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REPRESENTATION THEOREMS FOR INTEGRATED SEMIGROUPS

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ABSTRACT. In this paper $(S(t))_{t\geq 0}$ is an exponentially bounded integrated semigroup on a Banach space X, with generator A. We present some relations between an integrated semigroup and its generator A, or its resolvent.

1. Introduction

The theory of integrated semigroups of operators on a Banach space were introduced and developed during the last twenty years by Kellermann, Arendt, Thieme, Hieber, Neubrander and many other mathematicians (for example, see [1,2,4,5,6,8,10,12]). Representation theorems for C_0- semigroups of operators on a Banach space are given and proved in [7]. Some of these theorems were also proved by others authors (for example, see [3, 11]). The motivation for further investigation is Hille's first exponential formula and the Laplace inversion formula for C_0- semigroups.

2. Preliminaries from the theory of integrated semigroup and some applications

The theory of $\alpha-$times integrated semigroups ($\alpha \geq 0$) was introduced by Hieber in [4, 5, 6]. Some results were obtained also in [9]. Denote by X a Banach space with the norm $\|\cdot\|$; $L(X) = L(X, X)$ is the space of bounded linear operators from X into X.

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Definition 2.1. (in [9]) *Let $(S(t))_{t\geq 0}$ be a strongly continuous family of operators in $L(X)$ and $\alpha \in \mathbb{R}^+$. Then, $(S(t))_{t\geq 0}$ is called an $\alpha-$times integrated semigroup if $S(0) = 0$ and the following is true.*

$$S(t)S(s) = \frac{1}{\Gamma(\alpha)}\left[\int\limits_t^{t+s} (t+s-r)^{\alpha-1}\, S(r)\, dr - \int\limits_0^s (t+s-r)^{\alpha-1}\, S(r)\, dr\right]$$

for every $t,\ s \geq 0$. $(S(t))_{t\geq 0}$ is called non-degenerate if $S(t)x = 0$ for all $t \geq 0$ implies $x = 0$. If there exist constants $M \geq 0$ and $\omega \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, then $(S(t))_{t\geq 0}$ is called an $\alpha-$times integrated, exponentially bounded semigroup.

Theorem 2.2. (in [9]) *Let $\alpha \in \mathbb{R}^+$; $S : [0,\infty) \to L(X)$ be a strongly continuous, exponentially bounded at infinity (i.e. it satisfied $\|S(t)\| \leq Me^{\omega t}$ for $t \geq 0$ and some constants $M \geq 0$ and $\omega \in \mathbb{R}$), and $R(\lambda) = \lambda^\alpha \int\limits_0^\infty e^{-\lambda t} S(t) dt$, $\operatorname{Re}\lambda > \omega$. Then, $R(\lambda)$, $\operatorname{Re}\lambda > \omega$, is a pseudoresolvent (i.e. the resolvent equation $R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu)$) if and only if for every $t,\ s \geq 0$:*

$$S(t)S(s) = \frac{1}{\Gamma(\alpha)}\left[\int\limits_t^{t+s} (t+s-r)^{\alpha-1}\, S(r)\, dr - \int\limits_0^s (t+s-r)^{\alpha-1}\, S(r)\, dr\right].$$

Let $(S(t))_{t\geq 0}$ be an $\alpha-$times integrated semigroup, $\alpha \in \mathbb{R}^+$. Let

$$R(\lambda, A) = \lambda^\alpha \int\limits_0^\infty e^{-\lambda t} S(t) dt,\ \ \operatorname{Re}\lambda > \omega.$$

Here we take the function λ^α for which $1^\alpha := 1$. Then, by the resolvent equation, $\operatorname{Ker} R(\lambda)$ is independent of $\operatorname{Re}\lambda > \omega$. Hence, by the uniqueness theorem, $R(\lambda)$ is injective if and only if $(S(t))_{t\geq 0}$ is non-degenerate. In this case there exists a unique operator A satisfying $(\omega, \infty) \subset \rho(A)$ ($\rho(A)$ is the resolvent set of A) such that

$$R(\lambda) = (\lambda I - A)^{-1} \text{ for all } \lambda \text{ with } \operatorname{Re}\lambda > \omega.$$

This operator is called the generator of $(S(t))_{t\geq 0}$.

Definition 2.3. (in [4, 5, 6, 9]) *Let $\alpha \in \mathbb{R}^+$. An operator A is the generator of an $\alpha-$times integrated, exponentially bounded semigroup $(S(t))_{t\geq 0}$ if and only if $(a, \infty) \subset \rho(A)$ for some $a \in \mathbb{R}$ and*

$$R(\lambda, A)x = \lambda^\alpha \int\limits_0^\infty e^{-\lambda t} S(t)x\, dt, \quad x \in X,\ Re\,\lambda > a.$$

In [4] the author considered the initial value problem $u'(t) = Au(t)$, $u(0) = u_0$, for a differential operator A with constant coefficients on the function spaces $X = L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$, $C_0(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$ or $C_b(\mathbb{R}^n)$. He asked whether the given operator A generates a C_0-semigroup or an integrated semigroup on X. In [6] it is proved that , under suitable hypotheses, every homogeneous differential operator on $L^p(\mathbb{R}^n)^N$ corresponding to a system which is well-posed in $L^2(\mathbb{R}^n)^N$, generates an $\alpha-$times integrated semigroup on $L^p(\mathbb{R}^n)^N$ $(1 < p < \infty)$ whenever $\alpha > n\,|1/2 - 1/p|$. For some special systems of mathematical physics, such as the wave equation or Maxwell's equations, this constant can be improved to $(n-1)\,|1/2 - 1/p|$. In [9] it is shown that suitable differential operators generate $\alpha-$times integrated semigroups for $\alpha \in (1/2, 1)$. In [2] local $k-$times integrated semigroup $(k \in \mathbb{N})$ is defined as a solution $v \in C\left([0,\tau); D(A)\right) \cap C^1\left([0,\tau); X\right)$ of the $(k+1)-$times integrated Cauchy problem $v'(t) = Av(t) + (t^k/k!)x$, $v(0) = 0$ $(x \in X, \tau > 0)$.

3. Representation theorems for integrated semigroups

Here we give two results. The first theorem gives a representation formula for once integrated semigroup. The motivation for this theorem is the well-known Hille's first exponential formula for C_0-semigroups(see [7, Theorem 10.4.1.], [3, Theorem 1.2.2.], [11, Theorem 8.1.]). We need the following lemma.

Lemma 3.1. (in [3]) *For $N > 0$ and $u \ge 0$ we have*

$$e^{-u} \sum_{|k-u|>N} \frac{u^k}{k!} \le \frac{u}{N^2}.$$

The proof of this lemma is given in [3, Lemma 1.2.1.(a)].

Theorem 3.2. *Let $(S(t))_{t\ge0}$ be a once integrated, exponentially bounded semigroup on a Banach space X with generator A. Then for $x \in X$ and $t \ge 0$ we have*

$$S(t)x = \lim_{h\to0^+} \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{t}{h}\right)^n A^{n-1}S^n(h)x \tag{1}$$

with the limit existing uniformly with respect to t in any finite interval $[0, T]$.

Proof. It is known that for all $x \in X$ and all $t,\, s \geq 0$

$$S(t)S(s)x = \int_0^s [S(t+r)x - S(r)x]\, dr$$
$$= \int_0^{t+s} S(r)x\, dr - \int_0^t S(r)x\, dr - \int_0^s S(r)x\, dr$$

and $A\int_0^t S(r)xdr = S(t)x - tx$. Therefore,

$$AS^2(h)x = A\left[\int_0^{2h} S(r)x\, dr - 2\int_0^h S(r)x\, dr\right] = S(2h)x - 2S(h)x.$$

Also,

$$A^2S^3(h)x = A\left[AS^2(h)\right]S(h)x = AS(2h)S(h)x - 2AS^2(h)x$$
$$= A\left[\int_0^{3h} S(r)x\, dr - \int_0^{2h} S(r)x\, dr - \int_0^h S(r)x\, dr\right] - 2\left[S(2h)x - 2S(h)x\right]$$
$$= S(3h)x - 3S(2h)x + 3S(h)x.$$

Induction implies that for every $n \in \mathbb{N}$ we have

$$A^{n-1}S^n(h)x = \sum_{k=0}^{n-1}(-1)^k\binom{n}{k}S\left[(n-k)h\right]x.$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{t}{h}\right)^n A^{n-1}S^n(h)x = \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{t}{h}\right)^n\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}S\left[(n-k)h\right]x$$
$$= \underbrace{\left[1 - \frac{t}{h} + \frac{1}{2!}\left(\frac{t}{h}\right)^2 - \cdots\right]}_{=e^{-\frac{t}{h}}}\cdot$$
$$\cdot\left[\frac{t}{h}S(h)x + \frac{1}{2!}\left(\frac{t}{h}\right)^2 S(2h)x + \frac{1}{3!}\left(\frac{t}{h}\right)^3 S(3h)x + \cdots\right]$$
$$= e^{-\frac{t}{h}}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{t}{h}\right)^n S(nh)\, x.$$

Hence, we need to prove that

$$S(t)x = \lim_{h\to 0^+} e^{-\frac{t}{h}} \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{t}{h}\right)^n S(nh)x,\ \ x\in X. \tag{2}$$

The family $(S(t))_{t\geq 0}$ is exponentially bounded, i.e. there exist real constants M and ω such that $\|S(t)\| \leq Me^{\omega t}$ for all $t\geq 0$. Thus, we have

$$\left\| e^{-\frac{t}{h}} \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{t}{h}\right)^n S(nh)\right\| \leq Me^{-\frac{t}{h}} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{t}{h}e^{\omega h}\right)^n = Me^{\frac{t}{h}\left(e^{\omega h}-1\right)}.$$

By the inequality

$$\frac{t}{h} \leq \frac{t}{h}e^{\omega h} \leq \frac{t}{h} + t\left(e^{\omega}-1\right)\ \ (0<h\leq 1)$$

the norm of $e^{-\frac{t}{h}} \sum\limits_{n=1}^{\infty} \frac{1}{n!}\left(\frac{t}{h}\right)^n S(nh)$ is uniformly bounded by $Me^{T(e^{\omega}-1)}$ for $0\leq t\leq T$ and $0<h\leq 1$. We have for every $x\in X$:

$$\left\| S(t)x - e^{-\frac{t}{h}} \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{t}{h}\right)^n S(nh)x\right\| = e^{-\frac{t}{h}} \left\| \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{t}{h}\right)^n \left[S(t)x - S(nh)x\right]\right\|$$
$$\leq \sum\nolimits_1 + \sum\nolimits_2$$

where

$$\sum\nolimits_1 = e^{-\frac{t}{h}} \sum_{\left|n-\frac{t}{h}\right|\leq h^{-\frac{2}{3}}} \frac{1}{n!}\left(\frac{t}{h}\right)^n \|S(t)x - S(nh)x\|$$

and

$$\sum\nolimits_2 = e^{-\frac{t}{h}} \sum_{\left|n-\frac{t}{h}\right|> h^{-\frac{2}{3}}} \frac{1}{n!}\left(\frac{t}{h}\right)^n \|S(t)x - S(nh)x\|.$$

Hence, in $\sum_1$ we put natural numbers n such that $\left|n-\frac{t}{h}\right|\leq h^{-\frac{2}{3}}$, in $\sum_2$ we put natural numbers n such that $\left|n-\frac{t}{h}\right| > h^{-\frac{2}{3}}$. Fix $x\in X$ and $T>0$. Let

$$\varepsilon(\delta) = \sup\left\{\|S(t)x - S(s)x\| :\ \ 0\leq t,\ s\leq T,\ \ \ |t-s|\leq\delta\right\}.$$

If $\left|n-\frac{t}{h}\right|\leq h^{-\frac{2}{3}}$, then $\|S(t)x - S(nh)x\| \leq \varepsilon(h^{\frac{1}{3}})$ and $\sum_1 \leq \varepsilon(h^{\frac{1}{3}})$. Because $S(t)x$ is a strongly continuous function, $\varepsilon(h^{\frac{1}{3}}) \to 0$ as $h\to 0^+$.

Therefore $\sum_1 \to 0$ as $h\to 0^+$.

Let us estimate now $\sum_2$.

$$\sum\nolimits_2 \leq Me^{-\frac{t}{h}}\|x\| \sum_{\left|n-\frac{t}{h}\right|> h^{-\frac{2}{3}}} \frac{1}{n!}\left(\frac{t}{h}\right)^n \left(e^{\omega t} + e^{\omega hn}\right).$$

By Lemma 3.1., $e^{-\frac{t}{h}} \sum\limits_{\left|n-\frac{t}{h}\right|>h^{-\frac{2}{3}}} \frac{1}{n!}\left(\frac{t}{h}\right)^n \le \frac{t}{h}h^{\frac{4}{3}} = th^{\frac{1}{3}}$, and

$$e^{-\frac{t}{h}} \sum_{\left|n-\frac{t}{h}\right|>h^{-\frac{2}{3}}} \frac{1}{n!}\left(\frac{t}{h}e^{\omega h}\right)^n \le e^{-\frac{t}{h}} \sum_{\left|n-\frac{t}{h}e^{\omega h}\right|>\frac{h^{-\frac{2}{3}}}{2}} \frac{1}{n!}\left(\frac{t}{h}e^{\omega h}\right)^n$$
$$\le 4e^{t(e^{\omega}-1)}h^{\frac{1}{3}}.$$

Therefore, $\sum_2 \to 0$ as $h \to 0^+$. Hence, (2) holds, and also (1), uniformly with respect to t in any finite interval $[0,T]$. □

In [13] it is proved that for an $\alpha-$times integrated semigroup $S(t)$ ($\alpha \in \mathbb{R}^+$) and any $\beta > 0$ the following holds

$$S(t)x = (C,\beta) - \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} \frac{R(\lambda,A)x}{\lambda^{\alpha}} d\lambda.$$

($x \in X, t \ge 0, \gamma > \max(0,\omega_0)$), where $(C,\beta) - \lim$ denotes the Cesàro-β limit defined as in [7].

The next theorem shows that classical convergency holds for $x \in D(A)$.

Theorem 3.3. *Let $(S(t))_{t\ge 0}$ be an $\alpha-$times integrated, exponentially bounded semigroup on a Banach space X ($\alpha \in \mathbb{R}^+$), and let A be a generator of $(S(t))_{t\ge 0}$. Let $M \ge 0$ and $\omega_0 \in \mathbb{R}$ be constants such that $\|S(t)\| \le Me^{\omega_0 t}$ for $t \ge 0$. Then for all $x \in X$, $t \ge 0$ and $\gamma > \max(0,\omega_0)$*

$$\int_0^t S(s)x\,ds = \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} \frac{R(\lambda,A)x}{\lambda^{1+\alpha}} d\lambda. \tag{3}$$

Also, for all $x \in D(A)$, $t > 0$ and $\gamma > \max(0,\omega_0)$

$$S(t)x = \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} \frac{R(\lambda,A)x}{\lambda^{\alpha}} d\lambda. \tag{4}$$

Proof. Fix any $\gamma > \max(0,\omega_0)$, $x \in X$ and $t \ge 0$. Then

$$\lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} \frac{R(\lambda,A)x}{\lambda^{1+\alpha}} d\lambda = \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} \frac{e^{\lambda t}}{\lambda} d\lambda \int_0^{\infty} e^{-\lambda s} S(s)x\,ds.$$

We interchange the order of integration and obtain the expression :

$$\lim_{\omega\to\infty}\frac{1}{2\pi i}\int_0^{\infty} S(s)x\,ds\int_{\gamma-i\omega}^{\gamma+i\omega}\frac{e^{\lambda(t-s)}}{\lambda}d\lambda=$$

$$=\int_0^t S(s)x\,ds\lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega}\frac{e^{\lambda(t-s)}}{\lambda}d\lambda+\int_t^{\infty} S(s)x\,ds\lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega}\frac{e^{\lambda(t-s)}}{\lambda}d\lambda.$$

It is well-known that for $\gamma>0$ it holds :

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda z}}{\lambda}d\lambda=\begin{cases}1, & \text{for } z>0\\ 0, & \text{for } z\le 0.\end{cases}$$

Therefore, the limit above equals $\int_0^t S(s)x\,ds$. Hence, (3) holds.

It is known that for $x\in D(A)$ and $t\ge 0$ it holds :

$$S(t)x=\frac{t^{\alpha}}{\Gamma(\alpha+1)}x+\int_0^t S(s)Ax\,ds.$$

Here Γ denote the Gamma-function. From (3) we have for $\gamma>\max(0,\omega_0)$:

$$S(t)x=\frac{t^{\alpha}}{\Gamma(\alpha+1)}x+\lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega}e^{\lambda t}\frac{R(\lambda,A)Ax}{\lambda^{1+\alpha}}d\lambda.$$

Because of $R(\lambda,A)Ax=(\lambda I-A)^{-1}(A-\lambda I+\lambda I)\,x=\lambda R(\lambda,A)x-x$, we have :

$$S(t)x=\frac{t^{\alpha}}{\Gamma(\alpha+1)}x+\lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega}e^{\lambda t}\frac{\lambda R(\lambda,A)x-x}{\lambda^{1+\alpha}}d\lambda$$

$$=\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega}\frac{e^{\lambda t}}{\lambda^{1+\alpha}}d\lambda\right]x+\lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega}e^{\lambda t}\frac{R(\lambda,A)x}{\lambda^{\alpha}}d\lambda.$$

For all $t>0$

$$\lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega}\frac{e^{\lambda t}}{\lambda^{1+\alpha}}d\lambda=L^{-1}\left[\frac{1}{\lambda^{1+\alpha}}\right]=\frac{t^{\alpha}}{\Gamma(\alpha+1)}\,.$$

Here L^{-1} denote the inverse Laplace transform. Therefore, for $t>0$:

$$S(t)x = \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} \frac{R(\lambda, A)x}{\lambda^{\alpha}} d\lambda,$$

i.e. (4) holds. □

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