SARAJEVO JOURNAL OF MATHEMATICS
Vol.1 (14) (2005), 235–242

A THEOREM ON THE COMMUTATIVE NEUTRIX PRODUCT OF DISTRIBUTIONS

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Abstract. The commutative neutrix products $f_+(x)\cdot\delta^{(r)}(x)$ and $f_-(x)\cdot\delta^{(r)}(x)$ are evaluated for $r = 0, 1, 2, \dots$, where f is a function which is infinitely differentiable on an open interval containing the origin and $f_+(x) = H(x)f(x)$ and $f_-(x) = H(-x)f(x)$, H denoting Heaviside's function.

1. Introduction

The technique of neglecting appropriately defined infinite quantities was devised by J. Hadamard and the resulting finite value extracted from a divergent integral is usually referred to as the Hadamard Finite Part. In fact, Hadamard's method can be regarded as a particular application of the neutrix calculus developed by J. P. van der Corput, see [1]. This is a very general principle for the discarding of unwanted infinite quantities from asymptotic expansions and has been widely exploited in the context of distributions, by B. Fisher in connection with the problem of products of distributions, see [6], [7] or [9].

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

The definition of the product of a distribution and an infinitely differentiable function is the following, see for example [8].

Definition 1. *Let f be a distribution in $\mathcal{D}'$ and let g be an infinitely differentiable function. The product fg is defined by*

$$\langle fg, \varphi\rangle = \langle f, g\varphi\rangle$$

for all functions φ in $\mathcal{D}$.

2000 *Mathematics Subject Classification.* 46F10.

Key words and phrases. Distribution, Delta function, neutrix, neutrix limit, commutative neutrix product.

This research was supported by TUBITAK.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [2].

Definition 2. *Let f and g be distributions in $\mathcal{D}'$ for which on the interval (a, b), f is the k-th derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$ is a locally summable function in $L^q(a, b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by*

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [F g^{(i)}]^{(k-i)}.$$

Definition 2 was extended in [2]. In order to define this extension, we first of all let ρ be a fixed infinitely differentiable function in $\mathcal{D}$, having the properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

We now define the function $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$. It is obvious that $\{\delta_n\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let f be an arbitrary distribution and define the function f_n by

$$f_n(x) = f * \delta_n = \langle f(x - t), \delta_n(t) \rangle.$$

Then $\{f_n\}$ is a sequence of infinitely differentiable functions converging to the distribution f.

In order to define further products of distributions, a neutrix product was given in [6].

Definition 3. *Let f and g be arbitrary distributions and let $f_n = f * \delta_n$ and $g_n = g * \delta_n$. We say that the commutative neutrix product $f \cdot g$ of f and g exists and is equal to h on the open interval (a, b) ($-\infty \leq a < b \leq \infty$) if*

$$\operatorname*{N-lim}_{n \to \infty} \langle f_n g_n, \varphi \rangle = \langle h, \varphi \rangle$$

for all $\varphi \in \mathcal{D}$, where N is the neutrix having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^{\lambda} \ln^{r-1} n,\ \ln^r n \quad (\lambda > 0,\ r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as n tends to infinity.

If

$$\lim_{n\to\infty} \langle f_n g_n, \varphi\rangle = \langle h, \varphi\rangle,$$

we simply say that the *commutative product* $f.g$ of f and g exists and is equal to h on the open interval (a, b) $(-\infty \le a < b \le \infty)$.

Note that if $\text{f.p.}\,\langle f_n g_n, \varphi\rangle$ denotes the finite part of $\langle f_n g_n, \varphi\rangle$, then taking the neutrix limit as n tends to infinity of the sequence $\{\langle f_n g_n, \varphi\rangle\}$, is equivalent to taking the usual limit as n tends to infinity of the sequence $\{\text{f.p.}\,\langle f_n g_n, \varphi\rangle\}$, that is

$$\underset{n\to\infty}{\text{N}-\lim}\langle f_n g_n, \varphi\rangle = \lim_{n\to\infty} \text{f.p.}\,\langle \mathrm{f_n g_n}, \varphi\rangle.$$

The following example of a commutative product was proved in [4].

$$x^{-r}.\delta^{(r-1)}(x) = \frac{(-1)^r (r-1)!}{2(2r-1)!}\,\delta^{(2r-1)}(x),$$

for $r = 1, 2, \ldots$.

The next example is of a commutative neutrix product which does not exist as a commutative product and was proved in [6].

$$x_+^{-r} \cdot \delta^{(r-1)}(x) = \frac{(-1)^r (r-1)!}{4(2r-1)!}\,\delta^{(2r-1)}(x),$$

for $r = 1, 2, \ldots$.

For further results on the commutative product, see [2], [3] and [5] and for further results on the commutative neutrix product, see [7] and [9].

The proof of the next theorem is immediate.

Theorem 1. *Let f and g be distributions for which the commutative product $f.g$ exists. Then the commutative neutrix product $f \cdot g$ exists and defines the same distribution.*

2. Results

We now prove the following theorem.

Theorem 2. *Let f be a function which is infinitely differentiable on an ope interval containing the origin. Then the commutative neutrix products $f_+(x) \cdot \delta^{(r)}(x)$ and $f_-(x) \cdot \delta^{(r)}(x)$ exist and*

$$f_+(x) \cdot \delta^{(r)}(x) = \sum_{k=0}^{r} \frac{(-1)^r}{2} \binom{r}{k} f^{(r-k)}(0)\,\delta^{(k)}(x) \tag{1}$$

$$f_-(x) \cdot \delta^{(r)}(x) = \sum_{k=0}^{r} \frac{(-1)^r}{2} \binom{r}{k} f^{(r-k)}(0)\,\delta^{(k)}(x) \tag{2}$$

for $r = 0, 1, 2, \ldots$.

Proof. By Taylor's Theorem, we have

$$f_+(x) = \sum_{i=0}^{r} \frac{f^{(i)}(0)}{i!}\, x_+^i + \frac{f^{(r+1)}(\xi x)}{(r+1)!}\, x_+^{r+1},$$

where $0 < \xi < 1$. Putting $(x_+^i)_n = x_+^i * \delta_n(x)$ for $i = 0, 1, 2, \ldots$, $[f_+(x)]_n = f_+(x) * \delta_n(x)$ and $[f^{(r+1)}(\xi x) x_+^{r+1}]_n = [f^{(r+1)}(\xi x) x_+^{r+1}] * \delta_n(x)$ for large enough n to ensure that f is infinitely differentiable on the interval $(-1/n, 1/n)$, we have

$$[f_+(x)]_n = \sum_{i=0}^{r} \frac{f^{(i)}(0)}{i!}\, (x_+^i)_n + \frac{[f^{(r+1)}(\xi x)\, x_+^{r+1}]_n}{(r+1)!} \tag{3}$$

and then the support of $[f_+(x)]_n \delta_n^{(r)}(x)$ is equal to $[-1/n, 1/n]$.

Now

$$\begin{aligned}
\int_{-1/n}^{1/n} x^k (x_+^i)_n\, \delta_n^{(r)}(x)\, dx &= \int_{-1/n}^{1/n} \int_{-1/n}^{x} x^k [(x-t)^i \delta_n(t)\, \delta_n^{(r)}(x)]\, dt\, dx \\
&= \int_{-1/n}^{1/n} \delta_n(t) \int_{t}^{1/n} x^k (x-t)^i \delta_n^{(r)}(x)\, dx\, dt \\
&= n^{r-k-i} \int_{-1}^{1} \rho(u) \int_{u}^{1} v^k (v-u)^i \rho^{(r)}(v)\, dv\, du
\end{aligned}$$

on making the substitutions $nt = u$ and $nx = v$. It follows that

$$\operatorname*{N-lim}_{n\to\infty} \int_{-1/n}^{1/n} x^k [(x_+^i)_n\, \delta_n^{(r)}(x)]\, dx = 0 \tag{4}$$

for $k \neq r - i$.

We now consider the case $k = r - i$ and prove that

$$\int_{-1/n}^{1/n} x^{r-i} (x_+^i)_n\, \delta_n^{(r)}(x)\, dx = \int_{-1}^{1} \rho(u) \int_{u}^{1} v^{r-i} (v-u)^i \rho^{(r)}(v)\, dv\, du = \frac{(-1)^r r!}{2} \tag{5}$$

for $r = 0, 1, 2, \ldots$ and $i = 0, 1, 2, \ldots, r$.

In the particular case $i = 0$, we have

$$\begin{aligned}
\int_{-1}^{1} \rho(u) \int_{u}^{1} v^r \rho^{(r)}(v)\, dv\, du &= \int_{-1}^{1} v^r \rho^{(r)}(v) \int_{-1}^{v} \rho(u)\, du\, dv \\
&= \int_{-1}^{1} v^r \rho^{(r)}(v) H(v)\, dv \\
&= \int_{0}^{1} v^r \rho^{(r)}(v)\, dv
\end{aligned}$$

$$
\begin{aligned}
&= -r\int_0^1 v^{r-1}\rho^{(r-1)}(v)\,dv \\
&= \ \cdots \\
&= \frac{(-1)^r r!}{2}
\end{aligned}
$$

on integration by parts, proving equation (5) for the case $i = 0$ and $r = 0, 1, 2, \ldots$.

We now note that equation (5) is trivially true when $r = 0$. Suppose that equation (5) is true for some positive integer r and $i = 0, 1, 2, \ldots, r$. Then, with $i \neq 0$, we have on integration by parts,

$$
\begin{aligned}
&\int_{-1}^1 \rho(u)\int_u^1 v^{r-i+1}(v-u)^i\rho^{(r+1)}(v)\,dv\,du = \\
&= -\int_{-1}^1 \rho(u)\int_u^1 [(r-i+1)v^{r-i}(v-u)^i + iv^{r-i+1}(v-u)^{i-1}]\,\rho^{(r)}(v)\,dv\,du \\
&= -\frac{(r-i+1)(-1)^r r!}{2} - \frac{i(-1)^r r!}{2} \\
&= \frac{(-1)^{r+1}(r+1)!}{2}
\end{aligned}
$$

on using our assumption. Equation (5) now follows by induction for $r = 1, 2, \ldots$ and $i = 1, 2, \ldots$. This completes the proof of equation (5).

Next let ψ be an arbitrary continuous function. Then

$$
\begin{aligned}
&\int_{-1/n}^{1/n} x^{r+1}\,\psi(x)(x_+^i)_n\,\delta_n^{(r)}(x)\,dx \\
&\qquad = n^{-i-1}\int_{-1}^1 \rho(u)\int_u^1 v^k\psi(v)(v-u)^i\rho^{(r)}(v)\,dv\,du
\end{aligned}
$$

and it follows that

$$
\lim_{n\to\infty}\int_{-1/n}^{1/n} x^{r+1}\,\psi(x)(x_+^i)_n\,\delta_n^{(r)}(x)\,dx = 0 \tag{6}
$$

for $i = 0, 1, 2, \ldots$.

Finally we have

$$
\begin{aligned}
&\int_{-1/n}^{1/n} \psi(x)[f^{(r+1)}(\xi x)x_+^{r+1}]_n\delta_n^{(r)}(x)\,dx \\
&\qquad = n^{-1}\int_{-1}^1 \rho(u)\int_u^1 v^k\psi(v)f^{r+1}(\xi v-\xi u)(v-u)^{r+1}\rho^{(r)}(v)\,dv\,du
\end{aligned}
$$

and it follows that

$$\lim_{n\to\infty}\int_{-1/n}^{1/n}\psi(x)[f^{(r+1)}(\xi x)x_+^{r+1}]_n\delta_n^{(r)}(x)\,dx = 0. \tag{7}$$

Now let φ be an arbitrary function in $\mathcal{D}$. Then

$$\varphi(x) = \sum_{k=0}^{r}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{\varphi^{(r+1)}(\eta x)}{(r+1)!}x^{r+1},$$

where $0 < \eta < 1$. Then

$$\begin{aligned}\langle[f_+(x)]_n\,\delta_n^{(r)}(x),\varphi(x)\rangle &= \sum_{k=0}^{r}\sum_{i=0}^{r}\frac{f^{(i)}(0)\,\varphi^{(k)}(0)}{k!i!}\int_{-1/n}^{1/n}x^k(x_+^i)_n\,\delta_n^{(r)}(x)\,dx\\ &\quad+\frac{1}{(r+1)!}\sum_{i=0}^{r}\int_{-1/n}^{1/n}x^{r+1}\varphi^{(r+1)}(\eta x)(x_+^i)_n\,\delta_n^{(r)}(x)\,dx\\ &\quad+1(r+1)!\int_{-1/n}^{1/n}\varphi(x)[f^{(r+1)}(\xi x)\,x_+^{r+1}]_n\,dx\end{aligned}$$

and it follows from equations (4), (5), (6) and (7) that

$$\begin{aligned}\mathrm{N}\!-\!\lim_{n\to\infty}\langle[f_+(x)]_n\,\delta_n^{(r)}(x),\phi(x)\rangle &= \sum_{k=0}^{r}\frac{(-1)^r}{2}\binom{r}{k}f^{(r-k)}(0)\,\varphi^{(k)}(0)\\ &= \sum_{k=0}^{r}\frac{(-1)^{r-k}}{2}\binom{r}{k}f^{(r-k)}(0)\langle\delta^{(k)}(x),\varphi(x)\rangle,\end{aligned}$$

proving equation (1) for $r = 0, 1, 2, \ldots$.

Since f is an infinitely differentiable equation, we know that

$$f(x)\delta^{(r)}(x) = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}f^{(r-k)}(0)\,\delta^{(k)}(x)$$

and since

$$f(x) = f_+(x) + f_-(x),$$

equation (2) follows immediately. This completes the proof of the theorem. □

Example 1. With $f(x) = \sin x$ and $f_+(x) = \sin_+ x$ we have

$$\sin_+ x\cdot\delta^{(r)}(x) = \sum_{k=0}^{r}\frac{(-1)^r\sin[\frac{1}{2}(r-k)\pi]}{2}\binom{r}{k}\delta^{(k)}(x),$$

for $r = 0, 1, 2, \ldots$.

Example 2. With $f(x) = \cos x$ and $f_+(x) = \cos_+ x$ we have

$$\cos_+ x \cdot \delta^{(r)}(x) = \sum_{k=0}^{r} \frac{(-1)^r \cos[\frac{1}{2}(r-k)\pi]}{2} \binom{r}{k} \delta^{(k)}(x),$$

for $r = 0, 1, 2, \ldots$.

Example 3. With $f(x) = e^x \sin x$ and $f_+(x) = (e^x \sin x)_+$ we have

$$(e^x \sin x)_+ \cdot \delta^{(r)}(x) = \sum_{k=0}^{r} \frac{(-1)^r 2^{(r-k)/2} \sin[\frac{1}{4}(r-k)\pi]}{2(r-k)!} \binom{r}{k} \delta^{(k)}(x), \tag{8}$$

for $r = 0, 1, 2, \ldots$.

Equation (8) follows on noting that

$$e^x \sin x = \sum_{k=0}^{\infty} \frac{2^{k/2} \sin(\frac{1}{4}k\pi)}{k!} x^k.$$

Example 4. With $f(x) = x(e^x - 1)^{-1}$ and $f_+(x) = [x(e^x - 1)^{-1}]_+$ we have

$$[x(e^x - 1)^{-1}]_+ \cdot \delta^{(r)}(x) = \sum_{k=0}^{r} \frac{(-1)^r B_{r-k}}{2(r-k)!} \binom{r}{k} \delta^{(k)}(x), \tag{9}$$

for $r = 0, 1, 2, \ldots$. where the B_r denote the Bernoulli numbers.

Equation (9) follows on noting that

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k.$$

Acknowledgement. The authors would like to thank the referee for his help in the improvement of this paper.

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(Received: October 21, 2004)
(Revised: February 26, 2005)

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