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WARPED PRODUCT LIGHTLIKE SUBMANIFOLDS

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Abstract. We study a new class of lightlike submanifolds M, called warped product lightlike submanifolds, of a semi-Riemann manifold. We show that the null geometry of M reduces to the corresponding non-degenerate geometry of its semi-Riemann submanifold.

1. Introduction

The main purpose of this paper is to contribute to the study of the following problem:

Find a class of lightlike submanifolds whose geometry is essentially the same as that of their chosen screen distribution.

This problem was proposed by K.L. Duggal in [5], [6] and he also emphasized that it has several physical applications. Actually, this problem has been studied in many papers. In [7], K.L. Duggal and A. Bejancu showed that the geometry of a Monge lightlike surface reduces to the geometry of a leaf of its screen distribution. The same result was obtained for a hypersurface with canonical distribution in [2] and [3] by A. Bejancu and by A. Bejancu et al., respectively. Moreover, K.L. Duggal showed that this result is true for a half-lightlike submanifold of a semi-Euclidean space with integrable screen distribution [6]. Also, C. Atindogbe and K.L. Duggal introduced the notion of screen conformal lightlike hypersurface and they obtained that the geometry of such hypersurfaces reduces to the geometry of a leaf of its screen distribution [1]. Furthermore, K.L. Duggal and the present author showed that this notion is well defined for half-lightlike submanifolds and the geometry of screen conformal half-lightlike submanifolds has a close relation with non-degenerate geometry of a leaf of its screen distribution [9]. Here, note that the radical distribution has rank $r = 1$ in all those papers, mentioned above.

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On the other hand, warped product manifolds are defined in [4] as follows: Let $(B, \bar{g}_1)$ and $(F, \bar{g}_2)$ be two Riemannian manifolds, $f : B \to (0, \infty)$ and $\pi : B \times F \to B$, $\eta : B \times F \to F$ the projection maps given by $\pi(p, q) = p$ and $\eta(p, q) = q$ for every $(p, q) \in B \times F$. The warped product $\bar{M} = B \times F$ is the manifold $B \times F$ equipped with the Riemannian structure such that

$$\bar{g}(X, Y) = \bar{g}_1(\pi_* X, \pi_* Y) + (f o \pi)^2 \bar{g}_2(\eta_* X, \eta_* Y)$$

for every X and Y of $\bar{M}$ and $*$ is the symbol for the tangent map.

In this paper, we present a new class of lightlike submanifolds, using warped products, such that its radical distribution has rank $r \geq 1$. Roughly speaking, our main result is that the geometry of coisotropic warped product lightlike submanifolds of a semi-Riemann manifold reduces to the non-degenerate geometry of a leaf of its screen distribution.

2. Preliminaries

We follow [7] for the notation and formulas used in this paper. A submanifold M^m immersed in a semi-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ is called a *lightlike submanifold* if it is a lightlike manifold w.r.t. the metric g induced from $\bar{g}$ and the radical distribution $Rad(TM)$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, i.e., $TM = \mathrm{Rad}\,(TM) \perp S(TM)$.

Consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $\mathrm{Rad}(TM)$ in $TM^{\perp}$. Since, for any local basis $\{\xi_i\}$ of the $\mathrm{Rad}(TM)$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$, it follows that there exists a *lightlike transversal vector bundle* $\mathrm{ltr}(TM)$ locally spanned by $\{N_i\}$ [7, page 144]. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$. Then,

$$\mathrm{tr}(TM) = \mathrm{ltr}(TM) \perp S(TM^{\perp}),$$
$$T\bar{M}|_M = S(TM) \perp [\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)] \perp S(TM^{\perp}).$$

The following are four subcases of a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$.

Case 1: r - lightlike if $r < min\{m, n\}$;
Case 2: Co - isotropic if $r = n < m$; $S(TM^{\perp}) = \{0\}$;
Case 3: Isotropic if $r = m < n$; $S(TM) = \{0\}$;
Case 4: Totally lightlike if $r = m = n$; $S(TM) = \{0\} = S(TM^{\perp})$.

The Gauss and Weingarten equations are:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \forall X, Y \in \Gamma(TM), \tag{2.1}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V, \ \forall X \in \Gamma(TM), \ V \in \Gamma(\text{tr}(TM)), \tag{2.2}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X,Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(\text{ltr}(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $\text{ltr}(TM)$, respectively. The second fundamental form h is a symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(\text{tr}(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. Then we have

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.3}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X(N) + D^s(X,N), \tag{2.4}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X(W) + D^l(X,W), \ \forall X,Y \in \Gamma(TM), \tag{2.5}$$

$N \in \Gamma(\text{ltr}(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Denote the projection of TM on $S(TM)$ by $\bar{P}$. We set

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \tag{2.6}$$

$$\nabla_X \xi = -A^*_\xi X + {\nabla^*}^t_X \xi, \tag{2.7}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\text{Rad}\, TM)$.

In general, the induced connection ∇ on M is not metric connection. Since $\bar{\nabla}$ is a metric connection, by using (2.3) we get

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y). \tag{2.8}$$

Finally, we will give a brief review of the notion of lifting which is of crucial importance for computations on product manifolds, details can be found in [10]. Consider a product manifold $M \times N$. If $f \in C^\infty(M, \mathbf{R})$ the lift of f to $M \times N$ is $\tilde{f} = fo\pi \in C^\infty(M, \mathbf{R})$. If $x \in T_p(M), p \in M$ and $q \in N$ then the lift $\tilde{x}$ to (p,q) is the unique vector in $T_{(p,q)}M$ such that $\pi_*(\tilde{x}) = x$. If $X \in \Gamma(TM)$ the lift of X to $M \times N$ is the vector field $\tilde{X}$ whose value at each (p,q). Product coordinate systems show that $\tilde{X}$ is smooth. Let us denote vector fields on M (resp.N), lifted to $M \times N$, by $\Im(M)$ (resp. $\Im(N)$.) Then we have:

Lemma 2.1. [10] 1) *If* $\tilde{X}, \tilde{Y} \in \Im(M)$ *then* $[\tilde{X}, \tilde{Y}] = \widetilde{[X,Y]} \in \Im(M)$*, and similarly for* $\Im(N)$*.*
2) *If* $\tilde{X} \in \Im(M)$ *and* $\tilde{V} \in \Im(N)$*, then* $[\tilde{X}, \tilde{V}] = 0$.

Throughout this article we will use the same notation for a vector field and for its lift. Also we will assume that the manifolds are para-compact and every object in hand is smooth.

3. Warped product lightlike submanifolds

In this section we present a new class of lightlike submanifold of semi-Riemannian manifold and investigate the geometry of this class by using warped products.

Definition 1. *Let (M_1, g_1) be a totally lightlike submanifold of dimension r and (M_2, g_2) be a semi-Riemannian submanifold of dimension m of a semi-Riemann manifold $(\bar{M}, \bar{g})$. Then the product manifold $M = M_1 \times_f M_2$ is said to be a warped product lightlike submanifold of $\bar{M}$ with the degenerate metric g defined by*

$$g(X,Y) = g_1(\pi_* X, \pi_* Y) + (fo\pi)^2 g_2(\eta_* X, \eta_* Y) \tag{3.1}$$

for every $X, Y \in \Gamma(TM)$ and $*$ is the symbol for the tangent map. Here, $\pi : M_1 \times M_2 \to M_1$ and $\eta : M_1 \times M_2 \to M_2$ denote the projection maps given by $\pi(x,y) = x$ and $\eta(x,y) = y$ for $(x,y) \in M_1 \times M_2$.

It follows that the radical distribution $\mathrm{Rad}\,TM$ of M has rank r and its screen distribution $S(TM)$ has rank m. Thus M is an r-lightlike submanifold of $\bar{M}$. From now on we consider warped product lightlike submanifolds in the form $M_1 \times_f M_2$, where M_1 is a totally lightlike submanifold and M_2 is a semi-Riemann submanifold of $\bar{M}$. We say that M is a proper warped product lightlike submanifold if $M_1 \neq \{0\}$, $M_2 \neq \{0\}$ and f is non-constant on M.

Example 1. Let $\bar{M} = (\mathbf{R}^7_2, \bar{g})$ be a semi-Riemannian manifold, where $\mathbf{R}^7_2$ is semi-Euclidean space of signature $(-,-,+,+,+,+,+)$ with respect to the canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7\}.$$

Let M be a submanifold of $\mathbf{R}^7_2$ given by

$$x^1 = u^1, \qquad x^2 = u^2, \qquad x^3 = \frac{u^1}{\sqrt{2}} \sin u^3, \quad x^4 = \frac{u^1}{\sqrt{2}} \cos u^3$$
$$x^5 = \frac{u^1}{\sqrt{2}} sin\, u^4, \quad x^6 = \frac{u^1}{\sqrt{2}} \cos u^4, \quad x^7 = u^2,$$

where $u^3 \in \mathbf{R} - \{k\frac{\pi}{2}\}$ and $u^4 \in \mathbf{R} - \{k\pi, k \in Z\}$. Then TM is spanned by

$$Z_1 = \partial x_1 + \frac{1}{\sqrt{2}} \sin u^3\, \partial x_3 + \frac{1}{\sqrt{2}} \cos u^3\, \partial x_4 + \frac{1}{\sqrt{2}} \sin u^4\, \partial x_5 + \frac{1}{\sqrt{2}} \cos u^4\, \partial x_6$$
$$Z_2 = \partial x_2 + \partial x_7$$

$$Z_3 = \frac{1}{\sqrt{2}}u^1 \cos u^3\, \partial x_3 - \frac{1}{\sqrt{2}}u^1 \sin u^3 \partial\, x_4$$
$$Z_4 = \frac{1}{\sqrt{2}}u^1 \cos u^4\, \partial x_5 - \frac{1}{\sqrt{2}}u^1 \sin u^4\, \partial\, x_6$$

Thus M is $2-$ lightlike submanifold with $\operatorname{Rad} TM = \operatorname{Span}\{Z_1, Z_2\}$ Choose $S(TM) = \operatorname{Span}\{Z_3, Z_4\}$. Then a screen transversal bundle $S(TM^{\perp})$ is spanned by

$$W = \sin u^3\, \partial x_3 + \cos u^3\, \partial x_4 - \sin u^4\, \partial x_5 - \cos u^4\, \partial x_6,$$

and a lightlike transversal bundle $\operatorname{ltr}(TM)$ is spanned by

$$N_1 = \frac{1}{2\sqrt{2}}\{-\sqrt{2}\partial\, x_1 + \sin u^3\, \partial x_3 + \cos u^3\, \partial x_4 + \sin u^4\, \partial x_5 + \cos u^4\, \partial x_6\}$$
$$N_2 = \frac{1}{2}\{-\partial\, x_2 + \partial\, x_7\}.$$

It is easy to see that $\operatorname{Rad} TM$ and $S(TM)$ are integrable. Now, we denote the leaves of $\operatorname{Rad} TM$ and $S(TM)$ by M_1 and M_2, respectively. Then, the induced metric tensor of M is given by

$$\begin{aligned} ds^2 &= 0(du_1^2 + du_2^2) + \frac{(u^1)^2}{2}\,(du_3^2 + du_4^2) \\ &= \frac{(u^1)^2}{2}\,(du_3^2 + du_4^2). \end{aligned}$$

Hence M is a proper warped product lightlike submanifold $M_1 \times_f M_2$ with $f = \frac{u^1}{\sqrt{2}}$.

Proposition 3.1. *There exist no proper isotropic or totally lightlike warped product submanifolds of a semi-Riemann manifold $\bar{M}$.*

Proof. Let M be a isotropic warped product lightlike submanifold. Then $S(TM) = \{0\}$. Hence $M_2 = 0$. The other assertion can be proved in a similar way. □

Proposition 3.2. *Let $M = M_1 \times_f M_2$ be a proper warped product lightlike submanifold of a semi-Riemannian manifold $\bar{M}$ Then M_1 is totally geodesic in M as well as in $\bar{M}$.*

Proof. Let ∇ be a linear connection on M induced from $\bar{\nabla}$. We know that ∇ is not a metric connection. From the Kozsul formula we have

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_X Y, Z) = X\bar{g}(Y, Z) + Y\bar{g}(X, Z) - Z\bar{g}(X, Y) \\ + \bar{g}([X, Y], Z) + \bar{g}([Z, X], Y) - \bar{g}([Y, Z], X) \end{aligned}$$

for $X, Y \in \Gamma(TM_1)$ and $Z \in \Gamma(S(TM))$. On the other hand, from Lemma 2.1, we have $[X, Z] = 0$ for $X \in \Gamma(\mathrm{Rad}\, TM)$ and $Z \in \Gamma(S(TM))$. Thus we get

$$2\bar{g}(\bar{\nabla}_X Y, Z) = \bar{g}([X, Y], Z).$$

Using again Lemma 2.1, we get $[X, Y] \in \Gamma(\mathrm{Rad}\, TM)$. Hence we derive $2\bar{g}(\bar{\nabla}_X Y, Z) = 0$. Thus, from (2.3) we have $g(\nabla_X Y, Z) = 0$. this shows that M_1 is totally geodesic in M. On the other hand, from [7], Corollary 2.5, p. 167, we know that any totally lightlike submanifold of a semi-Riemann manifold $\bar{M}$ is totally geodesic in $\bar{M}$. □

Definition 2. [8] *A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $\mathbf{H} \in \Gamma(\mathrm{tr}(TM))$ on M, called the transversal curvature vector field of M, such that, for all $X, Y \in \Gamma(TM)$,*

$$h(X, Y) = \mathbf{H} g(X, Y). \tag{3.2}$$

Using (2.1) *and* (2.3) *it is easy to see that M is totally umbilical if and only if on each coordinate neighborhood $\mathcal{U}$ there exist smooth vector fields $H^l \in \Gamma(\mathrm{ltr}(TM))$ and $H^s \in \Gamma(S(TM^{\perp})$ such that*

$$h^l(X, Y) = H^l g(X, Y), \quad D^l(X, W) = 0, \tag{3.3}$$

$$h^s(X, Y) = H^s g(X, Y), \quad \forall X, Y \in \Gamma(TM), \quad W \in \Gamma(S(TM^{\perp}) \tag{3.4}$$

The above definition does not depend on the $S(TM)$ and $S(TM^{\perp})$ of M.

Corollary 3.1. *Let $M = M_1 \times_f M_2$ be a proper warped product lightlike submanifold of a semi-Riemann manifold $\bar{M}$. Then we have*

$$h^l(X, Z) = 0,\ h^*(X, Z) = 0,\ \forall X \in \Gamma(\mathrm{Rad}\, TM),\ Z \in \Gamma(S(TM)) \tag{3.5}$$

Proof. From equation (2.3) we have $\bar{g}(h^l(X, Z), Y) = \bar{g}(\bar{\nabla}_X Z, Y)$ for $X, Y \in \Gamma(\mathrm{Rad}\, TM)$ and $Z \in \Gamma(S(TM))$. Hence we have that $\bar{g}(h^l(X, Z), Y) = -\bar{g}(\bar{\nabla}_X Y, Z) = -g(\nabla_X Y, Z)$. From Proposition 3.2, we have known that M_1 is totally geodesic in M. Hence we get $\bar{g}(h^l(X, Z), Y) = 0$, thus we obtain the first equation of (3.5). In a similar way, we obtain the second equation. □

Proposition 3.3. *Let $M = M_1 \times_f M_2$ be a proper warped product lightlike submanifold of a semi-Riemann manifold $\bar{M}$. Then M is totally umbilical in $\bar{M}$ if and only if $h^s(X, Y) = g(X, Y) H^s$ for $X, Y \in \Gamma(TM)$, where H^s is a smooth vector field on coordinate neighborhood $\mathbf{U} \subset M$.*

Proof. First, we claim that $\nabla_X Z \in \Gamma(S(TM))$ for $X \in \Gamma(\mathrm{Rad}\, TM)$ and $Z \in \Gamma(S(TM))$. Let us suppose that $\nabla_X Z \in \Gamma(\mathrm{Rad}\, TM))$ for $X \in \Gamma(\mathrm{Rad}\, TM)$ and $Z \in \Gamma(S(TM))$. Then from Kozsul formula we have

$$2\bar{g}(\bar{\nabla}_X Z, W) = X\bar{g}(Z, W)$$

for $W \in \Gamma(S(TM))$. Since by assumption $\nabla_X Z \in \Gamma(\mathrm{Rad}\,TM)$ and the definition of warped metric tensor, using (2.3) we get

$$g(\nabla_X Z, W) = 0,$$

hence we derive

$$X(fo\pi)^2 g_2(Z, W) = 0.$$

Since g_2 is constant on M_1 we obtain

$$\frac{X(f)}{f} g(Z, W) = 0,$$

here we have put f for $fo\pi$. Thus $X(f) = 0$ or $g_2(Z, W) = 0$. Since g_2 is non-degenerate and f is not constant, we get a contradiction, so $\nabla_X Z \in \Gamma(S(TM))$. Now, since $\bar{\nabla}$ is a metric connection, we have

$$\bar{g}(\bar{\nabla}_Z W, X) = -\bar{g}(W, \bar{\nabla}_Z X)$$

for $X \in \Gamma(\mathrm{Rad}\,TM)$ and $Z, W \in \Gamma(S(TM))$. Using (2.3) we have

$$\bar{g}(h^l(Z, W), X) = -g(W, \nabla_Z X) = -g(W, \nabla_X Z).$$

Hence

$$\bar{g}(h^l(Z, W), X) = -\frac{X(f)}{f} g(Z, W). \tag{3.6}$$

Thus proof follows from (3.6), Corollary 3.1 and the definition of totally umbilical lightlike submanifold. □

Theorem 3.1. *Let $M = M_1 \times_f M_2$ be a proper warped product lightlike submanifold of a semi-Riemann manifold $\bar{M}$. Then the induced connection ∇ is never a metric connection.*

Proof. Let us suppose that ∇ is a metric connection on M. Then from [7] we know that $h^l = 0$. Thus from (3.6) we obtain $\frac{X(f)}{f} g(Z, W) = 0$ for $X \in \Gamma(\mathrm{Rad}\,TM)$ and $Z, W \in \Gamma(S(TM)$. Hence $X(f) f g_2(Z, W) = 0$. Thus, $f \neq 0$ implies that $X(f) = 0$ or $g_2(Z, W) = 0$. Since M is a proper warped product ligtlike submanifold and g_2 is non-degenerate,this is a contradiction. □

Corollary 3.2. *There exist no totally geodesic proper warped product lightlike submanifolds of a semi-Riemann manifold $\bar{M}$.*

We note that from Lemma 2.1, it follows that the radical distribution and the screen distribution of M are integrable. Now, we are ready to prove our main two results.

Theorem 3.2. *Let $M = M_1 \times_f M_2$ be a proper warped product lightlike submanifold of a semi-Riemann manifold $\bar{M}$. Then M is totally umbilical if any leaf of screen distribution is so immersed as a submanifold of $\bar{M}$*

and $\bar{g}(D^l(Z,W),X) = 0$ *for* $X \in \Gamma(\mathrm{Rad}\,TM), Z \in \Gamma(S(TM)$ *and* $W \in \Gamma(S(TM^\perp)$.

Proof. We note that from (3.1), we get $g(X,Y) = (fo\pi)^2 g_2(\bar{P}X, \bar{P}Y)$ for $X, Y \in \Gamma(TM)$ From Proposition 3.2 we know that $\mathrm{Rad}\,TM$ defines a totally geodesic foliation in $\bar{M}$, hence $h^l(X,Y) = h^s(X,Y) = 0$ for $X, Y \in \Gamma(\mathrm{Rad}\,TM)$. Moreover from Corollary 3.1, we have that $h^l(X,Z) = 0$ and $h^*(X,Z) = 0$ for $X \in \Gamma(\mathrm{Rad}\,TM)$ and $Z \in \Gamma(S(TM))$. Now, from (2.3) and (2.5) we obtain

$$\bar{g}(h^s(X,Z),W) = \bar{g}(X, D^l(Z,W)) \tag{3.7}$$

for $X \in \Gamma(\mathrm{Rad}\,TM)$, $Z \in \Gamma(S(TM)$ and $W \in \Gamma(S(TM^\perp)$. On the other hand, from (2.3) we write

$$\bar{\nabla}_Z V = \nabla'_Z V + h'(Z,V)$$

for $Z, V \in \Gamma(S(TM))$, where h' is the second fundamental form of M_2 in $\bar{M}$ and ∇' is the metric connection of M_2 in $\bar{M}$. Hence we have

$$h'(Z,V) = h^*(Z,V) + h^l(Z,V) + h^s(Z,V). \tag{3.8}$$

The proof follows from (3.7) and (3.8). □

Theorem 3.3. *Let* $M = M_1 \times_f M_2$ *be a coisotropic warped product lightlike submanifold of a semi-Riemann manifold* $\bar{M}$. *Then* M *is totally umbilical if any leaf of screen distribution is so immersed as a submanifold of* $\bar{M}$.

Proof. If M is coisotropic, then $S(TM^\perp) = \{0\}$. Thus, $h^s = 0$ on M. Then the proof follows from Proposition 3.2, Corollary 3.1 and (3.8). □

Example 2. Let $\bar{M} = (\mathbf{R}^6_3, \bar{g})$ be a semi-Riemannian manifold, where $\mathbf{R}^6_3$ is semi-Euclidean space of signature $(-,-,-,+,+,+)$ with respect to the canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6\}.$$

Let M be a submanifold of $\mathbf{R}^6_3$ given by

$$x^1 = u^1, \qquad x^2 = u^2, \qquad x^3 = u^1 \cos u^3 \sinh u^4$$
$$x^4 = u^1 \cos u^3 \cosh u^4, \qquad x^5 = u^2, \qquad x^6 = u^1\, sin\, u^3,$$

where $u^3 \in \mathbf{R} - \{k\frac{\pi}{2} \quad k \in Z\}$. Then TM is spanned by

$$Z_1 = \partial x_1 + \cos u^3 \sinh u^4 \,\partial x_3 + \cos u^3 \cosh u^4 \,\partial x_4 + \sin u^3 \,\partial x_6$$
$$Z_2 = \partial x_2 + \partial x_5$$
$$Z_3 = -u^1 \sin u^3 \sinh u^4 \,\partial x_3 - u^1 \sin u^3 \cosh u^4 \,\partial x_4 + u^1 \cos u^3 \,\partial x_6$$
$$Z_4 = u^1 \cos u^3 \cosh u^4 \,\partial x_3 + u^1 \cos u^3 \sinh u^4 \,\partial x_4.$$

Thus M is a $2-$ lightlike submanifold $\mathbf{R}^6_3$. Choose $S(TM) = \text{span}\{Z_3, Z_4\}$, then it follows that a lightlike transversal vector bundle $\text{ltr}(TM)$ is spanned by

$$N_1 = \frac{1}{2}\{-\partial x_1 + \cos u^3 \sinh u^4 \partial x_3 + \cos u^3 \cosh u^4 \partial x_4 + \sin u^3 \partial x_6\}$$
$$N_2 = \frac{1}{2}\{-\partial x_2 + \partial x_5\}$$

hence M is a coisotropic submanifold. By direct calculations, we conclude that $\text{Rad}\, TM$ and $S(TM)$ are integrable in M. Now denote the leaves of $\text{Rad}\, TM$ and $S(TM)$ by M_1 and M_2. We also obtain that the induced induced metric tensor is

$$ds^2 = (u^1)^2(du_3^2 - \cos^2 u^3\, du_4^2).$$

Thus M is a coisotropic warped product submanifold of $\mathbf{R}^6_3$ with $f = u^1$. By direct calculations, using Gauss formulas (2.3) and (2.6) we obtain

$$\begin{aligned} &h^l(X, Z_1) = h^l(X, Z_2) = h^l(Z_3, Z_4) = 0, \ \forall X \in \Gamma(TM) \\ &h^l(Z_3, Z_3) = -u^1 N_1, \ h^l(Z_4, Z_4) = u^1 \cos^2 u^3 N_1 \end{aligned} \tag{3.9}$$

and

$$h^*(X, Z_1) = h^*(X, Z_2) = h^*(Z_3, Z_4) = h^*(Z_4, Z_3) = 0, \ \forall X \in \Gamma(TM)$$
$$h^*(Z_3, Z_3) = -\frac{1}{2}u^1\, Z_1, \ h^*(Z_4, Z_4) = \frac{1}{2}\, u^1 \cos^2 u^3\, Z_1.$$

Now, we denote the second fundamental form of M_2 in $\bar{M}$ by h'. Then we obtain

$$h'(Z_3, Z_4) = 0$$

and

$$h'(Z_3, Z_3) = -u^1(N_1 + \frac{1}{2}Z_1), \, h'(Z_4, Z_4) = u^1 \cos^2 u^3(N_1 + \frac{1}{2}Z_1).$$

Hence we have

$$h'(X, Y) = g(X, Y)H',$$

where $H' = -N_1 - \frac{1}{2}Z_1$ for $X, Y \in \Gamma(S(TM))$. Thus, it follows that M_2 is totally umbilical in $\bar{M}$. On the other hand, from (3.9) we have

$$h^l(X, Y) = g(X, Y)H^l,$$

where $H^l = -N_1$ for $X, Y \in \Gamma(TM)$. Thus, from Definition 2, it follows that M is also totally umbilical in $\bar{M}$.

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