^{\infty}$, and $q := \{q_k\}_{k=0}^{\infty}$ be two sequences of non-negative numbers such that $p_0 > 0$, $q_0 > 0$ and*

$$\text{st–}\liminf \frac{P_{\lambda_m}}{P_m} > 1 \quad \textit{and} \quad \text{st–}\liminf \frac{Q_{\lambda_n}}{Q_n} > 1 \quad \textit{for every } \lambda > 1, \tag{2.1}$$

where $\lambda_m := [\lambda m]$, $\lambda_n := [\lambda n]$, and let (x_{jk}) be a sequence of real numbers, which is statistically summable $(\overline{N}, p, q)$ to a finite number L. Then (x_{jk}) is statistically convergent to the same L if and only if the following two conditions hold: for every $\epsilon > 0$,

$$\inf_{\lambda>1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Big|\Big\{ m \le M \ \textit{and} \ n \le N : \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}) \le -\epsilon \Big\}\Big| = 0 \tag{2.2}$$

and

$$\inf_{0<\lambda<1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Big|\Big\{ m \le M \ \textit{and} \ n \le N : \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k (x_{mn} - x_{jk}) \le -\epsilon \Big\}\Big| = 0. \tag{2.3}$$

Remark 1. Conditions (2.2) and (2.3) are independent of one another. We show this but - by the simplicity - in one dimensional case and let $p_j \equiv 1$. Then (2.2) and (2.3) can be replaced by

$$\inf_{\lambda>1} \limsup_{M\to\infty} \frac{1}{M+1} \Big|\Big\{ m \le M : \ \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} (x_j - x_m) \le -\epsilon \Big\}\Big| = 0, \tag{2.2'}$$

and

$$\inf_{0<\lambda<1} \limsup_{M\to\infty} \frac{1}{M+1} \Big|\Big\{ m \le M : \ \frac{1}{m - \lambda_m} \sum_{j=\lambda_m+1}^{m} (x_m - x_j) \le -\epsilon \Big\}\Big| = 0. \tag{2.3'}$$

We construct a statistically not convergent sequence (x_j) such that (x_j) is statistically summable to zero and condition (2.2') is satisfied but (2.3') is violated. Let the sequence (x_j) given by

$$x_j := \begin{cases} 1 & \text{if } l^2 - l + 1 \le j \le l^2, \\ -l & \text{if } j = l^2 + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $x_j = \{1,\ -1,\ 1,\ 1,\ -2,\ 0,\ 1,\ 1,\ 1,\ -3,\ 0,\ 0,\ 1,\ 1,\ 1,\ 1,\ -4,\ \dots\}$. It is clear that $\text{st–}\liminf x_j = 0$ and $\text{st–}\limsup x_j = 1$. Therefore x_j is not statistically convergent. It is easy to check that (x_j) is statistically summable to zero. (Even more is true, (x_j) is (C,1) summable to zero.) On the other hand, by virtue of Lemma 4 [see the one dimensional case], we have for every $\lambda > 1$,

$$\text{st–}\limsup \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} (x_j - x_m) = \text{st–}\lim \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} x_j - \text{st–}\liminf x_m = 0,$$

while for every $0 < \lambda < 1$,

$$\text{st–}\limsup \frac{1}{m - \lambda_m} \sum_{j=\lambda_m+1}^{m} (x_m - x_j) = \text{st–}\limsup x_m - \text{st–}\lim \frac{1}{m - \lambda_m} \sum_{j=\lambda_m+1}^{m} x_j \ = 1.$$

Remark 2. It is easy to check that conditions (2.1) imply (1.2).

Remark 3. If conditions (1.1), (1.4) and (2.1) hold, then we necessarily have

$$\text{st–}\lim \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}) = 0 \qquad (2.4)$$

for every $\lambda > 1$, and

$$\text{st–}\lim \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k (x_{mn} - x_{jk}) = 0 \qquad (2.5)$$

for every $0 < \lambda < 1$.

Following Schmidt [8], we say that a double sequence (x_{jk}) is *statistically slowly decreasing* with respect to the first index if, for every $\epsilon > 0$,

$$\inf_{\lambda>1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Big|\Big\{m \le M \text{ and } n \le N : \min_{m<j\le\lambda_m} (x_{jn} - x_{mn}) \le -\epsilon\Big\}\Big| = 0. \qquad (2.6)$$

We say that (x_{jk}) is *statistically slowly decreasing in the strong sense* with respect to the first index if (2.6) is satisfied with

$$\min_{\substack{m<j\leq\lambda_m\\ n<k\leq\lambda_n}} (x_{jk} - x_{mk}) \text{ in place of } \min_{m<j\leq\lambda_m} (x_{jn} - x_{mn}). \tag{2.6'}$$

Analogously, we say that (x_{jk}) is statistically slowly decreasing with respect to the second index if, for every $\epsilon > 0$,

$$\inf_{\lambda>1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Big|\Big\{m \leq M \text{ and } n \leq N : \min_{n<k\leq\lambda_n} (x_{mk} - x_{mn}) \leq -\epsilon\Big\}\Big| = 0; \tag{2.7}$$

and (x_{jk}) is said to enjoy this property in the strong sense if (2.7) is satisfied with

$$\min_{\substack{m<j\leq\lambda_m\\ n<k\leq\lambda_n}} (x_{jk} - x_{jn}) \text{ in place of } \min_{n<k\leq\lambda_n} (x_{mk} - x_{mn}). \tag{2.7'}$$

Remark 4. It is not difficult to check that (2.6) implies

$$\inf_{0<\lambda<1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Big|\Big\{m \leq M \text{ and } n \leq N : \min_{\lambda_m<j\leq m} (x_{mn} - x_{jn}) \leq -\epsilon\Big\}\Big| = 0,$$

and vice versa. The same equivalence hold in the case of (2.7) and in the cases where (2.6) and (2.7) are meant in the strong sense (that is, in the cases when (2.6) is modified by (2.6'), and (2.7) is modified by (2.7')). Taking into account that for the expression in (2.2), we have

$$\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}) \geq \min_{\substack{m<j\leq\lambda_m\\ n<k\leq\lambda_n}} (x_{jk} - x_{mk}) + \min_{n<k\leq\lambda_n} (x_{mk} - x_{mn})$$

and an analogous one for the corresponding expression in (2.3). The following corollary is an immediate consequence of Theorem 1.

Corollary 1. *Let* $p := \{p_j\}_{j=0}^\infty$, *and* $q := \{q_k\}_{k=0}^\infty$ *be two sequences of nonnegative numbers such that* $p_0 > 0$, $q_0 > 0$ *and conditions in* (2.1) *are satisfied, and let* (x_{jk}) *be a statistically slowly decreasing sequence with respect to both indices and, in addition, in the strong sense with respect to one of the indices. Then*

$$\text{st--lim } t_{mn} = L \textit{ implies } \text{st--lim } x_{jk} = L.$$

Second, we consider sequences (x_{jk}) of complex numbers and give *two–sided Tauberian conditions.*

Theorem 2. *Let* $p := \{p_j\}_{j=0}^{\infty}$, *and* $q := \{q_k\}_{k=0}^{\infty}$ *be two sequences of non-negative numbers such that* $p_0 > 0$, $q_0 > 0$ *and conditions in* (2.1) *are satisfied. Let* (x_{jk}) *be a sequence of complex numbers which is statistically summable* $(\overline{N}, p, q)$ *to* L. *Then* (x_{jk}) *is statistically convergent to the same* L *if and only if one of the following two conditions holds: for every* $\epsilon > 0$, *either*

$$\inf_{\lambda>1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Big| \Big\{ m \le M \text{ and } n \le N : \Big| \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}) \Big| \ge \epsilon \Big\} \Big| = 0 \tag{2.8}$$

or

$$\inf_{0<\lambda<1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Big| \Big\{ m \le M \text{ and } n \le N : \Big| \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k (x_{mn} - x_{jk}) \Big| \ge \epsilon \Big\} \Big| = 0. \tag{2.9}$$

Remark 5. Again even more is true: If conditions (1.1), (1.4) and (2.1) are satisfied, then we necessarily have (2.4) for every $\lambda > 1$, and (2.5) for every $0 < \lambda < 1$.

Following Hardy [4], a double sequence (x_{jk}) of complex numbers is said to be *statistically slowly oscillating* with respect to the first index if, for every $\epsilon > 0$,

$$\inf_{\lambda>1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Big| \Big\{ m \le M \text{ and } n \le N : \max_{m<j\le\lambda_m} |(x_{jn} - x_{mn})| \ge \epsilon \Big\} \Big| = 0. \tag{2.10}$$

We say that (x_{jk}) is *statistically slowly oscillating in the strong sense* with respect to the first index if (2.10) is satisfied with

$$\max_{\substack{m<j\le\lambda_m \\ n<k\le\lambda_n}} |(x_{jk} - x_{mk})| \text{ in place of } \max_{m<j\le\lambda_m} |(x_{jn} - x_{mn})|.$$

The statistically slow oscillation property with respect to the second index is defined analogously.

Remark 6. Similarly to Remark 4, condition (2.10) is equivalent to the following one: for every $\epsilon > 0$,

$$\inf_{0<\lambda<1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Big|\Big\{m \le M \text{ and } n \le N : \max_{\lambda_m<j\le m} |(x_{mn} - x_{jn})| \ge \epsilon\Big\}\Big| = 0,$$

and an analogous equivalence holds in the strong sense, as well.

Corollary 2. *Let $p := \{p_j\}_{j=0}^{\infty}$, and $q := \{q_k\}_{k=0}^{\infty}$ be two sequences of nonnegative numbers such that $p_0 > 0$, $q_0 > 0$ and conditions in* (2.1) *are satisfied, and let (x_{jk}) be a statistically slowly oscillating sequence with respect to both indices and, in addition, in the strong sense with respect to one of the indices. Then*

$$\text{st–lim } t_{mn} = L \textit{ implies } \text{st–lim } x_{jk} = L.$$

In the special case of summability (C, 1, 1) when $p_j \equiv 1$ and $q_k \equiv 1$, our theorems and corollaries were proved in [7].

3. Proofs

Lemma 1. *If P_m and Q_n are nondecreasing sequences of positive numbers, then conditions in* (2.1) *are equivalent with*

$$\text{st–}\liminf \frac{P_m}{P_{\lambda_m}} > 1 \textit{ and } \text{st–}\liminf \frac{Q_n}{Q_{\lambda_n}} > 1 \textit{ for every } 0 < \lambda < 1.$$

Proof. The proof is the same as in [6, Lemma 1]. □

Lemma 2. *Let $p := \{p_j\}_{j=0}^{\infty}$, and $q := \{q_k\}_{k=0}^{\infty}$ be two sequences of nonnegative numbers such that $p_0 > 0$, $q_0 > 0$ and conditions in* (2.1) *are satisfied, and let (x_{jk}) be a sequence of complex numbers which is statistically summable $(\overline{N}, p, q)$ to a finite number L. Then for every $\lambda > 0$,*

$$\text{st–lim } t_{\lambda_m,\lambda_n} = L. \tag{3.1}$$

Proof. Case $\lambda > 1$. For each $M \ge 1$, $N \ge 1$ and $\epsilon > 0$, we have

$$\{m \le M,\, n \le N : |t_{\lambda_m,\lambda_n} - L| \ge \epsilon\} \subseteq \{m \le \lambda_M,\, n \le \lambda_N : |t_{mn} - L| \ge \epsilon\},$$

whence we find

$$\frac{1}{(M+1)(N+1)} \big|\{m \le M,\, n \le N : |t_{\lambda_m,\lambda_n} - L| \ge \epsilon\}\big| \le \frac{\lambda^2}{(\lambda_M+1)(\lambda_N+1)} \big|\{m \le \lambda_M,\, n \le \lambda_N : |t_{mn} - L| \ge \epsilon\}\big|,$$

since $\frac{\lambda_M+1}{M+1} \leq \frac{\lambda M+1}{M+1} < \lambda$ and a similar inequality holds for $\frac{\lambda_N+1}{N+1}$. Since st–lim $t_{mn} = L$, therefore the left term of this inequality tends to zero, that is st–lim $t_{\lambda_m, \lambda_n} = L$.

Case $0 < \lambda < 1$. We claim that the same term t_{jk} cannot occur more than $(1+\frac{1}{\lambda})^2$ times in the sequence t_{λ_m, λ_n}. In fact, for fixed k let p and q some integers such that

$$j = \lambda_p = \lambda_{p+1} = \cdots = \lambda_{p+q-1} < \lambda_{p+q}$$

or equivalently,

$$j \leq \lambda p < \lambda(p+1) < \cdots < \lambda(p+q-1) < j+1 \leq \lambda(p+q),$$

then

$$j + \lambda(q-1) \leq \lambda(p+q-1) < j+1,$$

whence it follows that $\lambda(q-1) < 1$, that is $q < 1 + \frac{1}{\lambda}$.

Similarly for fixed j we can prove that t_{jk} cannot occur more than $(1+\frac{1}{\lambda})$ times in the sequence t_{λ_m, λ_n}. Consequently,

$$\frac{1}{(M+1)(N+1)}\left|\left\{m \leq M,\, n \leq N :\, |t_{\lambda_m, \lambda_n} - L| \geq \epsilon\right\}\right|$$

$$\leq (1+\frac{1}{\lambda})^2 \frac{\lambda_M+1}{M+1}\frac{\lambda_N+1}{N+1}\frac{1}{(\lambda_M+1)(\lambda_N+1)}\left|\left\{m \leq \lambda_M,\, n \leq \lambda_N : |t_{mn} - L| \geq \epsilon\right\}\right|$$

$$\leq \frac{(\lambda+1)^2}{\lambda^2} 2\lambda^2 \frac{1}{(\lambda_M+1)(\lambda_N+1)}\left|\left\{m \leq \lambda_M,\, n \leq \lambda_N : |t_{mn} - L| \geq \epsilon\right\}\right| \to 0, \text{ as } M,\, N \to \infty.$$

We used that

$$\frac{(\lambda_M+1)(\lambda_N+1)}{(M+1)(N+1)} \leq 2\lambda^2.$$

Therefore st–lim $t_{\lambda_m, \lambda_n} = L$.

□

The next two representations are very important in the proof of Theorems 1 and 2.

Lemma 3. *If* $\lambda > 1$*, then*

$$\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k x_{jk}$$

$$= t_{\lambda_m, \lambda_n} + \frac{P_m}{P_{\lambda_m} - P_m}(t_{\lambda_m, \lambda_n} - t_{m, \lambda_n}) + \frac{Q_n}{Q_{\lambda_n} - Q_n}(t_{\lambda_m, \lambda_n} - t_{\lambda_m, n})$$

$$+\frac{P_m Q_n}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}(t_{\lambda_m,\lambda_n}-t_{m,\lambda_n}-t_{\lambda_m,n}+t_{mn}). \tag{3.2}$$

If $0<\lambda<1$, *then*

$$\frac{1}{(P_m-P_{\lambda_m})(Q_n-Q_{\lambda_n})}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}p_j q_k x_{jk}=$$
$$=t_{mn}+\frac{P_{\lambda_m}}{P_m-P_{\lambda_m}}(t_{mn}-t_{\lambda_m,n})+\frac{Q_{\lambda_n}}{Q_n-Q_{\lambda_n}}(t_{mn}-t_{m,\lambda_n})$$
$$+\frac{P_{\lambda_m}Q_{\lambda_n}}{(P_m-P_{\lambda_m})(Q_n-Q_{\lambda_n})}(t_{mn}-t_{\lambda_m,n}-t_{m,\lambda_n}+t_{\lambda_m,\lambda_n}). \tag{3.3}$$

Proof. Case $\lambda>1$. By definition

$$\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}p_j q_k x_{jk}=$$
$$=P_{\lambda_m}Q_{\lambda_n}t_{\lambda_m,\lambda_n}-P_{\lambda_m}Q_n t_{\lambda_m,n}-P_m Q_{\lambda_n}t_{m,\lambda_n}+P_m Q_n t_{mn}.$$

Using this, a simple rearranging gives that

$$\frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}p_j q_k x_{jk}$$
$$=t_{\lambda_m,\lambda_n}+\frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\big\{(P_{\lambda_m}Q_n-P_m Q_n)(t_{\lambda_m,\lambda_n}-t_{\lambda_m,n})$$
$$+(P_m Q_{\lambda_n}-P_m Q_n)(t_{\lambda_m,\lambda_n}-t_{m,\lambda_n})+P_m Q_n(t_{\lambda_m,\lambda_n}-t_{m,\lambda_n}-t_{\lambda_m,n}+t_{mn})\big\}$$
$$=t_{\lambda_m,\lambda_n}+\frac{Q_n}{Q_{\lambda_n}-Q_n}(t_{\lambda_m,\lambda_n}-t_{\lambda_m,n})+\frac{P_m}{P_{\lambda_m}-P_m}(t_{\lambda_m,\lambda_n}-t_{m,\lambda_n})$$
$$+\frac{P_m Q_n}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}(t_{\lambda_m,\lambda_n}-t_{m,\lambda_n}-t_{\lambda_m,n}+t_{mn}).$$

The proof of (3.3) is similar to that of (3.2). □

Lemma 4. *Let* $p:=\{p_j\}_{j=0}^{\infty}$, *and* $q:=\{q_k\}_{k=0}^{\infty}$ *be two sequences of nonnegative numbers such that* $p_0>0$, $q_0>0$ *and conditions in* (2.1) *are satisfied, and let* (x_{jk}) *be a sequence of complex numbers which is statistically summable* $(\overline{N},p,q)$ *to a finite number* L. *Then for every* $\lambda>1$,

$$\text{st–lim}\,\frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}p_j q_k x_{jk}=L, \tag{3.4}$$

and for every $0 < \lambda < 1$,

$$\text{st–lim}\, \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k x_{jk} = L. \tag{3.5}$$

Proof. Case $\lambda > 1$. We use representation (3.2) and the fact that

$$\text{st–lim sup}\, \frac{P_m}{P_{\lambda_m} - P_m} = \text{st–lim sup}\, \frac{1}{\frac{P_{\lambda_m}}{P_m} - 1} = \frac{1}{\text{st–lim inf}\, \frac{P_{\lambda_m}}{P_m} - 1} < \infty, \tag{3.6}$$

due to (2.1). Similarly, we have

$$\text{st–lim sup}\, \frac{Q_n}{Q_{\lambda_n} - Q_n} < \infty. \tag{3.7}$$

Now (3.4) follows from the statistical summability $(\overline{N}, p, q)$ of (x_{jk}) and from Lemma 2.

Case $0 < \lambda < 1$. We use representation (3.3). Since by Lemma 1, we have

$$\text{st–lim sup}\, \frac{P_{\lambda_m}}{P_m - P_{\lambda_m}} = \frac{1}{\text{st–lim inf}\, \frac{P_m}{P_{\lambda_m}} - 1} < \infty \tag{3.8}$$

and

$$\text{st–lim sup}\, \frac{Q_{\lambda_n}}{Q_n - Q_{\lambda_n}} < \infty, \tag{3.9}$$

(3.5) follows again from the statistical summability $(\overline{N}, p, q)$ of (x_{jk}) and from Lemma 2. □

Proof of Theorem 1. *Necessity.* Assume that (x_{jk}) is both statistically convergent and statistically summable $(\overline{N}, p, q)$ of (x_{jk}) to the same number. In case $\lambda > 1$ applying Lemma 4 we get

$$\begin{aligned} &\text{st–lim}\, \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}) \\ &= \text{st–lim}\, \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k x_{jk} - \text{st–lim}\, x_{mn} \\ &= L - L = 0 \text{ as } m, n \to \infty. \end{aligned}$$

This proves (2.4). Similarly, Lemma 4 yields (2.5) for every $0 < \lambda < 1$.

Sufficiency. Assume that st–lim $t_{mn} = L$ and conditions (2.1)-(2.3) are satisfied. In order to prove that st–lim $x_{jk} = L$, it suffices to prove that

$$\text{st–lim}(x_{mn} - t_{mn}) = 0. \tag{3.10}$$

Case $\lambda > 1$. By Lemma 3, we have

$$x_{mn} - t_{\lambda_m, \lambda_n} = \frac{P_m}{P_{\lambda_m} - P_m}(t_{\lambda_m, \lambda_n} - t_{m, \lambda_n}) + \frac{Q_n}{Q_{\lambda_n} - Q_n}(t_{\lambda_m, \lambda_n} - t_{\lambda_m, n})$$
$$+ \frac{P_m Q_n}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)}(t_{\lambda_m, \lambda_n} - t_{m, \lambda_n} - t_{\lambda_m, n} + t_{mn})$$
$$- \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}) \qquad (3.11)$$

whence, for any $\epsilon > 0$,

$$\{m \le M,\, n \le N : x_{mn} - t_{\lambda_m, \lambda_n} \ge \epsilon\} \subseteq$$
$$\Big\{m \le M, n \le N : \frac{P_m}{P_{\lambda_m} - P_m}(t_{\lambda_m,\lambda_n} - t_{m,\lambda_n}) + \frac{Q_n}{Q_{\lambda_n} - Q_n}(t_{\lambda_m,\lambda_n} - t_{\lambda_m,n})$$
$$+ \frac{P_m Q_n}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)}(t_{\lambda_m, \lambda_n} - t_{m, \lambda_n} - t_{\lambda_m, n} + t_{mn}) \ge \frac{\epsilon}{2}\Big\}$$
$$\cup \Big\{m \le M,\, n \le N : \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn})$$
$$\le -\frac{\epsilon}{2}\Big\} =: A_{MN}(\epsilon) \,\cup\, B_{MN}(\epsilon), \text{ say.} \qquad (3.12)$$

By virtue of Lemma 2 and (3.6), (3.7), for every $\epsilon > 0$, we have

$$\lim_{M,N\to\infty} \frac{1}{(M+1)(N+1)} |A_{MN}(\epsilon)| = 0. \qquad (3.13)$$

On the other hand, given any $\delta > 0$, by (2.2) there exists some $\lambda > 1$ such that

$$\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} |B_{MN}(\epsilon)| \le \delta. \qquad (3.14)$$

Combining (3.12)-(3.14) gives

$$\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} |\{m \le M,\, n \le N : x_{mn} - t_{\lambda_m, \lambda_n} \ge \epsilon\}| \le \delta.$$

Since $\delta > 0$ is arbitrary, for every $\epsilon > 0$, we have

$$\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} |\{m \le M,\, n \le N : x_{mn} - t_{\lambda_m, \lambda_n} \ge \epsilon\}| = 0. \quad (3.15)$$

Applying Lemma 2 gives

$$\lim_{M,N\to\infty} \frac{1}{(M+1)(N+1)} |\{m \le M,\, n \le N : x_{mn} - t_{mn} \ge \epsilon\}| = 0. \quad (3.16)$$

Case $0 < \lambda < 1$. By (3.3), we have

$$\begin{aligned} x_{mn} - t_{mn} &= \frac{P_{\lambda_m}}{P_m - P_{\lambda_m}}(t_{mn} - t_{\lambda_m, n}) + \frac{Q_{\lambda_n}}{Q_n - Q_{\lambda_n}}(t_{mn} - t_{m, \lambda_n}) \\ &+ \frac{P_{\lambda_m} Q_{\lambda_n}}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})}(t_{mn} - t_{\lambda_m, n} - t_{m, \lambda_n} + t_{\lambda_m, \lambda_n}) \\ &+ \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k (x_{mn} - x_{jk}), \end{aligned} \tag{3.17}$$

whence for any $\epsilon > 0$

$$\begin{aligned} &\{m \le M,\, n \le N : \, x_{mn} - t_{mn} \le -\epsilon\} \subseteq \\ &\Big\{m \le M,\, n \le N : \, \frac{P_{\lambda_m}}{P_m - P_{\lambda_m}}(t_{mn} - t_{\lambda_m, n}) + \frac{Q_{\lambda_n}}{Q_n - Q_{\lambda_n}}(t_{mn} - t_{m, \lambda_n}) \\ &+ \frac{P_{\lambda_m} Q_{\lambda_n}}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})}(t_{mn} - t_{\lambda_m, n} - t_{m, \lambda_n} + t_{\lambda_m, \lambda_n}) \le -\frac{\epsilon}{2}\Big\} \\ &\cup \Big\{m \le M, n \le N : \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \cdot \\ &\qquad\qquad \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k (x_{mn} - x_{jk}) \le -\frac{\epsilon}{2}\Big\}. \end{aligned}$$

By virtue of Lemma 2, (2.3) and (3.8), (3.9), for every $\epsilon > 0$, we conclude

$$\lim_{M,N\to\infty} \frac{1}{(M+1)(N+1)} |\{m \le M,\, n \le N : \, x_{mn} - t_{mn} \le -\epsilon\}| = 0. \tag{3.18}$$

Combining (3.16) and (3.18) yields for every $\epsilon > 0$,

$$\lim_{M,N\to\infty} \frac{1}{(M+1)(N+1)} |\{m \le M,\, n \le N : \, |x_{mn} - t_{mn}| \ge \epsilon\}| = 0.$$

This proves (3.10) and the proof of Theorem 1 is complete. □

Proof of Theorem 2. *Necessity.* Assume that (x_{jk}) is both statistically convergent and statistically summable $(\overline{N}, p, q)$ of (x_{jk}) to the same number. Applying Lemma 4 yields (2.4) for every $\lambda > 1$ and (2.5) for every $0 < \lambda < 1$.

Sufficiency. Assume that st–lim $t_{mn} = L$ and one of the conditions (2.8) and (2.9) is satisfied. In order to prove that (x_{jk}) is statistically convergent to the same number, again it is enough to prove (3.10).

Let $\epsilon > 0$ be given. In case $\lambda > 1$, by (3.11) we have

$$\{m \le M,\, n \le N : \, |x_{mn} - t_{\lambda_m, \lambda_n}| \ge \epsilon\} \subseteq \Big\{m \le M,\, n \le N :$$

$$\Big|\frac{P_m}{P_{\lambda_m}-P_m}(t_{\lambda_m,\lambda_n}-t_{m,\lambda_n})+\frac{Q_n}{Q_{\lambda_n}-Q_n}(t_{\lambda_m,\lambda_n}-t_{\lambda_m,n})$$
$$+\frac{P_mQ_n}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}(t_{\lambda_m,\lambda_n}-t_{m,\lambda_n}-t_{\lambda_m,n}+t_{mn})\Big|\geq\frac{\epsilon}{2}\Big\}$$
$$\cup\Big\{m\leq M,\,n\leq N:\frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\Big|\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}p_jq_k(x_{jk}-x_{mn})\Big|$$
$$\geq\frac{\epsilon}{2}\Big\}:=A^{(1)}_{MN}(\epsilon)\cup B^{(1)}_{MN}(\epsilon). \tag{3.19}$$

Given $\delta>0$, by (2.8) there exist some $\lambda>1$ such that

$$\limsup_{M,N\to\infty}\frac{1}{(M+1)(N+1)}|B^{(1)}_{MN}(\epsilon)|\leq\delta.$$

In case $0<\lambda<1$, by (3.17) we have

$$\{m\leq M,\,n\leq N:\,|x_{mn}-t_{mn}|\leq\epsilon\}\subseteq\Big\{m\leq M,\,n\leq N:$$
$$\Big|\frac{P_{\lambda_m}}{P_m-P_{\lambda_m}}(t_{mn}-t_{\lambda_m,n})+\frac{Q_{\lambda_n}}{Q_n-Q_{\lambda_n}}(t_{mn}-t_{m,\lambda_n})$$
$$+\frac{P_{\lambda_m}Q_{\lambda_n}}{(P_m-P_{\lambda_m})(Q_n-Q_{\lambda_n})}(t_{mn}-t_{\lambda_m,n}-t_{m,\lambda_n}+t_{\lambda_m,\lambda_n})\Big|\geq\frac{\epsilon}{2}\Big\}$$
$$\cup\Big\{m\leq M,n\leq N:\frac{1}{(P_m-P_{\lambda_m})(Q_n-Q_{\lambda_n})}\Big|\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}p_jq_k(x_{mn}-x_{jk})\Big|$$
$$\geq\frac{\epsilon}{2}\Big\}:=A^{(2)}_{MN}(\epsilon)\cup B^{(2)}_{MN}(\epsilon). \tag{3.20}$$

Given $\delta>0$, by (2.9) there exist some $0<\lambda<1$ such that

$$\limsup_{M,N\to\infty}\frac{1}{(M+1)(N+1)}|B^{(2)}_{MN}(\epsilon)|\leq\delta.$$

By (3.19), (3.20) and Lemma 2, in either case we conclude that

$$\limsup_{M,N\to\infty}\frac{1}{(M+1)(N+1)}|\{m\leq M,\,n\leq N:\,|x_{mn}-t_{mn}|\geq\epsilon\}|\leq\delta.$$

Since $\delta>0$ is arbitrary, it follows that for every $\epsilon>0$,

$$\lim_{M,N\to\infty}\frac{1}{(M+1)(N+1)}|\{m\leq M,\,n\leq N:\,|x_{mn}-t_{mn}|\geq\epsilon\}|=0.$$

This proves (3.10) and the proof of Theorem 2 is complete. □

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