SARAJEVO JOURNAL OF MATHEMATICS
Vol.1 (14) (2005), 261–270

GENERAL COMMON FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS OF TYPE (C)

HAKIMA BOUHADJERA

Abstract. In this note, we prove a common fixed point theorem for four compatible mappings of type (C) satisfying an implicit relation. This theorem generalizes, improves and extends the result of Popa [6] and others.

1. Introduction

In 1986, G. Jungck [1] introduced more generalized commuting mappings, called compatible mappings, which are more general than commuting and weakly commuting maps. Several authors proved common fixed point theorems using the concept of compatible maps.

Further, G. Jungck, P. P. Murthy and Y. J. Cho [2] gave a generalization of compatible mappings called compatible mappings of type (A) which is equivalent to the concept of compatible maps under some conditions and proved a common fixed point theorem for compatible maps of type (A) in a metric space.

Extending type (A) mappings, H. K. Pathak and M. S. Khan [4] introduced the concept of compatible maps of type (B) and they gave some examples to show that compatible maps of type (B) need not be compatible of type (A).

Recently, in 1998, H. K. Pathak, Y. J. Cho, S. M. Kang, B. Madharia [3] introduced an other extension of compatible mappings of type (A) in normed spaces, called compatible mappings of type (C) and with some examples they compared these mappings with compatible maps, compatible maps of type (A) and compatible maps of type (B). Further they derived some relations between these mappings.

2000 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. Compatible and compatible mappings of type (A), (B) and (C), implicit relations, common fixed point theorem.

The subject of the present note is to prove a common fixed point theorem for compatible maps of type (C) satisfying an implicit relation. This theorem generalizes, improves and extends some results of Popa and others.

2. Preliminaries

Definition 2.1. *Let $\mathcal{S}$ and $\mathcal{T}$ be mappings from a metric space $(\mathcal{X}, d)$ into itself. The mappings $\mathcal{S}$ and $\mathcal{T}$ are said to be:*

(i) *compatible if*

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n) = 0,$$

(ii) *compatible of type (A) if*

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{T}x_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}\mathcal{S}x_n) = 0,$$

(iii) *compatible of type (B) if*

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{T}x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{S}t) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{S}\mathcal{S}x_n)\right]$$

and

$$\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}\mathcal{S}x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{T}t) + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{T}\mathcal{T}x_n)\right],$$

(iv) *compatible of type (C) if*

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{T}x_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{S}t) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{S}\mathcal{S}x_n) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{T}\mathcal{T}x_n)\right]$$

and

$$\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}\mathcal{S}x_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{T}t) + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{T}\mathcal{T}x_n) + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{S}\mathcal{S}x_n)\right]$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} \mathcal{S}x_n = \lim_{n\to\infty} \mathcal{T}x_n = t$ for some $t \in \mathcal{X}$.

It is clear to see that compatible maps of type (A) are compatible of type (C), but the converse is not true.

Moreover, as we shall show in the following example, there exist compatible maps of type (C) which are neither compatible nor compatible of type (A) (resp. compatible of type (B)).

Example 2.1. Let $\mathcal{X} = [1, 20]$ with the usual metric. Define

$$\mathcal{S}x = \begin{cases} 1 & \text{if } x = 1 \\ 3 & \text{if } 1 < x \leq 7 \\ x - 6 & \text{if } 7 < x \leq 20 \end{cases} \quad ; \quad \mathcal{T}x = \begin{cases} 1 & \text{if } x \in \{1\} \cup (7, 20] \\ 2 & \text{if } 1 < x \leq 7. \end{cases}$$

Let $\{x_n\}$ be the sequence defined by $x_n = 7 + \dfrac{1}{n}$, $n \in \mathbb{N}^*$. Then

$$\mathcal{S}x_n = x_n - 6 \to 1 = t; \ \ \mathcal{T}x_n = 1 \to 1 = t \ \text{ as } \ n \to \infty$$
$$\lim_{n\to\infty} |\mathcal{S}\mathcal{T}x_n - \mathcal{T}\mathcal{S}x_n| = 1 \neq 0$$

and it follows that $\mathcal{S}$ and $\mathcal{T}$ are not compatible. We have

$$\lim_{n\to\infty} |\mathcal{T}\mathcal{S}x_n - \mathcal{S}\mathcal{S}x_n| = 1 \neq 0$$

which tells us that $\mathcal{S}$ and $\mathcal{T}$ are not compatible of type (A). We also have

$$1 \not\leq \frac{1}{2}\left[\lim_{n\to\infty} |\mathcal{T}\mathcal{S}x_n - \mathcal{T}t| + \lim_{n\to\infty} |\mathcal{T}t - \mathcal{T}\mathcal{T}x_n|\right] = \frac{1}{2}$$

thus $\mathcal{S}$ and $\mathcal{T}$ are not compatible of type (B). Further,

$$0 \leq \frac{1}{3}\left[\lim_{n\to\infty} |\mathcal{S}\mathcal{T}x_n - \mathcal{S}t| + \lim_{n\to\infty} |\mathcal{S}t - \mathcal{S}\mathcal{S}x_n| + \lim_{n\to\infty} |\mathcal{S}t - \mathcal{T}\mathcal{T}x_n|\right] = \frac{2}{3}$$
$$1 \leq \frac{1}{3}\left[\lim_{n\to\infty} |\mathcal{T}\mathcal{S}x_n - \mathcal{T}t| + \lim_{n\to\infty} |\mathcal{T}t - \mathcal{T}\mathcal{T}x_n| + \lim_{n\to\infty} |\mathcal{T}t - \mathcal{S}\mathcal{S}x_n|\right] = 1$$

hence $\mathcal{S}$ and $\mathcal{T}$ are compatible of type (C).

The following proposition, whose proof is immediate, shows that definitions (i)∼(iv) are equivalent under some conditions:

Proposition 2.1. *Let $\mathcal{S}$ and $\mathcal{T}$ be continuous mappings from a metric space $(\mathcal{X}, d)$ into itself. Then the following are equivalent:*

(i) *$\mathcal{S}$ and $\mathcal{T}$ are compatible,*
(ii) *$\mathcal{S}$ and $\mathcal{T}$ are compatible of type (A),*
(iii) *$\mathcal{S}$ and $\mathcal{T}$ are compatible of type (B),*
(iv) *$\mathcal{S}$ and $\mathcal{T}$ are compatible of type (C).*

For our main result we need the following proposition for compatible mappings of type (C).

Proposition 2.2. *Let $\mathcal{S}$ and $\mathcal{T}$ be compatible mappings of type (C) from a metric space $(\mathcal{X}, d)$ into itself. Suppose that $\lim_{n\to\infty} \mathcal{S}x_n = \lim_{n\to\infty} \mathcal{T}x_n = t$ for some $t \in \mathcal{X}$. Then we have the following:*

(1) *$\lim_{n\to\infty} \mathcal{T}\mathcal{T}x_n = \mathcal{S}t$ if $\mathcal{S}$ is continuous at t,*
(2) *$\lim_{n\to\infty} \mathcal{S}\mathcal{S}x_n = \mathcal{T}t$ if $\mathcal{T}$ is continuous at t,*
(3) *$\mathcal{S}\mathcal{T}t = \mathcal{T}\mathcal{S}t$ and $\mathcal{S}t = \mathcal{T}t$ if $\mathcal{S}$ and $\mathcal{T}$ are continuous at t.*

3. Implicit relations

As in [5], we denote by $\mathcal{F}$ the set of all real continuous functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F_1) : F is non-increasing in the variables t_5 and t_6,

(F_2) : there exists $h \in (0,1)$ such that for every $u, v \geq 0$ with

(F_a) : $F(u,v,v,u,u+v,0) \leq 0$ or

(F_b) : $F(u,v,u,v,0,u+v) \leq 0$

we have $u \leq hv$,

(F_3) : $F(u,u,0,0,u,u) > 0, \ \forall u > 0$.

The next examples of functions in $\mathcal{F}$ are given in [5].

Example 3.1.
$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - k\max\left\{t_2,t_3,t_4,\frac{1}{2}(t_5+t_6)\right\}$,
where $k \in (0,1)$.

Example 3.2.
$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - c_1\max\left\{t_2^2,t_3^2,t_4^2\right\} - c_2\max\{t_3t_5,t_4t_6\} - c_3t_5t_6$,
where $c_1 > 0, c_2, c_3 \geq 0, c_1 + 2c_2 < 1$ and $c_1 + c_3 < 1$.

Example 3.3.
$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6$, where $a > 0, b, c, d \geq 0, a + b + c < 1$ and $a + d < 1$.

Example 3.4.
$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 - at_1^2t_2 - bt_1t_3t_4 - ct_5^2t_6 - dt_5t_6^2$,
where $a > 0, b, c, d \geq 0, a + b < 1$ and $a + c + d < 1$.

Example 3.5.
$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 - c\dfrac{t_3^2t_4^2 + t_5^2t_6^2}{t_2 + t_3 + t_4 + 1}$,
where $c \in (0,1)$.

Example 3.6.
$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - at_2^2 - \dfrac{bt_5t_6}{t_3^2 + t_4^2 + 1}$,
where $a > 0, b \geq 0$ and $a + b < 1$.

4. Main results

Now we state our first main result:

Theorem 4.1. *Let $\mathcal{S},\mathcal{T},\mathcal{I},\mathcal{J}$ be mappings from a complete metric space $(\mathcal{X},d)$ into itself satisfying the conditions*

(a) $\mathcal{S}(\mathcal{X}) \subset \mathcal{J}(\mathcal{X})$ *and* $\mathcal{T}(\mathcal{X}) \subset \mathcal{I}(\mathcal{X})$,

(b) *one of* $\mathcal{S},\mathcal{T},\mathcal{I},\mathcal{J}$ *is continuous,*

(c) *$\mathcal{S}$ and $\mathcal{I}$ as well as $\mathcal{T}$ and $\mathcal{J}$ are compatible of type (C),*

(d) *the inequality*

$$F(d(\mathcal{S}x,\mathcal{T}y),d(\mathcal{I}x,\mathcal{J}y),d(\mathcal{I}x,\mathcal{S}x),d(\mathcal{J}y,\mathcal{T}y),d(\mathcal{I}x,\mathcal{T}y),d(\mathcal{J}y,\mathcal{S}x))\le 0$$

holds for all x,y in $\mathcal{X}$, where $F\in\mathcal{F}$. Then $\mathcal{S},\mathcal{T},\mathcal{I}$ and $\mathcal{J}$ have an unique common fixed point.

Proof. Suppose x_0 is an arbitrary point in $\mathcal{X}$. Then, since (a) holds, we can define inductively a sequence

$$(e)\quad \{\mathcal{S}x_0,\mathcal{T}x_1,\mathcal{S}x_2,\mathcal{T}x_3,\dots,\mathcal{S}x_{2n},\mathcal{T}x_{2n+1},\dots\}$$

such that

$$y_{2n}=\mathcal{S}x_{2n}=\mathcal{J}x_{2n+1},y_{2n+1}=\mathcal{T}x_{2n+1}=\mathcal{I}x_{2n+2},\ \text{for } n\in\mathbb{N}.$$

Using inequality (d), we have successively

$$\begin{aligned}F(d(\mathcal{S}x_{2n},\mathcal{T}x_{2n+1}),d(\mathcal{I}x_{2n},\mathcal{J}x_{2n+1}),d(\mathcal{I}x_{2n},\mathcal{S}x_{2n}),\\ d(\mathcal{J}x_{2n+1},\mathcal{T}x_{2n+1}),d(\mathcal{I}x_{2n},\mathcal{T}x_{2n+1}),d(\mathcal{J}x_{2n+1},\mathcal{S}x_{2n}))\\ =F(d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n}),\\ d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n+1}),0)\le 0.\end{aligned}$$

By condition (F_1), we have

$$\begin{aligned}F(d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n}),\\ d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1}),0)\le 0.\end{aligned}$$

So we obtain, by (F_a),

$$d(y_{2n},y_{2n+1})\le h\,d(y_{2n-1},y_{2n}).$$

Similarly, by (F_1) and (F_b), one may get

$$d(y_{2n-1},y_{2n})\le h\,d(y_{2n-2},y_{2n-1})$$

therefore, we have by induction

$$d(y_{2n},y_{2n+1})\le h^{2n}d(y_0,y_1)$$

for $n\in\mathbb{N}$. By a routine calculation it follows that (e) is a Cauchy sequence. Since $\mathcal{X}$ is complete, sequence (e) converges, as its subsequences $\{\mathcal{S}x_{2n}\}=\{\mathcal{J}x_{2n+1}\},\{\mathcal{T}x_{2n-1}\}=\{\mathcal{I}x_{2n}\}$ do, to some element z in $\mathcal{X}$.

Let us now suppose that $\mathcal{I}$ is continuous, so that the sequence $\{\mathcal{IS}x_{2n}\}$ converges to $\mathcal{I}z$. Since $\mathcal{I}$ is continuous and $\mathcal{S}$ and $\mathcal{I}$ are compatible of type (C), by condition (2) of Proposition 2.2, we have the sequence $\{\mathcal{SS}x_{2n}\}$ also converges to $\mathcal{I}z$. Using estimation (d), we have

$$\begin{aligned}F(d(\mathcal{SS}x_{2n},\mathcal{T}x_{2n+1}),d(\mathcal{IS}x_{2n},\mathcal{J}x_{2n+1}),d(\mathcal{IS}x_{2n},\mathcal{SS}x_{2n}),\\ d(\mathcal{J}x_{2n+1},\mathcal{T}x_{2n+1}),d(\mathcal{IS}x_{2n},\mathcal{T}x_{2n+1}),d(\mathcal{J}x_{2n+1},\mathcal{SS}x_{2n}))\le 0.\end{aligned}$$

Letting $n \to \infty$ we have, by the continuity of F,

$$F(d(\mathcal{I}z, z), d(\mathcal{I}z, z), 0, 0, d(\mathcal{I}z, z), d(z, \mathcal{I}z)) \leq 0,$$

a contradiction of (F_3), if $d(\mathcal{I}z, z) \neq 0$. Thus $\mathcal{I}z = z$. Further, by (d), we have

$$F(d(\mathcal{S}z, \mathcal{T}x_{2n+1}), d(\mathcal{I}z, \mathcal{J}x_{2n+1}), d(\mathcal{I}z, \mathcal{S}z), \\ d(\mathcal{J}x_{2n+1}, \mathcal{T}x_{2n+1}), d(\mathcal{I}z, \mathcal{T}x_{2n+1}), d(\mathcal{J}x_{2n+1}, \mathcal{S}z)) \leq 0$$

consequently, we obtain at infinity

$$F(d(\mathcal{S}z, z), 0, d(\mathcal{S}z, z), 0, 0, d(\mathcal{S}z, z)) \leq 0$$

which implies, by (F_b), that $\mathcal{S}z = z$. This means that z is in the range of $\mathcal{S}$ and, since $\mathcal{S}(\mathcal{X}) \subset \mathcal{J}(\mathcal{X})$, there exists a point u in $\mathcal{X}$ such that $\mathcal{J}u = z$. Again using condition (d), we have successively

$$F(d(\mathcal{S}z, \mathcal{T}u), d(\mathcal{I}z, \mathcal{J}u), d(\mathcal{I}z, \mathcal{S}z), d(\mathcal{J}u, \mathcal{T}u), d(\mathcal{I}z, \mathcal{T}u), d(\mathcal{J}u, \mathcal{S}z)) \\ = F(d(z, \mathcal{T}u), 0, 0, d(z, \mathcal{T}u), d(z, \mathcal{T}u), 0) \leq 0$$

which implies by (F_a), that $z = \mathcal{T}u$. Since $\mathcal{J}$ and $\mathcal{T}$ are compatible of type (C) and $\mathcal{J}u = \mathcal{T}u = z$, by (3) of Proposition 2.1, $\mathcal{J}\mathcal{T}u = \mathcal{T}\mathcal{J}u$ and hence $\mathcal{J}z = \mathcal{J}\mathcal{T}u = \mathcal{T}\mathcal{J}u = \mathcal{T}z$. Moreover, by inequality (d), we may get

$$F(d(\mathcal{S}z, \mathcal{T}z), d(\mathcal{I}z, \mathcal{J}z), d(\mathcal{I}z, \mathcal{S}z), d(\mathcal{J}z, \mathcal{T}z), d(\mathcal{I}z, \mathcal{T}z), d(\mathcal{J}z, \mathcal{S}z)) \\ = F(d(z, \mathcal{T}z), d(z, \mathcal{T}z), 0, 0, d(z, \mathcal{T}z), d(\mathcal{T}z, z)) \leq 0,$$

contradicts (F_3), if $z \neq \mathcal{T}z$. Thus $z = \mathcal{T}z = \mathcal{J}z$. We have therefore proved that z is a common fixed point of $\mathcal{S}, \mathcal{T}, \mathcal{I}$ and $\mathcal{J}$. The same result holds, if we assume that $\mathcal{J}$ is continuous instead of $\mathcal{I}$.

Now suppose that $\mathcal{S}$ is continuous. Then the sequence $\{\mathcal{S}\mathcal{I}x_{2n}\}$ converges to $\mathcal{S}z$. Since $\mathcal{S}$ is continuous and $\mathcal{S}$ and $\mathcal{I}$ are compatible of type (C), condition (2) of Proposition 2.2, implies that the sequence $\{\mathcal{I}\mathcal{I}x_{2n}\}$ also converges to $\mathcal{S}z$. Using inequality (d), we have

$$F(d(\mathcal{S}\mathcal{I}x_{2n}, \mathcal{T}x_{2n+1}), d(\mathcal{I}\mathcal{I}x_{2n}, \mathcal{J}x_{2n+1}), d(\mathcal{I}\mathcal{I}x_{2n}, \mathcal{S}\mathcal{I}x_{2n}), \\ d(\mathcal{J}x_{2n+1}, \mathcal{T}x_{2n+1}), d(\mathcal{I}\mathcal{I}x_{2n}, \mathcal{T}x_{2n+1}), d(\mathcal{J}x_{2n+1}, \mathcal{S}\mathcal{I}x_{2n})) \leq 0.$$

Since F is continuous, then by letting $n \to \infty$, we obtain

$$F(d(\mathcal{S}z, z), d(\mathcal{S}z, z), 0, 0, d(\mathcal{S}z, z), d(z, \mathcal{S}z)) \leq 0,$$

which contradicts (F_3) if $z \neq \mathcal{S}z$. Thus $z = \mathcal{S}z$, which means that z is in the range of $\mathcal{S}$ and, since $\mathcal{S}(\mathcal{X}) \subset \mathcal{J}(\mathcal{X})$, there exists a point v in $\mathcal{X}$ such that $\mathcal{J}v = z$. Moreover, by (d), we can estimate

$$F(d(\mathcal{S}\mathcal{I}x_{2n}, \mathcal{T}v), d(\mathcal{I}\mathcal{I}x_{2n}, \mathcal{J}v), d(\mathcal{I}\mathcal{I}x_{2n}, \mathcal{S}\mathcal{I}x_{2n}), \\ d(\mathcal{J}v, \mathcal{T}v), d(\mathcal{I}\mathcal{I}x_{2n}, \mathcal{T}v), d(\mathcal{J}v, \mathcal{S}\mathcal{I}x_{2n})) \leq 0.$$

Therefore, by letting $n \to \infty$ we get the following,

$$F(d(z, \mathcal{T}v), 0, 0, d(z, \mathcal{T}v), d(z, \mathcal{T}v), 0) \leq 0$$

and, by (F_a), it follows that $z = \mathcal{T}v$. Since $\mathcal{J}$ and $\mathcal{T}$ are compatible of type (C) and $\mathcal{J}v = z = \mathcal{T}v$, it follows from (3) of Proposition 2.2 that, $\mathcal{J}\mathcal{T}v = \mathcal{T}\mathcal{J}v$ and hence $\mathcal{J}z = \mathcal{J}\mathcal{T}v = \mathcal{T}\mathcal{J}v = \mathcal{T}z$. Moreover, by assumption (d), we have

$$F(d(\mathcal{S}x_{2n}, \mathcal{T}z), d(\mathcal{I}x_{2n}, \mathcal{J}z), d(\mathcal{I}x_{2n}, \mathcal{S}x_{2n}), \\ d(\mathcal{J}z, \mathcal{T}z), d(\mathcal{I}x_{2n}, \mathcal{T}z), d(\mathcal{J}z, \mathcal{S}x_{2n})) \leq 0.$$

Taking the limit as $n \to \infty$ yields

$$F(d(z, \mathcal{T}z), d(z, \mathcal{T}z), 0, 0, d(z, \mathcal{T}z), d(\mathcal{T}z, z)) \leq 0$$

and, by (F_3), it follows that $z = \mathcal{T}z = \mathcal{J}z$. This means that z is in the range of $\mathcal{T}$ and, since $\mathcal{T}(\mathcal{X}) \subset \mathcal{I}(\mathcal{X})$ there exists $w \in \mathcal{X}$ such that $\mathcal{I}w = z$. Furthermore, estimation (d) gives

$$F(d(\mathcal{S}w, \mathcal{T}z), d(\mathcal{I}w, \mathcal{J}z), d(\mathcal{I}w, \mathcal{S}w), d(\mathcal{J}z, \mathcal{T}z), d(\mathcal{I}w, \mathcal{T}z), d(\mathcal{J}z, \mathcal{S}w)) \\ = F(d(\mathcal{S}w, z), 0, d(z, \mathcal{S}w), 0, 0, d(z, \mathcal{S}w)) \leq 0$$

and, by (F_b), we have $z = \mathcal{S}w = \mathcal{I}w$. Since $\mathcal{I}$ and $\mathcal{S}$ are compatible of type (C) and $\mathcal{S}w = \mathcal{I}w = z$, by (3) of Proposition 2.1, $\mathcal{S}\mathcal{I}w = \mathcal{I}\mathcal{S}w$ and hence $\mathcal{S}z = \mathcal{S}\mathcal{I}w = \mathcal{I}\mathcal{S}w = \mathcal{I}z$ and thus $z = \mathcal{I}z$. We have therefore proved that z is a common fixed point of $\mathcal{S}, \mathcal{T}, \mathcal{I}$ and $\mathcal{J}$. The same result holds, if we assume that $\mathcal{T}$ is continuous instead of $\mathcal{S}$.

Now, suppose that $\mathcal{S}, \mathcal{T}, \mathcal{I}$ and $\mathcal{J}$ have another common fixed point $z' \neq z$. Then by inequality (d) we have

$$F(d(\mathcal{S}z, \mathcal{T}z'), d(\mathcal{I}z, \mathcal{J}z'), d(\mathcal{I}z, \mathcal{S}z), d(\mathcal{J}z', \mathcal{T}z'), d(\mathcal{I}z, \mathcal{T}z'), d(\mathcal{J}z', \mathcal{S}z)) \\ = F(d(z, z'), d(z, z'), 0, 0, d(z, z'), d(z', z)) \leq 0,$$

which contradicts (F_3) if $z' \neq z$. Hence $z' = z$. Therefore z is the unique common fixed point of $\mathcal{S}, \mathcal{T}, \mathcal{I}$ and $\mathcal{J}$. □

Truly, Theorem 4.1 generalizes the work of Popa [6] and others. Indeed, by replacing the function F of our theorem by any function satisfying the conditions $(F_1), (F_2)$ and (F_3) we can obtain several corollaries. That is:

Corollary 4.1. *Let $\mathcal{S},\mathcal{T},\mathcal{I}$ and $\mathcal{J}$ be as in Theorem* 4.1. *If the inequality*

$$d(\mathcal{S}x,\mathcal{T}y)\le k\max\Big\{d(\mathcal{I}x,\mathcal{J}y),d(\mathcal{I}x,\mathcal{S}x),d(\mathcal{J}y,\mathcal{T}y),\\ \frac{1}{2}(d(\mathcal{I}x,\mathcal{T}y)+d(\mathcal{J}y,\mathcal{S}x))\Big\}$$

holds for all $x,y\in\mathcal{X}$, where $k\in(0,1)$, then $\mathcal{S},\mathcal{T},\mathcal{I}$ and $\mathcal{J}$ have an unique common fixed point.

Proof. It follows from Theorem 4.1 and Example 3.1. □

Corollary 4.2. *Let $\mathcal{S},\mathcal{T},\mathcal{I}$ and $\mathcal{J}$ be as in Theorem* 4.1. *If the inequality*

$$\begin{aligned} d^2(\mathcal{S}x,\mathcal{T}y)\le c_1\max\big\{&d^2(\mathcal{I}x,\mathcal{J}y),d^2(\mathcal{I}x,\mathcal{S}x),d^2(\mathcal{J}y,\mathcal{T}y)\big\}\\ &+c_2\max\{d(\mathcal{I}x,\mathcal{S}x)d(\mathcal{I}x,\mathcal{T}y),d(\mathcal{J}y,\mathcal{T}y)d(\mathcal{J}y,\mathcal{S}x))\}\\ &+c_3\,d(\mathcal{I}x,\mathcal{T}y)d(\mathcal{J}y,\mathcal{S}x)\end{aligned}$$

holds for all $x,y\in\mathcal{X}$, where $c_1>0$, $c_2,c_3\ge 0$, $c_1+2c_2<1$ and $c_1+c_3<1$, then $\mathcal{S},\mathcal{T},\mathcal{I}$ and $\mathcal{J}$ have an unique common fixed point.

Proof. Use Theorem 4.1 and Example 3.2. □

In a similar way as in Corollaries 4.1 and 4.2, one can obtain several corollaries using the examples given above.

Remark. If we put in Theorem 4.1 and its Corollaries $\mathcal{I}=\mathcal{J}=\mathcal{I}_{\mathcal{X}}$ (the identity mapping on $\mathcal{X}$) and also $\mathcal{S}=\mathcal{T}$ and $\mathcal{I}=\mathcal{J}=\mathcal{I}_{\mathcal{X}}$ then we can get additional corollaries.

Now, we give an example in support our result.

Example 4.1. Let $\mathcal{X}=[0,\infty)$ with the usual metric d and define $\mathcal{I},\mathcal{J},\mathcal{S}$ and $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ by

$$\mathcal{I}x=x^2,\qquad \mathcal{S}x=\begin{cases}3 & \text{if } x\in[0,1)\\ x & \text{if } x\in[1,\infty)\end{cases},$$

$$\mathcal{J}x=\begin{cases}0 & \text{if } x\in[0,1)\\ x^4 & \text{if } x\in[1,\infty)\end{cases},\qquad \mathcal{T}x=\begin{cases}3 & \text{if } x\in[0,1)\\ x^2 & \text{if } x\in[1,\infty)\end{cases}.$$

Clearly, $\mathcal{S}(\mathcal{X})=[1,\infty)\subset\mathcal{J}(\mathcal{X})=\{0\}\cup[1,\infty)$ and $\mathcal{T}(\mathcal{X})=[1,\infty)\subset\mathcal{I}(\mathcal{X})=[0,\infty)=\mathcal{X}$ and only $\mathcal{I}$ is continuous at 1. Further, let $\{x_n\}$ be the sequence in $\mathcal{X}$ defined by $x_n=1+\dfrac{1}{n}$ for all $n\in\mathbb{N}^*$. Then

$$\lim_{n\to\infty}\mathcal{I}x_n=\lim_{n\to\infty}x_n^2=1\quad\text{and}\quad\lim_{n\to\infty}\mathcal{S}x_n=\lim_{n\to\infty}x_n=1.$$

Consequently,

$$\begin{aligned} 0 &= \lim_{n\to\infty} |\mathcal{S}\mathcal{I}x_n - \mathcal{I}\mathcal{I}x_n| \\ &\le \frac{1}{3}\left[\lim_{n\to\infty} |\mathcal{S}\mathcal{I}x_n - \mathcal{S}(1)| + \lim_{n\to\infty} |\mathcal{S}(1) - \mathcal{S}\mathcal{S}x_n| + \lim_{n\to\infty} |\mathcal{S}(1) - \mathcal{I}\mathcal{I}x_n|\right] = 0 \end{aligned}$$

and

$$\begin{aligned} 0 &= \lim_{n\to\infty} |\mathcal{I}\mathcal{S}x_n - \mathcal{S}\mathcal{S}x_n| \\ &\le \frac{1}{3}\left[\lim_{n\to\infty} |\mathcal{I}\mathcal{S}x_n - \mathcal{I}(1)| + \lim_{n\to\infty} |\mathcal{I}(1) - \mathcal{I}\mathcal{I}x_n| + \lim_{n\to\infty} |\mathcal{I}(1) - \mathcal{S}\mathcal{S}x_n|\right] = 0. \end{aligned}$$

Thus, $\mathcal{I}$ and $\mathcal{S}$ are compatible of type (C) Similarly, since $x_n \to 1$ as $n \to \infty$, then

$$\lim_{n\to\infty} \mathcal{J}x_n = \lim_{n\to\infty} x_n^4 = 1 \quad \text{and} \quad \lim_{n\to\infty} \mathcal{T}x_n = \lim_{n\to\infty} x_n^2 = 1$$

and so

$$\begin{aligned} 0 = \lim_{n\to\infty} |\mathcal{J}\mathcal{T}x_n - \mathcal{T}\mathcal{T}x_n| \le \frac{1}{3}\Big[&\lim_{n\to\infty} |\mathcal{J}\mathcal{T}x_n - \mathcal{J}(1)| \\ &+ \lim_{n\to\infty} |\mathcal{J}(1) - \mathcal{J}\mathcal{J}x_n| + \lim_{n\to\infty} |\mathcal{J}(1) - \mathcal{T}\mathcal{T}x_n|\Big] = 0 \end{aligned}$$

and

$$\begin{aligned} 0 = \lim_{n\to\infty} |\mathcal{T}\mathcal{J}x_n - \mathcal{J}\mathcal{J}x_n| \le \frac{1}{3}\Big[&\lim_{n\to\infty} |\mathcal{T}\mathcal{J}x_n - \mathcal{T}(1)| \\ &+ \lim_{n\to\infty} |\mathcal{T}(1) - \mathcal{T}\mathcal{T}x_n| + \lim_{n\to\infty} |\mathcal{T}(1) - \mathcal{J}\mathcal{J}x_n|\Big] = 0. \end{aligned}$$

Hence $\mathcal{J}$ and $\mathcal{T}$ are also compatible of type (C). Now, define $F : \mathbb{R}_+^6 \to \mathbb{R}$ by the expression $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \frac{1}{2}t_2$. Clearly, $F \in \mathcal{F}$. Moreover, we have

$$\begin{aligned} d(\mathcal{S}x, \mathcal{T}y) &= \left|x - y^2\right| \\ &\le \left|x - y^2\right| \left|x + y^2\right| = \left|x^2 - y^4\right| \\ &= d(\mathcal{I}x, \mathcal{J}y), \end{aligned}$$

for all $x, y \ge 1$. Then, F satisfies condition (d). Obviously all hypotheses of Theorem 4.1 are satisfied and the point 1 is the unique common fixed point of the mappings $\mathcal{I}, \mathcal{J}, \mathcal{S}$ and $\mathcal{T}$.

As in [5], for a mapping $f : \mathcal{X} \to \mathcal{X}$ we denote

$$F_f = \{x \in \mathcal{X} : x = f(x)\}.$$

Theorem 4.2. [5] *Let $\mathcal{I}, \mathcal{J}, \mathcal{S}$ and $\mathcal{T}$ be mappings from a metric space $(\mathcal{X}, d)$ into itself. If inequality (d) holds for all x, y in $\mathcal{X}$ then*

$$(F_{\mathcal{I}} \cap F_{\mathcal{J}}) \cap F_{\mathcal{S}} = (F_{\mathcal{I}} \cap F_{\mathcal{J}}) \cap F_{\mathcal{T}}.$$

Now we present our second main result which is a generalization of Theorem 4.1.

Theorem 4.3. *Let $\mathcal{I}, \mathcal{J}$ and $\{\mathcal{T}_i\}_{i \in \mathbb{N}^*}$ be mappings from a complete metric space $(\mathcal{X}, d)$ into itself such that*

(i) $\mathcal{T}_1(\mathcal{X}) \subset \mathcal{J}(\mathcal{X})$ *and* $\mathcal{T}_2(\mathcal{X}) \subset \mathcal{I}(\mathcal{X})$,
(ii) *one of* $\mathcal{I}, \mathcal{J}, \mathcal{T}_1$ *and* $\mathcal{T}_2$ *is continuous,*
(iii) *the pairs* $\{\mathcal{T}_1, \mathcal{I}\}$ *and* $\{\mathcal{T}_2, \mathcal{J}\}$ *are compatible of type* (C),
(iv) *the inequality*

$$F(d(\mathcal{T}_i x, \mathcal{T}_{i+1} y), d(\mathcal{I}x, \mathcal{J}y), d(\mathcal{I}x, \mathcal{T}_i x), d(\mathcal{J}y, \mathcal{T}_{i+1} y), d(\mathcal{I}x, \mathcal{T}_{i+1} y), d(\mathcal{J}y, \mathcal{T}_i x)) \leq 0$$

holds for all x, y in $\mathcal{X}, \forall i \in \mathbb{N}^*$ and $F \in \mathcal{F}$. Then $\mathcal{I}, \mathcal{J}$ and $\{\mathcal{T}_i\}_{i \in \mathbb{N}^*}$ have a unique common fixed point.

Proof. It follows from Theorems 4.1 and 4.2. □

Acknowledgement. The author wishes to thank the referee for many valuable suggestions.

References

[1] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci., 9 (1986), 771–779.

[2] G. Jungck, P. P. Murthy and Y. J. Cho, *Compatible mappings of type (A) and common fixed points*, Math. Japon., 38 (1993), 381–390.

[3] H. K. Pathak, Y. J. Cho, S. M. Kang and B. Madharia, *Compatible mappings of type (C) and common fixed point theorems of Greguš type*, Demonstr. Math., 31 (3) 1998, 499–517.

[4] H. K. Pathak and M. S. Khan, *Compatible mappings of type (B) and common fixed point theorems of Greguš type*, Czechoslovak Math. J., 45 (120) (1995), 685–698.

[5] V. Popa, *Some fixed point theorems for compatible mappings satisfying an implicit relation*, Demonstr. Math., 32 (1) 1999, 157–163.

[6] V. Popa, *Common fixed point theorems for compatible mappings of type (A) satisfying an implicit relation*, Stud. Cercet. Ştiinţ. Ser. Mat. Univ. Bacău, 9 (1999), 165–172.

(Received: April 20, 2005)
(Revised: August 2, 2005)

Université de Annaba
Faculté des sciences
Département de mathématiques
B. P. 12, 23000 Annaba, Algerie
E–mail: b_hakima2000@yahoo.fr