SARAJEVO JOURNAL OF MATHEMATICS
Vol.1 (14) (2005), 211–234

THE KONTOROVICH - LEBEDEV TRANSFORMATION ON SOBOLEV TYPE SPACES

SEMYON B. YAKUBOVICH

Abstract. The Kontorovich-Lebedev transformation

$$(KLf)(x) = \int_0^\infty K_{i\tau}(x) f(\tau) d\tau, \ \ x \in \mathbf{R}_+$$

is considered as an operator, which maps the weighted space $L_p(\mathbf{R}_+; \omega(\tau)d\tau)$, $2 \le p \le \infty$ into the Sobolev type space $S_p^{N,\alpha}(\mathbf{R}_+)$ with the finite norm

$$||u||_{S_p^{N,\alpha}(\mathbf{R}_+)} = \left(\sum_{k=0}^{N} \int_0^\infty |A_x^k u|^p x^{\alpha_k p - 1} dx \right)^{1/p} < \infty,$$

where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_N), \alpha_k \in \mathbf{R}, k = 0, \ldots, N$, and A_x is the differential operator of the form

$$A_x u = x^2 u(x) - x \frac{d}{dx}\left[x \frac{du}{dx} \right],$$

and A_x^k means k-th iterate of A_x, $A_x^0 u = u$. Elementary properties for the space $S_p^{N,\alpha}(\mathbf{R}_+)$ are derived. Boundedness and inversion properties for the Kontorovich-Lebedev transform are studied. In the Hilbert case ($p = 2$) the isomorphism between these spaces is established for the special type of weights and Plancherel's type theorem is proved.

1. Introduction

The object of the present paper is to extend the theory of the Kontorovich-Lebedev transformation [8], [11]

$$(KLf)(x) = \int_0^\infty K_{i\tau}(x) f(\tau)\, d\tau, \tag{1.1}$$

on the so-called Sobolev type spaces, which will be defined below. In the following, $x \in \mathbf{R}_+ \equiv (0, \infty)$, $K_{i\tau}(x)$ is the modified Bessel function or the Macdonald function (cf. [1], [8, p. 355]), and the pure imaginary subscript

2000 *Mathematics Subject Classification.* 44A15, 46E35, 26D10.
Key words and phrases. Kontorovich-Lebedev transform, modified Bessel function, Sobolev spaces, Hardy inequality, Plancherel theorem, imbedding theorem.

(an index) $i\tau$ is such that τ is restricted to $\mathbf{R}_+$. The function $K_\nu(z)$ satisfies the differential equation

$$z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}-(z^2+\nu^2)u=0. \tag{1.2}$$

for which it is the solution that remains bounded as z tends to infinity on the real line. The modified Bessel function has the asymptotic behaviour (cf. [1], relations (9.6.8), (9.6.9), (9.7.2))

$$K_\nu(z)=\left(\frac{\pi}{2z}\right)^{1/2}e^{-z}[1+O(1/z)],\ \ z\to\infty, \tag{1.3}$$

and near the origin

$$K_\nu(z)=O\left(z^{-|\mathrm{Re}\,\nu|}\right),\ \ z\to 0, \tag{1.4}$$

$$K_0(z)=O(\log z),\ \ z\to 0. \tag{1.5}$$

Meanwhile, when x is restricted to any compact subset of $\mathbf{R}_+$ and τ tends to infinity we have the following asymptotic [11, p. 20]

$$K_{i\tau}(x)=\left(\frac{2\pi}{\tau}\right)^{1/2}e^{-\pi\tau/2}\sin\left(\frac{\pi}{4}+\tau\log\frac{2\tau}{x}-\tau\right)\left[1+O(1/\tau)\right],\ \ \tau\to\infty. \tag{1.6}$$

The modified Bessel function can be represented by the integrals of the Fourier and Mellin types [1], [8], [11]

$$K_\nu(x)=\int_0^\infty e^{-x\cosh u}\cosh\nu u\,du, \tag{1.7}$$

$$K_\nu(x)=\frac{1}{2}\left(\frac{x}{2}\right)^\nu\int_0^\infty e^{-t-\frac{x^2}{4t}}t^{-\nu-1}dt. \tag{1.8}$$

Hence it is not difficult to show that for positive values of x and τ $K_{i\tau}(x)$ is real-valued and infinitely times differentiable. We also note that the product of the modified Bessel functions of different arguments can be represented by the Macdonald formula [1], [6], [11]

$$K_{i\tau}(x)K_{i\tau}(y)=\frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\left(u\frac{x^2+y^2}{xy}+\frac{xy}{u}\right)}K_{i\tau}(u)\frac{du}{u}. \tag{1.9}$$

In this paper we deal with the Lebesgue weighted $L_p(\mathbf{R}_+;\omega(x)dx)$ spaces with respect to the measure $\omega(x)dx$ with the norm

$$||f||_p=\left(\int_0^\infty|f(x)|^p\omega(x)dx\right)^{1/p},\ \ 1\le p<\infty, \tag{1.10}$$

$$||f||_\infty=\text{ess sup}\,|f(x)|. \tag{1.11}$$

In particular, we will use the spaces $L_{\nu,p} \equiv L_p(\mathbf{R}_+; x^{\nu p-1}dx),\ 1 \leq p \leq \infty,\ \nu \in \mathbf{R}$, which are related to the Mellin transforms pair [7], [8], [9]

$$f^{\mathcal{M}}(s) = \int_0^\infty f(x)x^{s-1}dx, \tag{1.12}$$

$$f(x) = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} f^{\mathcal{M}}(s)x^{-s}ds,\ s = \nu + it,\ x > 0. \tag{1.13}$$

The integrals (1.12)- (1.13) are convergent, in particular, in mean with respect to the norm of the spaces $L_2(\nu - i\infty, \nu + i\infty; ds)$ and $L_2(\mathbf{R}_+; x^{2\nu-1}dx)$, respectively. In addition, the Parseval equality of the form

$$\int_0^\infty |f(x)|^2 x^{2\nu-1}dx = \frac{1}{2\pi}\int_{-\infty}^\infty |f^{\mathcal{M}}(\nu + it)|^2 dt \tag{1.14}$$

holds true.

As it is proved in [12], [13], the Kontorovich-Lebedev operator (1.1) is an isomorphism between the spaces $L_2(\mathbf{R}_+; [\tau \sinh \pi\tau]^{-1}d\tau)$ and $L_2(\mathbf{R}_+; x^{-1}dx)$ with the identity for the square of norms

$$\int_0^\infty |(KLf)(x)|^2 \frac{dx}{x} = \frac{\pi^2}{2}\int_0^\infty |f(\tau)|^2 \frac{d\tau}{\tau \sinh \pi\tau}, \tag{1.15}$$

and the Plancherel equality of type

$$\int_0^\infty (KLf)(x)\overline{(KLg(x)}\frac{dx}{x} = \frac{\pi^2}{2}\int_0^\infty f(\tau)\overline{g(\tau)}\frac{d\tau}{\tau \sinh \pi\tau}, \tag{1.16}$$

where $f, g \in L_2(\mathbf{R}_+; [\tau \sinh \pi\tau]^{-1}d\tau)$. We note that the convergence of the integral (1.1) in this case is with respect to the norm (1.10) for the space $L_2(\mathbf{R}_+; x^{-1}dx)$.

However, our goal is to study the Kontorovich-Lebedev transformation in the space $S_p^{N,\alpha}(\mathbf{R}_+), 1 \leq p < \infty$, which we call the Sobolev type space with the finite norm

$$||u||_{S_p^{N,\alpha}(\mathbf{R}_+)} = \left(\sum_{k=0}^N \int_0^\infty |A_x^k u|^p x^{\alpha_k p-1}dx\right)^{1/p} < \infty. \tag{1.17}$$

Here $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_N)$, $\alpha_k \in \mathbf{R}, k = 0, \ldots, N$, and A_x is the differential operator (1.2), which has eigenfunction $K_\nu(x)$ with eigenvalue $-\nu^2$ and can be written in the form

$$A_x u = x^2 u(x) - x\frac{d}{dx}\left[x\frac{du}{dx}\right],\ A_x K_\nu = -\nu^2 K_\nu(x). \tag{1.18}$$

As usual we denote by A_x^k the k-th iterate of A_x, $A_x^0 u = u$. The differential operator (1.18) was used for instance in [4], [16] in order to construct the spaces of testing functions to consider the Kontorovich-Lebedev transform on distributions (see also in [10]). Recently (see [15]) it is involved to investigate the corresponding class of the Kontorovich-Lebedev convolution integral equations.

In the sequel we will derive imbedding properties for the spaces $S_p^{N,\alpha}(\mathbf{R}_+)$ and we will find integral representations for the functions from $S_p^{N,\alpha}(\mathbf{R}_+)$. Finally we will study the boundedness and inversion properties for the Kontorovich-Lebedev transformation as an operator from the weighted L_p-space $L_p(\mathbf{R}_+;\omega(x)dx)$ into the space $S_p^{N,\alpha}(\mathbf{R}_+)$. When $p=2, \alpha=0$ we will prove the Plancherel type theorem and we will establish an isomorphism for the special type of weights between these spaces.

2. Elementary properties for the space $S_p^{N,\alpha}(\mathbf{R}_+)$

Let $\varphi(x)$ belong to the space $C_0^\infty(\mathbf{R}_+)$ of infinitely differentiable functions with a compact support on $\mathbf{R}_+$. Hence taking (1.18), we integrate by parts for any twice continuously differentiable function $u \in C^2(\mathbf{R}_+)$ and we derive the following equality

$$\int_0^\infty u(x) A_x \varphi \frac{dx}{x} = \int_0^\infty A_x u \; \varphi(x) \frac{dx}{x}. \tag{2.1}$$

Now if furthermore we suppose, that for any $\varphi \in C_0^\infty(\mathbf{R}_+)$ and some locally integrable function $v \in L_{loc}(\mathbf{R}_+)$ it satisfies

$$\int_0^\infty u(x) A_x \varphi \frac{dx}{x} = \int_0^\infty v(x)\varphi(x) \frac{dx}{x}$$

then subtracting these equalities we immediately obtain

$$\int_0^\infty [A_x u - v(x)] \;\; \varphi(x) \frac{dx}{x} = 0. \tag{2.2}$$

Consequently, via Du Bois-Reymond lemma we find that $v(x) = A_x u$ almost everywhere in $\mathbf{R}_+$. Equality (2.2) is used to define the so-called generalized derivative $v(x)$ for the function $u(x)$ in terms of the operator A_x. A k-th generalized derivative can be easily defined from (2.1). Indeed, for any $\varphi \in C_0^\infty(\mathbf{R}_+)$ we have that $A_x\varphi \in C_0^\infty(\mathbf{R}_+)$ and we will call $v_k(x) \in L_{loc}(\mathbf{R}_+)$ a k-th generalized derivative for $u \in L_{loc}(\mathbf{R}_+)$ ($v_k(x) \equiv A_x^k u$) if it satisfies the equality

$$\int_0^\infty u(x) A_x^k \varphi \frac{dx}{x} = \int_0^\infty v_k(x)\varphi(x) \frac{dx}{x}. \tag{2.3}$$

Further, from the norm definition (1.17) and elementary inequalities it follows that there are positive constants C_1, C_2 such that

$$C_1 \sum_{k=0}^N \left(\int_0^\infty |A_x^k u|^p x^{\alpha_k p - 1} dx \right)^{1/p} \le \left(\sum_{k=0}^N \int_0^\infty |A_x^k u|^p x^{\alpha_k p - 1} dx \right)^{1/p}$$

$$\leq C_2 \sum_{k=0}^{N} \left(\int_0^\infty |A_x^k u|^p x^{\alpha_k p-1} dx \right)^{1/p}. \tag{2.4}$$

Hence by (1.10) we have the equivalence of norms

$$C_1 \sum_{k=0}^{N} ||A_\cdot^k u||_{L_p(\mathbf{R}_+;x^{\alpha_k p-1}dx)} \leq ||u||_{S_p^{N,\alpha}(\mathbf{R}_+)} \leq C_2 \sum_{k=0}^{N} ||A_\cdot^k u||_{L_p(\mathbf{R}_+;x^{\alpha_k p-1}dx)}. \tag{2.5}$$

In order to show that $S_p^{N,\alpha}(\mathbf{R}_+), 1 \leq p < \infty$ is a Banach space we take a fundamental sequence $u_n(x)$, i.e. $||u_n - u_m||_{S_p^{N,\alpha}(\mathbf{R}_+)} \to 0,\ m,n \to \infty$. This will immediately imply that

$$||u_n - u_m||_{L_{\alpha_0,p}} \to 0,$$

$$||A_\cdot^k u_n - A_\cdot^k u_m||_{L_{\alpha_k,p}} \to 0,\ \ k = 1,\ldots,N,$$

when $m,n \to \infty$. Since spaces $L_{\alpha,p}, k = 0,1,\ldots,N$ are complete, there are functions $v_0 \in L_{\alpha_0,p}$, $v_k \in L_{\alpha_k,p}$ such that

$$||u_n - v_0||_{L_{\alpha_0,p}} \to 0, \tag{2.6}$$

$$||A_\cdot^k u_n - v_k||_{L_{\alpha_k,p}} \to 0,\ \ k = 1,\ldots,N, \tag{2.7}$$

when $n \to \infty$. If we show that v_k is a k-th generalized derivative of v_0 then we prove that the sequence u_n converges to $v_0 \in S_p^{N,\alpha}(\mathbf{R}_+)$ with respect to the norm (1.17). In fact, from (2.6), (2.7) for any $\varphi \in C_0^\infty(\mathbf{R}_+)$ we have the limit equalities

$$\lim_{n\to\infty} \int_0^\infty u_n(x)\varphi(x)\frac{dx}{x} = \int_0^\infty v_0(x)\ \varphi(x)\frac{dx}{x},$$
$$\lim_{n\to\infty} \int_0^\infty A_x^k u_n\ \varphi(x)\frac{dx}{x} = \int_0^\infty v_k(x)\ \varphi(x)\frac{dx}{x}.$$

But on the other hand,

$$\lim_{n\to\infty} \int_0^\infty A_x^k u_n\ \varphi(x)\frac{dx}{x} = \lim_{n\to\infty} \int_0^\infty u_n(x)\ A_x^k\varphi\frac{dx}{x} = \int_0^\infty v_0(x)\ A_x^k\varphi\frac{dx}{x}.$$

Therefore invoking (2.3) we get $v_k(x) = A_x^k v_0$ and we prove that $S_p^{N,\alpha}(\mathbf{R}_+)$ is a Banach space.

For the space $S_p^{1,\alpha}(\mathbf{R}_+)$ we establish an imbedding theorem into Sobolev's weighted space ${}_0W_p^1(\mathbf{R}_+;x^{\gamma p-1}dx)$ with the norm

$$||u||_{{}_0W_p^1(\mathbf{R}_+;x^{\gamma p-1}dx)} = \left(\int_0^\infty |u'(x)|^p x^{\gamma p-1} dx \right)^{1/p}.$$

Indeed, we have the following result.

Theorem 1. *Let* $1 < p < \infty$, $\alpha = (2-\beta, -\beta)$, $\beta > 0$. *The imbedding*

$$S_p^{1,\alpha}(\mathbf{R}_+) \subset {}_0W_p^1(\mathbf{R}_+; x^{(1-\beta)p-1}dx)$$

is true.

Proof. Appealing to the classical Hardy's inequality [2]

$$\int_0^\infty x^{-r} \left| \int_0^x f(t)dt \right|^p dx \le \text{const.} \int_0^\infty x^{p-r} |f(x)|^p \, dx, \tag{2.8}$$

where $1 < p < \infty$, $r > 1$ we put $f(x) = A_x u/x$, $r = \beta p + 1$, $\beta > 0$ and we have the estimate

$$\left(\int_0^\infty |A_x u|^p x^{-\beta p - 1} dx \right)^{1/p} \ge \text{const.} \left(\int_0^\infty x^{-\beta p-1} \left| \int_0^x \frac{A_t u}{t} dt \right|^p dx \right)^{1/p}$$

$$= \text{const.} \left(\int_0^\infty x^{-\beta p-1} \left| \int_0^x tu(t)dt - xu'(x) \right|^p dx \right)^{1/p}$$

$$\ge \text{const.} \left[\left(\int_0^\infty x^{p(1-\beta)-1} \left| u'(x) \right|^p dx \right)^{1/p} - \left(\int_0^\infty x^{-\beta p-1} \left| \int_0^x tu(t)dt \right|^p dx \right)^{1/p} \right].$$

Thus we get

$$\left(\int_0^\infty x^{p(1-\beta)-1} \left| u'(x) \right|^p dx \right)^{1/p} \le \text{const.} \left[\left(\int_0^\infty |A_x u|^p x^{-\beta p-1} dx \right)^{1/p} + \left(\int_0^\infty x^{-\beta p-1} \left| \int_0^x tu(t)dt \right|^p dx \right)^{1/p} \right]. \tag{2.9}$$

Invoking again Hardy's inequality (2.8) to estimate the latter term in (2.9) it becomes

$$\left(\int_0^\infty x^{-\beta p-1} \left| \int_0^x tu(t)dt \right|^p dx \right)^{1/p} \le \text{const.} \left(\int_0^\infty x^{p(2-\beta)-1} |u(x)|^p \, dx \right)^{1/p}.$$

Combining with (2.9) and (1.17) we obtain

$$\left(\int_0^\infty x^{p(1-\beta)-1} \left| u'(x) \right|^p dx \right)^{1/p} \le \text{const.} \left[\left(\int_0^\infty |A_x u|^p x^{-\beta p-1} dx \right)^{1/p} + \left(\int_0^\infty x^{p(2-\beta)-1} |u(x)|^p \, dx \right)^{1/p} \right] \le \text{const.} ||u||_{S_p^{1,\alpha}(\mathbf{R}_+)},$$

where $\alpha = (2-\beta, -\beta)$, $\beta > 0$. Theorem 1 is proved. □

Our goal now is to derive integral representations for functions from the space $S_p^{N,\alpha}(\mathbf{R}_+)$. For this we will use a technique from [14]. Precisely, let us introduce for any $u(x) \in L_{\nu,p}$, $\nu \in \mathbf{R}$ and $\varepsilon \in (0,\pi)$ the following regularization operator

$$u_\varepsilon(x) = \frac{x\sin\varepsilon}{\pi}\int_0^\infty \frac{K_1((x^2+y^2-2xy\cos\varepsilon)^{1/2})}{(x^2+y^2-2xy\cos\varepsilon)^{1/2}}u(y)dy,\ x>0. \tag{2.10}$$

We are ready to prove the Bochner type representation theorem.

We have

Theorem 2. *Let* $u(x) \in L_{\nu,p},\ 0<\nu<1,\ 1\le p<\infty$. *Then*

$$u(x) = \lim_{\varepsilon\to 0} u_\varepsilon(x), \tag{2.11}$$

with respect to the norm in $L_{\nu,p}$. *Besides, for* $1<p<\infty$ *the limit* (2.11) *exists for almost all* $x>0$.

Proof. We first show that (2.10) is a bounded operator in $L_{\nu,p}$ under conditions of the theorem. To do this we make the substitution $y = x(\cos\varepsilon + t\sin\varepsilon)$ in the corresponding integral and it becomes

$$u_\varepsilon(x) = \frac{x\sin\varepsilon}{\pi}\int_{-\cot\varepsilon}^\infty \frac{K_1(x\sin\varepsilon\sqrt{t^2+1})}{\sqrt{t^2+1}}u(x(\cos\varepsilon+t\sin\varepsilon))\,dt. \tag{2.12}$$

Hence owing to the generalized Minkowski inequality and elementary inequality for the modified Bessel function $xK_1(x)\le 1, x\ge 0$ (see (1.7)) we estimate the norm of the integral (2.12) as follows

$$\begin{aligned}||u_\varepsilon||_{L_{\nu,p}} &\le \frac{1}{\pi}\int_{-\cot\varepsilon}^\infty \frac{dt}{t^2+1}||u(\cdot(\cos\varepsilon+t\sin\varepsilon))||_{L_{\nu,p}}\\ &= \frac{1}{\pi}||u||_{L_{\nu,p}}\int_{-\cot\varepsilon}^\infty \frac{(\cos\varepsilon+t\sin\varepsilon)^{-\nu}}{t^2+1}\,dt\\ &= ||u||_{L_{\nu,p}}\frac{\sin\varepsilon}{\pi}\int_0^\infty \frac{\cosh\nu\xi}{\cosh\xi-\cos\varepsilon}\,d\xi,\ 0<\nu<1,\end{aligned}$$

where we have made the substitution $e^\xi = \cos\varepsilon + t\sin\varepsilon$ in the latter integral. However, via formula (2.4.6.6) in [5] we find accordingly,

$$\begin{aligned}\frac{\sin\varepsilon}{\pi}\int_0^\infty \frac{\cosh\nu\xi}{\cosh\xi-\cos\varepsilon}\,d\xi = \frac{\sin(\nu(\pi-\varepsilon))}{\sin\nu\pi} &\le 1+\frac{\sin\nu\varepsilon}{\sin\nu\pi}\\ &\le 1+\frac{\pi\nu}{\sin\nu\pi} = C_\nu,\ 0<\nu<1.\end{aligned}$$

Thus for all $\varepsilon\in(0,\pi)$ we get

$$||u_\varepsilon||_{L_{\nu,p}} \le C_\nu ||u||_{L_{\nu,p}}. \tag{2.13}$$

Further, by using the identity

$$\frac{1}{\pi}\int_{-\cot\varepsilon}^\infty \frac{dt}{t^2+1} = 1-\frac{\varepsilon}{\pi}$$

and denoting by

$$R(x,t,\varepsilon)=x\sin\varepsilon\sqrt{t^2+1}K_1(x\sin\varepsilon\sqrt{t^2+1}) \tag{2.14}$$

we find that

$$||u_\varepsilon-u||_{L_{\nu,p}}\le\frac{1}{\pi}\int\limits_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\left|\left|u(\cdot(\cos\varepsilon+t\sin\varepsilon))R(\cdot,t,\varepsilon)-\left(1-\frac{\varepsilon}{\pi}\right)^{-1}u\right|\right|_{L_{\nu,p}}$$

$$\le\frac{1}{\pi}\int\limits_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\left|\left|\left[u(\cdot(\cos\varepsilon+t\sin\varepsilon))-\left(1-\frac{\varepsilon}{\pi}\right)^{-1}u\right]R(\cdot,t,\varepsilon)\right|\right|_{L_{\nu,p}}$$

$$+\frac{1}{\pi-\varepsilon}\int\limits_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}||u[R(\cdot,t,\varepsilon)-1]||_{L_{\nu,p}}=I_1(\varepsilon)+I_2(\varepsilon).$$

But since [1]

$$\frac{d}{dx}[xK_1(x)]=-xK_0(x),$$

and $xK_1(x)\to 1$, $x\to 0$ we obtain the following representation

$$R(x,t,\varepsilon)-1=-\int_0^{x\sin\varepsilon(t^2+1)^{1/2}}yK_0(y)dy.$$

Hence appealing again to the generalized Minkowski inequality we deduce

$$I_2(\varepsilon)=\frac{1}{\pi-\varepsilon}\int\limits_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\left(\int\limits_0^{\infty}x^{\nu p-1}\left(\int_0^{x\sin\varepsilon(t^2+1)^{1/2}}yK_0(y)dy\right)^p|u(x)|^pdx\right)^{1/p}$$

$$\le\frac{1}{\pi-\varepsilon}\int\limits_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\int_0^{\infty}yK_0(y)\left(\int_{y/(\sin\varepsilon(t^2+1)^{1/2})}^{\infty}x^{\nu p-1}|u(x)|^pdx\right)^{1/p}dy$$

$$\le\frac{1}{\pi-\varepsilon}\int\limits_{-\cot\varepsilon}^{\infty}dt\int_0^{\infty}\xi K_0\left(\xi\sqrt{t^2+1}\right)\left(\int\limits_{\frac{\xi}{\sin\varepsilon}}^{\infty}x^{\nu p-1}|u(x)|^pdx\right)^{1/p}d\xi$$

$$=\frac{1}{\pi-\varepsilon}\int_{-\cot\varepsilon}^{\infty}dt\left(\int_0^{\sqrt{\varepsilon}}+\int_{\sqrt{\varepsilon}}^{\infty}\right)\xi K_0\left(\xi\sqrt{t^2+1}\right)\left(\int\limits_{\frac{\xi}{\sin\varepsilon}}^{\infty}x^{\nu p-1}|u(x)|^pdx\right)^{1/p}d\xi$$

$$\le\frac{1}{\pi-\varepsilon}\int\limits_{-\cot\varepsilon}^{\infty}dt\int_0^{\sqrt{\varepsilon}}\xi K_0\left(\xi\sqrt{t^2+1}\right)\left(\int\limits_{\frac{\xi}{\sin\varepsilon}}^{\infty}x^{\nu p-1}|u(x)|^pdx\right)^{1/p}d\xi$$

$$+\frac{1}{\pi-\varepsilon}\int\limits_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\int_0^{\infty}\xi K_0(\xi)d\xi\left(\int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty}x^{\nu p-1}|u(x)|^p dx\right)^{1/p}$$

$$\leq \frac{\varepsilon^{\nu/2}}{\pi-\varepsilon}||u||_{L_{\nu,p}}\int_{-\infty}^{\infty}(t^2+1)^{\frac{\nu}{2}-1}dt\int_0^{\infty}\xi^{1-\nu}K_0(\xi)d\xi$$

$$+\frac{\pi}{\pi-\varepsilon}\left(\int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty}x^{\nu p-1}|u(x)|^p dx\right)^{1/p}=\frac{\pi}{\pi-\varepsilon}\Bigg(\varepsilon^{\nu/2}\Gamma(1-\nu)||u||_{L_{\nu,p}}$$

$$+\left(\int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty}x^{\nu p-1}|u(x)|^p dx\right)^{1/p}\Bigg)\to 0,\ \ \varepsilon\to 0,\ 0<\nu<1.$$

Concerning the integral I_1 we first approximate $u \in L_{\nu,p}(\mathbf{R}_+)$ by a smooth function $\varphi \in C_0^{\infty}(\mathbf{R}_+)$. This implies that there exists a function $\varphi \in C_0^{\infty}(\mathbf{R}_+)$ such that $||f-\varphi||_{L_{\nu,p}} \leq \varepsilon$ for any $\varepsilon > 0$. Hence since the kernel (2.14) $R(x,t,\varepsilon) \leq 1$ then in view of the representation

$$\varphi(x(\cos\varepsilon+t\sin\varepsilon))-\varphi(x)$$
$$=\int_1^{\cos\varepsilon+t\sin\varepsilon}\frac{d}{dy}\left[\varphi(xy)\right]dy=\int_1^{\cos\varepsilon+t\sin\varepsilon}x\varphi'(xy)dy.$$

In a similar manner we have

$$I_1(\varepsilon)\leq\frac{1}{\pi}\int\limits_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\,||u(\cdot(\cos\varepsilon+t\sin\varepsilon))-\varphi(\cdot(\cos\varepsilon+t\sin\varepsilon))||_{L_{\nu,p}}$$

$$+\frac{1}{\pi}\int\limits_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\left|\left|\varphi(\cdot(\cos\varepsilon+t\sin\varepsilon))-\left(1-\frac{\varepsilon}{\pi}\right)^{-1}u\right|\right|_{L_{\nu,p}}$$

$$\leq ||u-\varphi||_{L_{\nu,p}}\frac{1}{\pi}\int\limits_{-\cot\varepsilon}^{\infty}\frac{(\cos\varepsilon+t\sin\varepsilon)^{-\nu}dt}{t^2+1}+\frac{1}{\pi}\int\limits_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\left|\left|\varphi-\frac{\pi}{\pi-\varepsilon}u\right|\right|_{L_{\nu,p}}$$

$$+||\varphi'||_{L_{1+\nu,p}}\frac{1}{\pi}\int\limits_{-\cot\varepsilon}^{\infty}\frac{dt}{t^2+1}\left|\int_1^{\cos\varepsilon+t\sin\varepsilon}y^{-\nu-1}dy\right|$$

$$\leq (C_\nu+1)||u-\varphi||_{L_{\nu,p}}+\frac{\varepsilon}{\pi}||u||_{\nu,p}+\frac{||\varphi'||_{L_{1+\nu,p}}}{\pi\nu}\int\limits_{-\cot\varepsilon}^{\infty}\frac{|1-(\cos\varepsilon+t\sin\varepsilon)^{-\nu}|}{t^2+1}\,dt.$$

The latter integral we treat by making the substitution $e^{\xi} = \cos\varepsilon + t\sin\varepsilon$. Then it takes the form

$$\int\limits_{-\cot\varepsilon}^{\infty} \frac{|1-(\cos\varepsilon + t\sin\varepsilon)^{-\nu}|}{t^2+1}\, dt = \sin\varepsilon \int_0^{\infty} \frac{\sinh\nu\xi}{\cosh\xi - \cos\varepsilon}\, d\xi$$

$$= \sin\varepsilon\left(\int_0^1 + \int_1^{\infty}\right)\frac{\sinh\nu\xi}{\cosh\xi-\cos\varepsilon}\, d\xi \le \sin\varepsilon\Bigg(\log(\cosh\xi - \cos\varepsilon)\ |_0^1$$

$$+ \int_1^{\infty}\frac{\sinh\nu\xi}{\cosh\xi - 1}\, d\xi\Bigg) \le \sin\varepsilon\left[\log\left(2^{-1}\sin^{-2}\frac{\varepsilon}{2}\right) + A_{\nu}\right],$$

where

$$A_{\nu} = 1 + \int_1^{\infty}\frac{\sinh\nu\xi}{\cosh\xi - 1}\, d\xi,\ \ 0 < \nu < 1.$$

Thus we immediately obtain that $\lim_{\varepsilon\to 0} I_1(\varepsilon) = 0$. Therefore by virtue of the above estimates $\lim_{\varepsilon\to 0}||u_{\varepsilon} - u||_{L_{\nu,p}} = 0$ and relation (2.11) is proved.

In order to verify the convergence almost everywhere we use the fact that any sequence of functions $\{\varphi_n\} \in C_0^{\infty}(\mathbf{R}_+)$ which converges to u in $L_{\nu,p}$-norm contains a subsequence $\{\varphi_{n_k}\}$ convergent almost everywhere, i.e. $\lim_{k\to\infty}\varphi_{n_k}(x) = u(x)$ for almost all $x > 0$. Then we find

$$|u_{\varepsilon}(x) - u(x)| \le \frac{1}{\pi}\int\limits_{-\cot\varepsilon}^{\infty}\left|u(x(\cos\varepsilon + t\sin\varepsilon))R(x,t,\varepsilon) - \left(1-\frac{\varepsilon}{\pi}\right)^{-1}u(x)\right|\frac{dt}{t^2+1}$$

$$\le \frac{1}{\pi}\int\limits_{-\cot\varepsilon}^{\infty}|u(x(\cos\varepsilon + t\sin\varepsilon)) - \varphi_{n_k}(x(\cos\varepsilon + t\sin\varepsilon))|\,\frac{dt}{t^2+1}$$

$$+\frac{1}{\pi}\int\limits_{-\cot\varepsilon}^{\infty}|\varphi_{n_k}(x(\cos\varepsilon + t\sin\varepsilon)) - \varphi_{n_k}(x)|\,\frac{dt}{t^2+1}$$

$$+\frac{1}{\pi}\int\limits_{-\cot\varepsilon}^{\infty}\left|\varphi_{n_k}(x)R(x,t,\varepsilon) - \left(1-\frac{\varepsilon}{\pi}\right)^{-1}u(x)\right|\frac{dt}{t^2+1}$$

$$= J_{1\varepsilon}(x) + J_{2\varepsilon}(x) + J_{3\varepsilon}(x).$$

But,

$$J_{3\varepsilon}(x) \le \frac{1}{\pi}\int\limits_{-\cot\varepsilon}^{\infty}\left|\varphi_{n_k}(x) - \left(1-\frac{\varepsilon}{\pi}\right)^{-1}u(x)\right|\frac{dt}{t^2+1}$$

$$+\frac{1}{\pi-\varepsilon}\int\limits_{-\cot\varepsilon}^{\infty}|u(x)\left[R(x,t,\varepsilon)-1\right]|\frac{dt}{t^2+1}$$

$$\leq|\varphi_{n_k}(x)-u(x)|+\frac{\varepsilon}{\pi}|u(x)|+\frac{|u(x)|}{\pi-\varepsilon}\int\limits_{-\cot\varepsilon}^{\infty}\left|\int_0^{x\sin\varepsilon(t^2+1)^{1/2}}yK_0(y)dy\right|\frac{dt}{t^2+1}$$

$$\leq|\varphi_{n_k}(x)-u(x)|+\frac{\varepsilon}{\pi}|u(x)|+\frac{|u(x)|\varepsilon^\nu x^\nu}{\pi-\varepsilon}\int\limits_{-\infty}^{\infty}(t^2+1)^{\nu/2-1}dt\int\limits_0^{\infty}y^{1-\nu}K_0(y)dy$$

$$=|\varphi_{n_k}(x)-u(x)|+\frac{\varepsilon}{\pi}|u(x)|+\frac{\pi\Gamma(1-\nu)\varepsilon^\nu x^\nu}{\pi-\varepsilon}|u(x)|\to 0,\ \ 0<\nu<1,$$

when $\varepsilon\to 0,\ k>k_0$ for almost all $x>0$. Similarly,

$$J_{2\varepsilon}(x)=\frac{1}{\pi}\int\limits_{-\cot\varepsilon}^{\infty}\left|\int_1^{\cos\varepsilon+t\sin\varepsilon}x\varphi'_{n_k}(xy)dy\right|\frac{dt}{t^2+1}$$

$$\leq\frac{x}{\pi\nu}\sup_{y\geq 0}y^{1+\nu}|\varphi'_{n_k}(xy)|\int\limits_{-\cot\varepsilon}^{\infty}\left|1-(\cos\varepsilon+t\sin\varepsilon)^{-\nu}\right|\frac{dt}{t^2+1}$$

$$\leq\sin\varepsilon\left[\log\left(2^{-1}\sin^{-2}\frac{\varepsilon}{2}\right)+A_\nu\right]\frac{x}{\pi\nu}\sup_{y\geq 0}y^{1+\nu}|\varphi'_{n_k}(xy)|,$$

which tends to zero almost for all $x>0$ when $\varepsilon\to 0$. Meantime, by taking $1<p<\infty$, $q=\frac{p}{p-1}$ for any $\varepsilon>0$ such that $||u-\varphi_{n_k}||_{L_{\nu,p}}<\varepsilon$ for $k>k_0$ we have

$$J_{1\varepsilon}(x)\leq\frac{x^{-\nu}||u-\varphi_{n_k}||_{L_{\nu,p}}}{\pi\sin^{1/p}\varepsilon}\left(\int\limits_{-\cot\varepsilon}^{\infty}\frac{(\cos\varepsilon+t\sin\varepsilon)^{q(1-\nu)-1}dt}{(t^2+1)^q}\right)^{1/q}$$

$$<x^{-\nu}\varepsilon\sin\varepsilon\left(\int_0^\infty\frac{\xi^{q(1-\nu)-1}d\xi}{(\xi^2-2\xi\cos\varepsilon+1)^q}\right)^{1/q}.$$

But the latter integral can be treated in terms of the Legendre functions [1] appealing to relation (2.2.9.7) from [5]. This gives the value

$$\int_0^\infty\frac{\xi^{q(1-\nu)-1}d\xi}{(\xi^2-2\xi\cos\varepsilon+1)^q}=$$

$$=\left(\frac{\sin\varepsilon}{2}\right)^{1/2-q}\Gamma(q+1/2)\frac{\Gamma(q(1-\nu))\Gamma(q(1+\nu))}{\Gamma(2q)}P^{1/2-q}_{-1/2-q\nu}(-\cos\varepsilon).$$

When $\varepsilon \to 0+$ we have

$$\int_0^\infty \frac{\xi^{q(1-\nu)-1}d\xi}{(\xi^2 - 2\xi\cos\varepsilon + 1)^q} \sim \sqrt{\pi}\frac{\Gamma(q-1/2)}{\Gamma(q)}\varepsilon^{1-2q}.$$

Thus

$$J_{1\varepsilon}(x) < \text{const. } x^{-\nu}\varepsilon^{1/q} \to 0, \;\; \varepsilon \to 0, x > 0$$

and we prove Theorem 2. □

Appealing to Theorem 2 we will approximate functions from $S_p^{N,\alpha}(\mathbf{R}_+)$ by regularization operator (2.10). Indeed we have

Corollary 1. *Operator* (2.10) *is defined on functions from* $S_p^{N,\alpha}(\mathbf{R}_+)$ *with* $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_N),\; 0 < \alpha_k < 1,\; k = 0, 1, \ldots, N$ *and* $1 \le p < \infty$. *Besides*

$$u(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x), \tag{2.15}$$

with respect to the norm in $S_p^{N,\alpha}(\mathbf{R}_+)$.

Proof. Indeed, taking some function $u \in S_p^{N,\alpha}(\mathbf{R}_+)$ we then choose a sequence $\{\varphi_n\} \in C_0^\infty(\mathbf{R}_+)$, which converges to u. This immediately implies (see (2.6), (2.7)) that $A_x^k\varphi_n \to A_x^k u,\; n \to \infty$ with respect to the norm in $L_{\alpha_k,p},\; k = 0, 1, \ldots, N$, respectively.

Defining by

$$\varphi_{\varepsilon,n}(x) = \frac{x\sin\varepsilon}{\pi}\int_0^\infty \frac{K_1((x^2+y^2-2xy\cos\varepsilon)^{1/2})}{(x^2+y^2-2xy\cos\varepsilon)^{1/2}}\varphi_n(y)dy, \;\; x > 0, \tag{2.16}$$

we employ the relation (2.16.51.8) in [6]

$$\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}(x)K_{i\tau}(y)d\tau$$

$$= \frac{\pi}{2}xy\sin\varepsilon\frac{K_1((x^2+y^2-2xy\cos\varepsilon)^{1/2})}{(x^2+y^2-2xy\cos\varepsilon)^{1/2}}, \;\; x, y > 0, \, 0 < \varepsilon \le \pi$$

and we substitute it in (2.16). Changing the order of integration by the Fubini theorem we find

$$\varphi_{\varepsilon,n}(x) = \frac{2}{\pi^2}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}(x)\int_0^\infty K_{i\tau}(y)\varphi_n(y)\frac{dy}{y}d\tau.$$

Meantime, we apply the operator $A_x^k,\; k = 0, 1 \ldots, N$ (1.18) through both sides of the latter integral. Then via its uniform convergence with respect to $x \in (x_0, X_0) \subset \mathbf{R}_+$ and by using the equalities (see (1.18)) $A_x^k K_{i\tau}(x) = \tau^{2k}K_{i\tau}(x)$, (2.1) we come out with

$$A_x^k\varphi_{\varepsilon,n} = \frac{2}{\pi^2}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}(x)\int_0^\infty \tau^{2k}K_{i\tau}(y)\varphi_n(y)\frac{dy}{y}d\tau$$

$$= \frac{2}{\pi^2}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}(x)\int_0^\infty K_{i\tau}(y)A_y^k\varphi_n\frac{dy}{y}d\tau.$$

This is equivalent to

$$A_x^k\varphi_{\varepsilon,n} = \frac{x\sin\varepsilon}{\pi}\int_0^\infty \frac{K_1((x^2+y^2-2xy\cos\varepsilon)^{1/2})}{(x^2+y^2-2xy\cos\varepsilon)^{1/2}}A_y^k\varphi_n dy.$$

Hence

$$A_x^k\varphi_{\varepsilon,n}-(A_x^k u)_\varepsilon = \frac{x\sin\varepsilon}{\pi}\int_0^\infty \frac{K_1((x^2+y^2-2xy\cos\varepsilon)^{1/2})}{(x^2+y^2-2xy\cos\varepsilon)^{1/2}}\left[A_y^k\varphi_n - A_y^k u\right]dy$$

and due to (2.13) we have that $\lim_{n\to\infty} A_x^k\varphi_{\varepsilon,n} = (A_x^k u)_\varepsilon$ with respect to the norm in $L_{\alpha_k,p}$ for each $\varepsilon\in(0,\pi)$. By Theorem 2 we derive that

$$\left|\left|(A_\cdot^k u)_\varepsilon - A_\cdot^k u\right|\right|_{L_{\alpha_k,p}} \to 0,\ \ \varepsilon\to 0,\ k=0,1,\dots,N.$$

If we show that almost for all $x>0$ $(A_x^k u)_\varepsilon = A_x^k u_\varepsilon$, $k=0,1,2,\dots,N$ then via (2.5) we complete the proof of Corollary 1. When $k=0$ it is defined by (2.10). At the same time according to Du Bois-Reymond lemma it is sufficient to show that for any $\psi\in C_0^\infty(\mathbf{R}_+)$

$$\int_0^\infty \left[(A_x^k u)_\varepsilon - A_x^k u_\varepsilon\right]\frac{\psi(x)}{x}dx = 0. \tag{2.17}$$

We have

$$\int_0^\infty \left[(A_x^k u)_\varepsilon - A_x^k u_\varepsilon\right]\frac{\psi(x)}{x}dx = \int_0^\infty \left[(A_x^k u)_\varepsilon - A_x^k\varphi_{\varepsilon,n}\right]\frac{\psi(x)}{x}dx$$
$$+\int_0^\infty \left[A_x^k\varphi_{\varepsilon,n} - A_x^k u_\varepsilon\right]\frac{\psi(x)}{x}dx = \int_0^\infty \left[(A_x^k u)_\varepsilon - A_x^k\varphi_{\varepsilon,n}\right]\frac{\psi(x)}{x}dx$$
$$+\int_0^\infty \left[\varphi_{\varepsilon,n} - u_\varepsilon\right]\frac{A_x^k\psi}{x}dx.$$

Now as it is easily seen the right-hand side of the last equality is less than an arbitrary $\delta>0$ when $n\to\infty$. Thus we prove (2.17) and we complete the proof of Corollary 1. □

3. The Kontorovich - Lebedev transformation in $S_2^{N,\alpha}(\mathbf{R}_+)$

Our goal in this section is to establish the boundedness of the Kontorovich-Lebedev transformation (1.1) as an operator $KL : L_2(\mathbf{R}_+;\omega_\alpha(\tau)d\tau) \to S_2^{N,\alpha}(\mathbf{R}_+)$, where the measure $\omega_\alpha(\tau)d\tau$ will be defined below. Finally, we will prove the Plancherel theorem and an analog of the Parseval equality (1.16) when $\alpha_k = 0$, $k=0,1,\dots,N$.

We begin with the use of the following inequality for the transformation (1.1), which is proved in [13]

$$\int_0^\infty |(KLf)(x)|^2 x^{2\nu-1}dx \le \frac{\pi^{3/2}2^{-2\nu-1}}{\Gamma(2\nu+1/2)} \int_0^\infty |f(\tau)|^2|\Gamma(2\nu+i\tau)|^2 d\tau, \ \nu > 0. \tag{3.1}$$

It gives the boundedness for the Kontorovich-Lebedev transformation as an operator $KL : L_2(\mathbf{R}_+; |\Gamma(2\nu+i\tau)|^2 d\tau) \to L_{\nu,2}$. Moreover, when $\nu \to 0+$ it attains equality (1.15) where the measure (see in [1]) $|\Gamma(i\tau)|^2 = \pi\,[\tau\sinh\pi\tau]^{-1}$.

Let $f \in L_2(\mathbf{R}_+; \omega_\alpha(\tau)d\tau)$, where the weighted function $\omega_\alpha(\tau)$ is defined by

$$\omega_\alpha(\tau) = \pi^{3/2}\sum_{k=0}^N \frac{2^{-2\alpha_k-1}\tau^{4k}|\Gamma(2\alpha_k+i\tau)|^2}{\Gamma(2\alpha_k+1/2)}, \ \alpha_k > 0, \ k = 0,1,\dots,N. \tag{3.2}$$

Considering a sequence $\{f_n\}_{n=1}^\infty$, where

$$f_n(\tau) = \begin{cases} f(\tau), & \text{if } \tau \in \left[\frac{1}{n}, n\right], \\ 0, & \text{if } \tau \notin \left[\frac{1}{n}, n\right], \end{cases}$$

and using the asymptotic formula (1.6) with Schwarz's inequality we find that integral (1.1) for (KLf_n) exists as a Lebesgue integral for any n. Moreover, since $K_{i\tau}(z)$ is analytic in the right half-plane $\mathrm{Re} z > 0$ (cf. in (1.7)) and integral (1.1) is uniformly convergent on every compact set of $\mathbf{R}_+$, we may repeatedly differentiate under the integral sign to obtain

$$A_x^k KLf_n = \int_{1/n}^n A_x^k K_{i\tau}(x)f(\tau)d\tau = \int_{1/n}^n \tau^{2k}K_{i\tau}(x)f(\tau)d\tau, \ k = 0,1,\dots,N. \tag{3.3}$$

Hence, invoking (3.1), (1.17) we deduce

$$\begin{aligned} ||KLf_n||_{S_2^{N,\alpha}(\mathbf{R}_+)} &= \left(\sum_{k=0}^N \int_0^\infty |A_x^k KLf_n|^2 x^{2\alpha_k-1}dx\right)^{1/2} \\ &\le \left(\int_{1/n}^n |f(\tau)|^2\omega_\alpha(\tau)d\tau\right)^{1/2} = ||f_n||_{L_2(\mathbf{R}_+;\omega_\alpha(\tau)d\tau)}. \end{aligned} \tag{3.4}$$

Meanwhile, we easily see that $||f - f_n||_{L_2(\mathbf{R}_+;\omega_\alpha(\tau)d\tau)} \to 0$, when $n \to \infty$. Moreover, from (3.4) we have

$$||KLf_n - KLf_m||_{S_2^{N,\alpha}(\mathbf{R}_+)} \le ||f_n - f_m||_{L_2(\mathbf{R}_+;\omega_\alpha(\tau)d\tau)} \to 0, \ n,m \to \infty.$$

Therefore the sequence $\{KLf_n\}$ converges to a function $g(x) \in S_2^{N,\alpha}(\mathbf{R}_+)$, which we call the Kontorovich-Lebedev transformation $(KLf)(x)$ of f. Thus integral (1.1) can be continuously extended on the whole space $L_2(\mathbf{R}_+;\omega_\alpha(\tau)d\tau)$. It is understood as a limit

$$g(x) \equiv (KLf)(x) = \lim_{n\to\infty} \int_{1/n}^{n} K_{i\tau}(x) f(\tau) d\tau \tag{3.5}$$

with respect to the norm (1.17) and it represents a bounded operator $KL : L_2(\mathbf{R}_+;\omega_\alpha(\tau)d\tau) \to S_2^{N,\alpha}(\mathbf{R}_+)$. Indeed, we pass to the limit through inequality (3.4) when $n \to \infty$ to obtain

$$||KLf||_{S_2^{N,\alpha}(\mathbf{R}_+)} \leq ||f||_{L_2(\mathbf{R}_+;\omega_\alpha(\tau)d\tau)}.$$

The case $\alpha = 0$ corresponds to the Plancherel type theorem, which will establish an isometric isomorphism between the corresponding L_2- spaces. Indeed, in this case we easily have from (3.2) that

$$\omega_0(\tau) = \frac{\pi^2}{2} \frac{1-\tau^{4(N+1)}}{(1-\tau^4)\tau \sinh \pi\tau}. \tag{3.6}$$

Theorem 3. *Let $f \in L_2(\mathbf{R}_+;\omega_0(\tau)d\tau)$, where the weighted function ω_0 is defined by* (3.6)*. Then the integral* (3.5) *for the Kontorovich-Lebedev transform converges to $(KLf)(x)$ with respect to the norm in the space $S_2^{N,0}(\mathbf{R}_+)$; and*

$$f_n(\tau) = \frac{2}{\pi^2} \tau \sinh \pi\tau \int_{1/n}^{n} K_{i\tau}(x)(KLf)(x) \frac{dx}{x} \tag{3.7}$$

converges in the mean to $f(\tau)$ with respect to the norm in $L_2(\mathbf{R}_+;\omega_0(\tau)d\tau)$. Moreover, the following Plancherel identity is true

$$\sum_{k=0}^{N} \int_0^\infty A_x^k KLf \; \overline{A_x^k KLh} \frac{dx}{x} = \frac{\pi^2}{2} \int_0^\infty f(\tau)\overline{h(\tau)} \; \frac{1-\tau^{4(N+1)}}{1-\tau^4} \; \frac{d\tau}{\tau \sinh \pi\tau}, \tag{3.8}$$

where $f, h \in L_2(\mathbf{R}_+;\omega_0(\tau)d\tau)$. In particular,

$$||KLf||^2_{S_2^{N,0}(\mathbf{R}_+)} = ||f||^2_{L_2(\mathbf{R}_+;\omega_0(\tau)d\tau)}$$

that is

$$\sum_{k=0}^{N} \int_0^\infty |A_x^k KLf|^2 \frac{dx}{x} = \frac{\pi^2}{2} \int_0^\infty |f(\tau)|^2 \; \frac{1-\tau^{4(N+1)}}{1-\tau^4} \frac{d\tau}{\tau \sinh \pi\tau}. \tag{3.9}$$

Finally, for almost all τ and x from $\mathbf{R}_+$ the reciprocal formulas take place

$$(KLf)(x) = g(x) = \frac{d}{dx} \int_0^\infty \int_0^x K_{i\tau}(y) f(\tau) dy \, d\tau, \tag{3.10}$$

$$f(\tau) = \frac{2}{\pi^2} \frac{(1-\tau^4)\sinh \pi\tau}{1-\tau^{4(N+1)}} \frac{d}{d\tau} \int_0^\infty \int_0^\tau y K_{iy}(x) \frac{1-y^{4(N+1)}}{1-y^4} \; (KLf)(x) \frac{dy\, dx}{x}. \tag{3.11}$$

Remark 1. When $N = 0$ we immediately obtain Plancherel identities (1.15), (1.16). Relations (3.10), (3.11) become then reciprocal formulas for the Kontorovich-Lebedev transformation in L_2- space with respect to the weight $\frac{\pi^2}{2}[\tau \sinh \pi\tau]^{-1}$ (see [11], [12]).

Proof of Theorem 3. Let $f \in L_2(\mathbf{R}_+; \omega_0(\tau)d\tau)$. Taking a sequence as in (3.3), which converges to f we find that $\tau^{2k} f_n(\tau) \in L_2\big(\mathbf{R}_+; \frac{\pi^2}{2}[\tau \sinh \pi\tau]^{-1} d\tau\big)$ for all $k = 0, 1, \ldots, N$. Hence from (3.3) via Parseval's equality (1.15) we obtain

$$\int_0^\infty |A_x^k KLf_n|^2 \frac{dx}{x} = \frac{\pi^2}{2} \int_0^\infty |f_n(\tau)|^2 \ \frac{\tau^{4k-1}}{\sinh \pi\tau} d\tau, \ \ k = 0, 1, \ldots, N.$$

Making elementary summations we immediately arrive at the equality (3.9) for f_n. Moreover passing to the limit we get that (3.9) is true for any $f \in L_2(\mathbf{R}_+; \omega_0(\tau)d\tau)$. Further, taking $x > 0$ we easily have

$$\int_0^x (KLf_n)(y)dy = \int_0^\infty \int_0^x K_{i\tau}(y) f_n(\tau) dy\, d\tau.$$

Hence we prove that

$$\lim_{n\to\infty} \int_0^x (KLf_n)(y)dy = \int_0^x (KLf)(y)dy = \int_0^\infty \int_0^x K_{i\tau}(y) f(\tau) dy\, d\tau. \tag{3.12}$$

The latter integral with respect to τ in (3.12) is absolutely convergent and therefore exists in Lebesgue's sense. Indeed, with Schwarz's inequality we derive (cf. in [11], [12], see (1.6))

$$\int_0^\infty \left| \int_0^x K_{i\tau}(y) dy \right| |f(\tau)| d\tau$$
$$\leq ||f||_{L_2(\mathbf{R}_+;\omega_0(\tau)d\tau)} \left(\int_0^\infty \left| \int_0^x K_{i\tau}(y)dy \right|^2 \frac{d\tau}{\omega_0(\tau)} \right)^{1/2} < \infty.$$

Consequently,

$$\left| \int_0^x [KLf_n)(y) - (KLf)(y)]\, dy \right| \leq \text{const.} ||f - f_n||_{L_2(\mathbf{R}_+;\omega_0(\tau)d\tau)} \to 0, \ n \to \infty$$

and we prove (3.12). Differentiating with respect to x almost for all $x > 0$ we arrive at (3.10).

In the meantime with the parallelogram identity we easily derive from (3.9) the Parseval equality (3.8). In particular, putting

$$h(y) = \begin{cases} y, & \text{if } y \in [0, \tau], \\ 0, & \text{if } y \in (\tau, \infty), \end{cases}$$

we have that $h \in L_2(\mathbf{R}_+;\omega_0(\tau)d\tau)$. Further, we find for the sequence $\{f_n\}$ that

$$\sum_{k=0}^{N}\int_0^\infty A_x^k KLf_n\ A_x^k \int_0^\tau yK_{iy}(x)dy\frac{dx}{x} = \frac{\pi^2}{2}\int_0^\tau f_n(y)\frac{1-y^{4(N+1)}}{1-y^4}\frac{dy}{\sinh \pi y}. \tag{3.13}$$

On the other hand we show that the left-hand side of (3.13) is equal to

$$\sum_{k=0}^{N}\int_0^\infty A_x^k KLf_n\ A_x^k \int_0^\tau yK_{iy}(x)dy\frac{dx}{x}$$
$$= \int_0^\infty (KLf_n)(x)\int_0^\tau \frac{1-y^{4(N+1)}}{1-y^4}K_{iy}(x)y\,dy\frac{dx}{x}. \tag{3.14}$$

Indeed, via (1.18) $(k = 1, 2, \ldots, N)$ we have

$$\int_0^\infty A_x^k KLf_n A_x^k \int_0^\tau yK_{iy}(x)dy\frac{dx}{x} = \int_0^\infty A_x KL\psi_{k-1,n}\int_0^\tau y^{2k+1}K_{iy}(x)dy\frac{dx}{x},$$

where $\psi_{k,n}(\tau) = \tau^{2k}f_n(\tau)$ and the relation $KL\psi_{k,n}(x) = A_x KL\psi_{k-1,n}$ holds. Then we use (2.1) and we integrate by parts to obtain

$$\int_0^\infty A_x KL\psi_{k-1,n}\int_0^\tau y^{2k+1}K_{iy}(x)dy\frac{dx}{x}$$
$$= \int_0^\infty KL\psi_{k-1,n}(x)\int_0^\tau y^{2k+3}K_{iy}(x)dy\frac{dx}{x},$$

where the integrated terms are vanishing due to the following limit equalities

$$\lim_{x\to\{{0\atop\infty}\}} x\frac{d}{dx}\left[KL\psi_{k-1,n}(x)\right]\int_0^\tau y^{2k+1}K_{iy}(x)dy = 0, \tag{3.15}$$

$$\lim_{x\to\{{0\atop\infty}\}} KL\psi_{k-1,n}(x)x\frac{d}{dx}\int_0^\tau y^{2k+1}K_{iy}(x)dy = 0, \tag{3.16}$$

for all $k = 1, 2, \ldots, N$. To verify (3.15), (3.16) when $x \to \infty$ we appeal to the relation [1]

$$x\frac{d}{dx}K_\mu(x) = \mu K_\mu(x) - xK_{\mu+1}(x) \tag{3.17}$$

and we employ the asymptotic formula (1.3). In the case $x \to 0$ we employ the definition of the Macdonald function $K_\mu(x)$ through the modified Bessel function $I_\mu(x)$ [1]

$$K_\mu(x) = \frac{\pi}{2\sin \pi\mu}\left[I_{-\mu}(x) - I_\mu(x)\right], \tag{3.18}$$

where

$$I_\mu(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{2m+\mu}}{k!\Gamma(m+\mu+1)} = \frac{e^{\mu \log x}}{2^\mu \Gamma(\mu+1)} + \sum_{m=1}^{\infty} \frac{(x/2)^{2m+\mu}}{m!\Gamma(m+\mu+1)}. \quad (3.19)$$

Hence putting $\mu = iy$ we substitute the right-hand side of the latter equality (3.19) into (3.18) and then we use this expression together with relation (3.17) to treat integrals with respect to y in (3.15), (3.16). Namely, with the integration by parts we derive the following asymptotic relations

$$\int_0^\tau y^{2k+1} K_{iy}(x) dy = O\left(\frac{1}{\log x}\right) + o(x^2), \ \ x \to 0,$$

$$x \frac{d}{dx} \int_0^\tau y^{2k+1} K_{iy}(x) dy = \int_0^\tau y^{2k+1} \left[iy K_{iy}(x) - x K_{iy+1}(x)\right] dy$$
$$= O\left(\frac{1}{\log x}\right) + o(x^2), \ \ x \to 0.$$

Thus it tends to zero, when $x \to 0$. Meanwhile since

$$KL\psi_{k-1,n}(x) = \int_{1/n}^{n} \tau^{2(k-1)} K_{i\tau}(x) f(\tau) d\tau = \int_{1/n}^{n} \frac{\tau^{2(k-1)}}{2^{i\tau+1}} f(\tau)$$
$$\times \Gamma(i\tau) e^{-i\tau \log x} d\tau + \int_{1/n}^{n} \frac{\tau^{2(k-1)}}{2^{1-i\tau}} f(\tau) \Gamma(-i\tau) e^{i\tau \log x} d\tau + o(x^2), \ \ x \to 0,$$

we have that $KL\psi_{k-1,n}(x) \to 0, \ x \to 0$ via the Riemann- Lebesgue lemma for the Fourier transform of integrable function. In a similar manner we get that $x \frac{d}{dx} \left[KL\psi_{k-1,n}(x)\right] \to 0, \ x \to 0$ and therefore relations (3.15), (3.16) are verified. Continuing this process of elimination of the operator A_x we come out with

$$\int_0^\infty KL\psi_{k-1,n}(x) \int_0^\tau y^{2k+3} K_{iy}(x) dy \frac{dx}{x}$$
$$= \int_0^\infty KL\psi_{k-2,n}(x) \int_0^\tau y^{2k+5} K_{iy}(x) dy \frac{dx}{x}$$
$$= \cdots = \int_0^\infty KL\psi_{0,n}(x) \int_0^\tau y^{4k+1} K_{iy}(x) dy \frac{dx}{x}$$
$$= \int_0^\infty (KLf_n)(x) \int_0^\tau y^{4k+1} K_{iy}(x) dy \frac{dx}{x}.$$

Hence making elementary summations we establish (3.14). Combining with (3.13) we find

$$\int_0^\infty (KLf_n)(x) \int_0^\tau \frac{1 - y^{4(N+1)}}{1 - y^4} K_{iy}(x) y \, dy \frac{dx}{x}$$

$$= \frac{\pi^2}{2}\int_0^\tau f_n(y)\frac{1-y^{4(N+1)}}{1-y^4}\frac{dy}{\sinh \pi y}. \tag{3.20}$$

Passing to the limit through (3.20) when $n \to \infty$ and differentiating with respect to τ we arrive at the reciprocal formula (3.11), where the corresponding integral exists in the Lebesgue sense, since it is not difficult to show (cf. (3.15), (3.16)) that for each $\tau > 0$

$$\int_0^\tau yK_{iy}(x)\frac{1-y^{4(N+1)}}{1-y^4}dy = \begin{cases} O(K_0(x)), & \text{if } x \geq x_0 > 0,\\ O\left(\frac{1}{\log x}\right), & \text{if } x \to 0,\end{cases}$$

and therefore it belongs to $L_{0,2}$.

Conversely, let $g(x) \in S_2^{N,0}(\mathbf{R}_+)$ be an arbitrary function. Taking a sequence $\{\varphi_n\}_{n=1}^\infty \in C_0^\infty(\mathbf{R}_+)$, which converges to g with respect to the norm in $S_2^{N,0}(\mathbf{R}_+)$ and denoting I_n the least segment which contains the support of the function φ_n we observe that the corresponding formula (3.11) will take the form

$$s_n(\tau) = \frac{2}{\pi^2}\frac{(1-\tau^4)\sinh \pi\tau}{1-\tau^{4(N+1)}}\frac{d}{d\tau}\int_{I_n}\int_0^\tau yK_{iy}(x)\frac{1-y^{4(N+1)}}{1-y^4}\varphi_n(x)\frac{dy\,dx}{x}. \tag{3.21}$$

Differentiating under the integral sign in (3.21) with respect to τ, which is indeed possible, we obtain

$$s_n(\tau) = \frac{2}{\pi^2}\tau\sinh \pi\tau\int_{I_n}K_{i\tau}(x)\varphi_n(x)\frac{dx}{x}. \tag{3.22}$$

But returning to Corollary 1 in Section 2 we see that functions φ_n and their generalized derivatives $A_x^k\varphi_n$ may be represented in terms of the regularization operator (2.10). Hence as a consequence of this fact we have the expansions

$$A_x^k\varphi_{\varepsilon,n}(x) = \frac{2}{\pi^2}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}(x)\int_{I_n}K_{i\tau}(y)A_y\varphi_n(y)\frac{dy}{y}d\tau, \tag{3.23}$$

where $A_x^k\varphi_{\varepsilon,n} \to A_x^k\varphi_n,\ \varepsilon \to 0$ with respect to the norm in the space $S_2^{N,\alpha}$ $(\mathbf{R}_+)$ via Corollary 1 for all $k = 0, 1, \ldots, N$. However, via Lemma 2.5 from [11] with (1.18), (3.22) we can pass to the limit in (3.23) when $\varepsilon \to 0$ pointwisely for all $x > 0$ to get

$$\begin{aligned} A_x^k\varphi_n(x) &= \frac{2}{\pi^2}\int_0^\infty \tau\sinh \pi\tau K_{i\tau}(x)\int_{I_n}K_{i\tau}(y)A_y\varphi_n(y)\frac{dy}{y}d\tau\\ &= \frac{2}{\pi^2}\int_0^\infty \tau^{2k+1}\sinh \pi\tau K_{i\tau}(x)\int_{I_n}K_{i\tau}(y)\varphi_n(y)\frac{dy}{y}d\tau \end{aligned}$$

$$= \int_0^\infty \tau^{2k} K_{i\tau}(x) s_n(\tau) d\tau.$$

Thus we obtain that $A_x^k \varphi_n(x) = (KLd_{k,n})(x)$, where $d_{k,n}(\tau) = \tau^{2k} s_n(\tau)$, $k = 0, 1, \ldots, N$. In particular, we have $\varphi_n(x) = (KLs_n)(x)$. Further, via Corollary 2.1 from [11] functions $A_x^k \varphi_n(x)$, $d_{k,n}(\tau)$ satisfy the Parseval equality (1.15) for the Kontorovich-Lebedev transform. Making elementary summations we derive then equality (3.9), which is written in the form

$$\sum_{k=0}^N \int_0^\infty |A_x^k \varphi_n|^2 \frac{dx}{x} = \frac{\pi^2}{2} \int_0^\infty |s_n(\tau)|^2 \frac{1 - \tau^{4(N+1)}}{1 - \tau^4} \frac{d\tau}{\tau \sinh \pi\tau}. \tag{3.24}$$

Hence for $m, n \to \infty$ we find

$$||s_m - s_n||^2_{L_2(\mathbf{R}_+; \omega_0(\tau) d\tau)} = ||\varphi_m - \varphi_n||^2_{S_2^{N,0}(\mathbf{R}_+)} \to 0.$$

Therefore the Cauchy sequence $\{s_n\}$ converges to a function $s(\tau) \in L_2(\mathbf{R}_+; \omega_0(\tau) d\tau)$. Passing to the limit through (3.24) when $n \to \infty$ we derive the corresponding identity (3.9) for functions $s(\tau)$ and its Kontorovich-Lebedev transform $g(x)$, which can be written by formula (3.10). In a similar manner we write (3.11) for this pair. In particular, defining

$$g_n(x) = \begin{cases} g(x), & \text{if } x \in \left[\frac{1}{n}, n\right], \\ 0, & \text{if } x \notin \left[\frac{1}{n}, n\right], \end{cases}$$

and differentiating with respect to τ we write it in the form (3.7), namely

$$r_n(\tau) = \frac{2}{\pi^2} \tau \sinh \pi\tau \int_{1/n}^n K_{i\tau}(x) g(x) \frac{dx}{x}, \tag{3.25}$$

where $r_n(\tau)$ is convergent to $r(\tau)$ when $n \to \infty$ due to (3.24). We will finally prove that $r(\tau) = s(\tau)$ almost for all $\tau \in \mathbf{R}_+$. Indeed, integrating through equalities (3.22), (3.25) with respect to τ we obtain

$$\int_0^\tau s_n(y) dy = \frac{2}{\pi^2} \int_0^\infty \int_0^\tau y \sinh \pi y K_{iy}(x) \varphi_n(x) \frac{dy\, dx}{x}, \tag{3.26}$$

$$\int_0^\tau r_n(y) dy = \frac{2}{\pi^2} \int_0^\infty \int_0^\tau y \sinh \pi y K_{iy}(x) g_n(x) \frac{dy\, dx}{x}, \tag{3.27}$$

where we change the order of integration by Fubini's theorem. In a similar manner as above we verify that for each $\tau > 0$ the function

$$\int_0^\tau y \sinh \pi y K_{iy}(x) dy \in L_{0,2}.$$

Therefore with the Schwarz inequality we show that integrals with respect to x in (3.26), (3.27) exist in a Lebesgue sense. Further, due to the following imbedding

$$L_2(\mathbf{R}_+;\omega_0(\tau)d\tau) \subseteq L_2\left(\mathbf{R}_+;[\tau\sinh\pi\tau]^{-1}\,d\tau\right) \subset L_1\left([0,\tau]\right)$$

we have that $r_n, r, s_n, s \in L_1\left([0,\tau]\right)$ and left-hand sides of (3.26), (3.27) represent continuous functionals. Moreover taking into account the convergence $g_n \to g, n \to \infty$ by virtue of (1.18) and since

$$||g-g_n||^2_{S_2^{N,0}(\mathbf{R}_+)} = \sum_{k=0}^{N}\left(\int_n^\infty + \int_0^{1/n}\right)|A_x^k g|^2\frac{dx}{x} \to 0,\ \ n\to\infty$$

one can pass to the limit through (3.26), (3.27) similar to (3.12) to derive

$$\int_0^\tau s(y)dy = \frac{2}{\pi^2}\int_0^\infty\int_0^\tau y\sinh\pi y K_{iy}(x)g(x)\frac{dy\,dx}{x},$$
$$\int_0^\tau r(y)dy = \frac{2}{\pi^2}\int_0^\infty\int_0^\tau y\sinh\pi y K_{iy}(x)g(x)\frac{dy\,dx}{x}.$$

Finally equating left-hand sides of latter equalities and differentiating with respect to τ we conclude that $r(\tau)=s(\tau)$ almost everywhere on $\mathbf{R}_+$. Theorem 3 is proved. $\square$

4. On the boundedness in $S_p^{N,\alpha}(\mathbf{R}_+),\ p\geq 2$

In this final section we will interpolate the norm of the Kontorovich-Lebedev transformation (1.1) as an operator $KL : L_p(\mathbf{R}_+;\rho_{p,\alpha}(\tau)d\tau) \to S_p^{N,\alpha}(\mathbf{R}_+)$, where $2\leq p\leq\infty$. The weighted function $\rho_{p,\alpha}(\tau)$ will be indicated below. In the case $p=\infty$ we understand the norm in the space $S_\infty^{N,\alpha}(\mathbf{R}_+)$ as (see (1.17))

$$||u||_{S_\infty^{N,\alpha}(\mathbf{R}_+)} = \lim_{p\to\infty}\left(\sum_{k=0}^{N}\int_0^\infty |A_x^k u|^p x^{\alpha_k p-1}dx\right)^{1/p}. \tag{4.1}$$

From the equivalence of norms (2.5) we immediately derive that

$$C_1\sum_{k=0}^{N}||A_\cdot^k u||_{L_{\alpha_k,\infty}} \leq ||u||_{S_\infty^{N,\alpha}(\mathbf{R}_+)} \leq C_2\sum_{k=0}^{N}||A_\cdot^k u||_{L_{\alpha_k,\infty}}, \tag{4.2}$$

where the norm in $L_{\nu,\infty}$ is defined by (see (1.10), (1.11))

$$||f||_{L_{\nu,\infty}} = \text{ess sup}\,|x^\nu f(x)| = \lim_{p\to\infty}\left(\int_0^\infty |f(x)|^p x^{\nu p-1}dx\right)^{1/p}. \tag{4.3}$$

We begin to derive an inequality for the modulus of the modified Bessel function $|K_{i\tau}(x)|$. We will apply it below to estimate the $L_{\nu,\infty}$-norm for the $(KLf)(x)$. Indeed, taking the Macdonald formula (1.9), we employ the

Schwarz inequality and invoke (1.8) with relation (2.16.33.2) from [6] to obtain

$$\begin{aligned} K_{i\tau}^2(x) &= \frac{1}{2}\int_0^\infty e^{-u-\frac{x^2}{2u}} K_{i\tau}(u)\frac{du}{u} \\ &\le \frac{1}{2}\left(\int_0^\infty e^{-2u-\frac{x^2}{u}} u^{-2\nu-1}du\right)^{1/2}\left(\int_0^\infty K_{i\tau}^2(u)u^{2\nu-1}du\right)^{1/2} \\ &= \pi^{1/4}2^{(\nu-3)/2}x^{-\nu}K_{2\nu}^{1/2}\left(2\sqrt{2}x\right)\left(\frac{\Gamma(\nu)}{\Gamma(\nu+1/2)}\right)^{1/2}|\Gamma(\nu+i\tau)|, \ \ \nu>0. \end{aligned} \tag{4.4}$$

Hence we get

$$|K_{i\tau}(x)| \le \pi^{1/8}2^{(\nu-3)/4}\left(\frac{\Gamma(\nu)}{\Gamma(\nu+1/2)}\right)^{1/4}|\Gamma(\nu+i\tau)|^{1/2}x^{-\nu/2}K_{2\nu}^{1/4}\left(2\sqrt{2}x\right).$$

Invoking inequality $x^\beta K_\beta(x) \le 2^{\beta-1}\Gamma(\beta), \beta > 0$ (see (1.8)) we derive an inequality

$$x^\nu |K_{i\tau}(x)| \le 2^{(2\nu-5)/4}\Gamma^{1/2}(\nu)|\Gamma(\nu+i\tau)|^{1/2}, \ \ x,\nu>0. \tag{4.5}$$

Thus from (1.1), (1.11), (4.5) we find that

$$\begin{aligned} x^\nu|(KLf)(x)| &\le ||f||_\infty x^\nu \int_0^\infty |K_{i\tau}(x)|d\tau \\ &\le 2^{(2\nu-5)/4}\Gamma^{1/2}(\nu)||f||_\infty \int_0^\infty |\Gamma(\nu+i\tau)|^{1/2}d\tau = C_\nu ||f||_\infty, \end{aligned}$$

where $C_\nu > 0$ is a constant

$$C_\nu = 2^{(2\nu-5)/4}\Gamma^{1/2}(\nu)\int_0^\infty |\Gamma(\nu+i\tau)|^{1/2}d\tau, \ \ \nu>0.$$

Therefore via (4.3) we obtain that the Kontorovich-Lebedev transformation is a bounded operator $KL : L_\infty(\mathbf{R}_+; d\tau) \to L_{\nu,\infty}$ of type (∞,∞) and

$$||KLf||_{L_{\nu,\infty}} \le C_\nu ||f||_\infty. \tag{4.6}$$

But inequality (3.1) says that this operator is of type $(2,2)$ too. Consequently, by the Riesz-Thorin convexity theorem [3] the Kontorovich-Lebedev transformation is of type (p,p), where $2 \le p \le \infty$ i.e. maps the space $L_p(\mathbf{R}_+; |\Gamma(2\nu+i\tau)|^2 d\tau)$ into $L_{\nu,p}$. Moreover for $2 \le p < \infty$ we arrive at the inequality

$$\int_0^\infty |(KLf)(x)|^p\, x^{\nu p-1}dx \le B_{p,\nu}\int_0^\infty |f(\tau)|^p|\Gamma(2\nu+i\tau)|^2 d\tau, \ \ \nu>0, \tag{4.7}$$

where we denoted by $B_{p,\nu}$ the constant

$$B_{p,\nu} = \pi^{3/2} 2^{-(3-p/2)\nu - 5p/4 + 3/2} \frac{\Gamma^{p/2-1}(\nu)}{\Gamma(2\nu + 1/2)} \left(\int_0^\infty |\Gamma(\nu + i\mu)|^{1/2} d\mu \right)^{p-2}.$$

Hence by the same method as in previous section we prove an analog of the inequality (3.4). Thus we obtain

$$||KLf||_{S_p^{N,\alpha}(\mathbf{R}_+)} \leq ||f||_{L_p(\mathbf{R}_+;\rho_{p,\alpha}(\tau)d\tau)}, \tag{4.8}$$

where

$$\rho_{p,\alpha}(\tau) = \sum_{k=0}^{N} B_{p,\alpha_k} \tau^{2kp} |\Gamma(2\alpha_k + i\tau)|^2, \;\; \alpha_k > 0, k = 0, 1, \ldots, N.$$

In particular, we have $\rho_{2,\alpha}(\tau) = \omega_\alpha(\tau)$ (see (3.2)). So the boundedness of the Kontorovich-Lebedev transformation (1.1) is proved. Finally we show that for all $x > 0$ it exists as a Lebesgue integral for any $f \in L_p(\mathbf{R}_+; \rho_{p,\alpha}(\tau)d\tau)$, $p > 2$. Indeed, it will immediately follow from the inequality

$$\int_0^\infty |K_{i\tau}(x) f(\tau)|\, d\tau \leq ||f||_{L_p(\mathbf{R}_+;|\Gamma(2\nu+i\tau)|^2 d\tau)} \times \left(\int_0^\infty |K_{i\tau}(x)|^q \, |\Gamma(2\nu + i\tau)|^{-2q/p} d\tau \right)^{1/q}, \;\; q = \frac{p}{p-1},$$

and from the convergence of the latter integral with respect to τ. This is easily seen from (1.6) and the Stirling asymptotic formula for gamma-functions [1] since the integrand behaves as $O\left(e^{\pi\tau q\left(\frac{1}{p}-\frac{1}{2}\right)} \tau^{\frac{q}{p}(1-4\nu)-\frac{q}{2}}\right), \tau \to +\infty$.

Acknowledgment. The present investigation was supported, in part, by the "Centro de Matemática" of the University of Porto.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1972.

[2] E. M. Dynkin and B. P. Osilenker, *Weighted estimates for singular integrals and their applications*, Mathematical analysis, Itogi Nauki i Tekhniki, 21 (1983), 42–129 (in Russian).

[3] R. E. Edwards, *Fourier Series II*, Holt, Rinehart and Winston, New York, 1967.

[4] B. Lisena, *On the generalized Kontorovich-Lebedev transform*, Rend. Mat. Appl., Ser. YII, 9 (1) (1989), 87–101.

[5] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series: Elementary Functions*, Gordon and Breach, New York, 1986.

[6] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series: Special Functions*, Gordon and Breach, New York, 1986.

[7] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series: More Special Functions*, Gordon and Breach, New York, 1989.
[8] I. N. Sneddon, *The Use of Integral Transforms*, McGraw-Hill, New York, 1972.
[9] E. C. Titchmarsh, *An Introduction to the Theory of Fourier Integrals*, Clarendon Press, Oxford, 1937.
[10] S. B. Yakubovich and B. Fisher, *On the theory of the Kontorovich-Lebedev transformation on distributions*, Proc. Amer. Math. Soc., 122 (3) (1994), 773–777.
[11] S. B. Yakubovich, *Index Transforms*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1996.
[12] S. B. Yakubovich and J. de Graaf, *On Parseval equalities and boundedness properties for Kontorovich-Lebedev type operators*, Novi Sad J. Math., 29 (1) (1999), 185–205.
[13] S. B. Yakubovich, *On the integral transfomation associated with the product of gamma-functions*, Port. Math., 60 (3) (2003), 337–351.
[14] S. B. Yakubovich, *On the Kontorovich-Lebedev transformation*, J. Integral Equations Appl., 15 (1) (2003), 95–112.
[15] S. B. Yakubovich, *Integral transforms of the Kontorovich-Lebedev convolution type*, Collect. Math., 54 (2) (2003), 99–110.
[16] A. H. Zemanian, *The Kontorovich-Lebedev transformation on distributions of compact support and its inversion*, Math. Proc. Cambridge Philos. Soc., 77 (1975), 139– 143.

(Received: May 16, 2005)
(Revised: August 24, 2005)

Department of Pure Mathematics,
Faculty of Sciences,
University of Porto,
Campo Alegre st., 687
4169-007 Porto, Portugal
E–mail: syakubov@fc.up.pt