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FIXED POINT PROPERTIES OF DECOMPOSABLE ISOTONE OPERATORS IN POSETS

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Abstract. A known theorem of R.M. Dacić, involving increasing operators decomposable into a finite product of monotone mappings, is extended from a complete lattice to a poset by using our previous results.

1. Introduction

Let $(P, \leq)$ be a poset and f be a selfmap of P. We recall in Klimes [5] that the ordered pair $(x, y) \in P^2$ is called a fixed edge for f if and only if $x \leq y$ implies $f(x) = y$ and $f(y) = x$. Modifying slightly the notation of Dacić [2], we denote by $I_P(f)$ the set of fixed points of f in P and by $E_p(f)$ the set of the fixed edges for f in P. We define f to be antitone (resp., isotone) in P if for all $x, y \in P, x \leq y$ implies $f(x) \geq f(y)$ (resp., $f(x) \leq f(y)$) and moreover we say that f is monotone in P if it is antitone or isotone in P. In the sequel we also shall use the set $P(f) = \{x \in P \colon x \leq f(x)\}$ and let $I_n = \{1, \ldots, n\}$ be the set of the n first natural numbers. Further, we say that $F = \{f_i : i \in I_n\}$ is a *commutative family* of selfmaps of P if $f_i \cdot f_j(x) = f_j \cdot f_i(x)$ for all $i, j \in I_n$ and $x \in P$.

The literature contains results on the existence of fixed edges for antitone selfmaps of P (see, e.g., [1], [2], [4], [5]). In particular, by considering decomposable isotone operators, Dacić [2] proved the following theorem.

Theorem 1. *Let $(P, \leq)$ be a complete lattice and $F = \{f_i : i \in I_{2n}\}$ be a commutative family of monotone selfmaps of P. Let $f = f_1 \cdot f_2 \cdots f_{2n}$ be isotone in P and suppose that $P(f) \neq \emptyset$. Then $P(f_i) \neq \emptyset$ for every $i \in I_{2n}, \cap I_P(f_i) \neq \emptyset$ for all $f_i \in F$ isotone and $\cap E_P(f_i) \neq \emptyset$ for all $f_i \in F$ antitone.*

Our aim is to extend Theorem 1 to a result for posets, without necessarily assuming that P is a complete lattice. The following Theorem 2 [3], whose proof (based on Zorn's lemma) is given in [4], shall play a crucial role in the proof of the main result.

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Theorem 2. *Let $(P, \leq)$ be a poset and F be a commutative family of isotone selfmaps of P satisfying the following properties:*

(1') *There exists an element $x_0 \in P$ such that $x_0 \geq f(x_0)$ for every $f \in F$,*

(2') *If C is a chain of $(P, \leq)$ not having an infimum in P, then there exists an $h \in F$ such that $h(C)$ has an infimum.*

Then there exists an element $m_0 \in P$ such that $m_0 = \max\{m \colon m \in \cap_{f \in F} I_P(f) \cap P^-(x_0)\}$, where $P^-(x_0) = \{x \in P \colon x \leq x_0\}$.

2. Main theorem

In order to establish our main result, we make use of the following Lemma of Dacić [2].

Lemma. *Let $P \neq \emptyset$ be a poset and $\{f_i \colon i \in I_n\}$ be a commutative family of selfmaps of P. Assume that $f = f_1 \cdot f_2 \cdots f_n$. Then $f_k(I_P(f)) \subseteq I_P(f)$ for every $k \in I_n$.*

We now prove our main theorem.

Theorem 3. *Let $(P, \leq)$ be a poset and $F = \{f_i : i \in I_n\}$ be a commutative family of monotone selfmaps of P. Let $f = f_1 \cdot f_2 \cdots f_n$ be isotone in P and suppose that:*

(1") *There exists an element $x_0 \in P$ such that $x_0 \geq f_i(x)$ for every $i \in I_n$ and $x \in P^-(x_0)$,*

(2") *If C is a chain of $(P, \leq)$ not having an infimum in P, then $f(C)$ has an infimum.*

Then $P(f_i) \neq \emptyset$ for every $i \in I_n$, $\cap I_P(f_i) \neq \emptyset$ for all $f_i \in F$ isotone and $\cap E_P(f_i) \neq \emptyset$ for all $f_i \in F$ antitone.

Proof. Since $f_i \cdot f_j = f_j \cdot f_i$ for all $i, j \in I_n$, we can write $f = f_1 \cdot f_2 \cdots f_k \cdot f_{k+1} \cdot f_{k+2} \cdots f_n$, where all $f_1, f_2, \ldots, f_k$ are antitone and all $f_{k+1}, f_{k+2}, \ldots, f_n$ are isotone. By property (1"), we observe that $f_i(x) \in P^-(x_0)$ for every $i \in I_n$ and $x \in P^-(x_0)$. Thus property (1") implies also that $x_0 \geq f_i(f_j(x))$ for every $i, j \in I_n$ and $x \in P^-(x_0)$, hence we deduce that $x_0 \geq f_1(f_2(\ldots(f_n(x))\ldots)) = f(x)$ for every $x \in P^-(x_0)$. In particular, we have that $x_0 \geq f_i(x_0)$ for every $i \in \{k+1, k+2, \ldots, n\}$, $x_0 \geq f_i(f_j(x_0))$ for every $i, j \in I_k$ and $x_0 \geq f(x_0)$. Furthermore, we have that $f_i \cdot f = f \cdot f_i$ for all $i \in I_n$. Since property (2") holds, we obtain that $F = \{f, f_{k+1}, f_{k+2}, \cdots, f_n, \{f_i \cdot f_j : i, j \in I_k\}\}$ is a commutative family of isotone selfmaps of P satisfying properties (1') and (2') of Theorem 2. Hence there exists an element $m_0 \in P$ such that $m_0 = \max\{m \colon m \in \cap_{g \in F} I_P(g) \cap P^-(x_0)\}$. Clearly $m_0 \in \cap_{i=k+1,\ldots,n} I_P(f_i) \subseteq \cap_{i=k+1,\ldots,n} P(f_i)$.

We note that $f_j(m_0) = f_j(f_i(f_j(m_0))) = f_i(f_j(f_j(m_0))) = f_i(m_0)$ for every $i, j \in I_k$ and then we can put $z_0 = f_i(m_0)$ for every $i \in I_k$. It follows that $z_0 \in \cap\{I_P(f_i \cdot f_j) : i, j \in I_k\}$ and $z_0 \in I_P(f)$ by the above Lemma. Furthermore we have $f_h(z_0) = f_h(f_j(m_0)) = f_j(f_h(m_0)) = f_j(m_0) = z_0$ for every $h \in \{k+1, k+2, \ldots, n\}$, that is $z_0 \in \cap_{h=k+1,\ldots,n} I_P(f_h)$. Since property (1") gives $z_0 = f_i(m_0) \in P^-(x_0)$, then we deduce that $z_0 \leq m_0$, hence $(z_0, m_0) \in \cap_{j=1,\ldots,k} E_P(f_j)$. Finally, we have $f_j(z_0) \geq f_j(m_0) = z_0$ for every $j \in I_k$ and therefore $z_0 \in \cap_{j=1,\ldots,k} P(f_j)$. □

Corollary 1. *Let $(P, \leq)$ be a poset with maximum M and $F = \{f_i : i \in I_n\}$ be a commutative family of monotone selfmaps of P. Let $f = f_1 \cdot f_2 \cdots f_n$ be isotone in P and suppose that property* (2") *holds. Then the conclusions of Theorem* 3 *hold.*

Proof. It suffices to assume $x_0 = M$ in Theorem 3, thus property (1") holds. □

Remark 1. Any complete lattice P has a maximum M and any chain in P has an infimum, thus Theorem 1 follows from Corollary 1.

Motivated by Example 2 of [4], we now give the following:

Example 1. Let $P = \{a, b, c, d, e\}$ be a set in which we define the following partial ordering: $a \geq b$, $a \geq c$, $a \geq d$, $a \geq e$, $b \geq c$, $b \geq e$, $d \geq e$. Then $(P, \leq)$ is a finite poset with maximum $M = a$. Let $F = \{f_1, f_2, f_3, f_4\}$, where $f_1 = f_2, f_3, f_4 \colon P \to P$ are defined as $f_1(a) = e$, $f_1(b) = f_1(d) = b$, $f_1(c) = f_1(e) = a$, $f_3(a) = a$, $f_3(b) = f_3(d) = b$, $f_3(c) = f_3(e) = e$ and $f_4(a) = f_4(b) = f_4(c) = f_4(d) = f_4(e) = b$, respectively. It is easily verified that f_1 is antitone and f_3, f_4 are isotone with respect to the given partial ordering. A simple calculation proves that F is a family of commuting selfmaps of P. Property (2") of Theorem 3 holds trivially since any chain in P is finite and hence has a minimum. Then all the assumptions of the Corollary 1 are satisfied and we find that $I_P(f_3) \cap I_P(f_4) = \{b\}$, $E_P(f_1) = \{e, a\}$. Theorem 1 is not applicable because P is not a complete lattice with respect to the given partial ordering: indeed the subset $\{c, e\}$ has no an infimum.

Remark 2. Property (2") is necessary in Theorem 2 since some conclusion may fail if it is omitted. Indeed, let $P = [0, 1] \backslash \{1/2\}$ with its natural ordering and $F = \{f_1, f_2, f_3, f_4\}$, where $f_1 = f_2, f_3, f_4 \colon P \to P$ are given as $f_1(x) = 1 - x$, $f_3(x) = (x + 1)/3$ and $f_4(x) = (x + 2)/5$ for every $x \in P$. Note that $f_1(f_3(x)) = (2 - x)/3 = f_3(f_1(x))$, $f_1(f_4(x)) = (3 - x)/5 = f_4(f_1(x))$ and $f_3(f_4(x)) = (x + 7)/15 = f_4(f_3(x))$ for every $x \in P$. It is also easily verified that f_1 is antitone and f_3, f_4, f_5 are isotone. Thus F is a commuting family of monotone selfmaps of P and we have $f(x) = f_1(f_2(f_3(f_4(x)))) = (x+7)/15$ for every $x \in P$. Take a chain $C = \{c_n\}$ of P such that $1/2 < c_n \leq 1$ and

$\inf c_n = 1/2$. Then we have $\inf f(C) = 1/2 \notin P$. By assuming $x_0 = 1$, then all the assumptions of Theorem 2 hold except property (2"). In fact we have $I_P(f_3) \cap I_P(f_4) = \{\emptyset\}$ since $1/2$ is the unique common fixed point of f_3 and f_4 in the reals but $1/2 \notin P$.

3. Other theorems

The following theorem is also given in [4]:

Theorem 4. *Let $(P, \leq)$ be a poset and F be a commutative family of isotone selfmaps of P satisfying the following properties:*

(1"') *There exists an element $x_0 \in P$ such that $x_0 \leq f(x_0)$ for every $f \in F$,*

(2"') *If C is a chain in $(P, \leq)$ not having a supremum (in P), then there exists an $h \in F$ such that $h(C)$ has a supremum.*

Then there exists an element $q_0 \in P$ such that $q_0 = \min\{q\colon q \in \cap_{f\in F} I_P(f) \cap P^+(x_0)\}$, where $P^+(x_0) = \{x \in P\colon x \geq x_0\}$.

As in Section 2, from Theorem 4 we can deduce the following result.

Theorem 5. *Let $(P, \leq)$ be a poset and $F = \{f_i\colon i \in I_n\}$ be a commutative family of monotone selfmaps of P. Let $f = f_1 \cdot f_2 \cdots f_n$ be isotone in P and suppose that the following properties hold:*

(1"") *There exists an element $x_0 \in P$ such that $x_0 \leq f_i(x)$ for every $i \in I_n$ and $x \in P^+(x_0)$,*

(2"") *If C is a chain in $(P, \leq)$ not having a supremum in P, then $f(C)$ has a supremum.*

Then $P(f_i) \neq \emptyset$ for every $i \in I_n$, $\cap I_P(f_i) \neq \emptyset$ for all $f_i \in F$ isotone and $\cap E_P(f_i) \neq \emptyset$ for all $f_i \in F$ antitone.

Remark 3. If F is defined as in the proof of Theorem 3, then there exists a point $q_0 = \min\{q\colon q \in \cap_{g\in F} I_P(g) \cap P^+(x_0)\}$. Clearly $q_0 \in \cap_{i=k+1,\dots,n} I_P(f_i) \subseteq \cap_{i=k+1,\dots,n} P(f_i)$. It is also easily seen that $f_i(q_0) = f_j(q_0) \in \cap_{f\in F} I_P(f) \cap P^+(x_0)$ for every $i, j \in I_k$ and $q_0 \leq f_i(q_0)$ for every $i \in I_k$, that is $q_0 \in P(f_i)$ for every $i \in I_k$.

Corollary 2. *Let $(P, \leq)$ be a poset with minimum m and $F = \{f_i\colon i \in I_n\}$ be a commutative family of monotone selfmaps of P. Let $f = f_1 \cdot f_2 \cdots f_n$ be isotone in P and suppose it satisfies property (2""). Then the conclusions of Theorem 5 hold.*

A complete lattice P has a minimum m and a maximum M. Further, any chain of P has an infimum and a supremum in P. Using the techniques in the proofs of Theorems 3 and 5, we can obtain the following:

Theorem 6. *Let* $(P, \leq)$ *be a complete lattice and* $F = \{f_i : i \in I_n\}$ *be a commutative family of monotone selfmaps of* P*. Let* $f = f_1 \cdot f_2 \cdots f_k \cdot f_{k+1} \cdot f_{k+2} \cdots f_n$ *be isotone in* P*, where* $f_1, f_2, \ldots, f_k$ *are antitone and* $f_{k+1}, f_{k+2}, \ldots, f_n$ *are isotone. Then there exists a point* $q_0 = \min G$ *and a point* $m_0 = \max G$*, where* $G = \{x \in P : x \in I_P(f) \cap I_P(f_{k+1}) \cap \cdots \cap I_P(f_n) \cap \{\cap I_P(f_i \cdot f_j) : i, j \in I_k\}\}$.

We conclude with the following example, illustrative of Theorem 6:

Example 2. Let $P = [0, 1]$ with natural ordering and $F = \{f_1, f_2, f_3, f_4, f_5\}$, where $f_1 = f_2, f_3, f_4, f_5 : P \to P$ are given as $f_1(x) = 1 - x$ for every $x \in P, f_3(x) = (x + 1)/3$, $f_4(x) = (x + 2)/5$ and $f_5(x) = (x + 3)/7$ if $0 \leq x \leq 1/2$, $f_3(x) = f_4(x) = f_5(x) = x$ if $1/2 \leq x \leq 1$. Note that F is a commuting family of monotone selfmaps of P (see Remark 2) and $f(x) = f_1(f_2(f_3(f_4(f_5(x))))) = (x + 52)/105$ if $0 \leq x \leq 1/2$ and $f(x) = f_1(f_2(f_3(f_4(f_5(x))))) = x$ if $1/2 \leq x \leq 1$. In this case we have $q_0 = 1/2$ and $m_0 = 1$ since $G = [1/2, 1]$, and $I_P(f) = I_P(f_3) = I_P(f_4) = I_P(f_5) = [1/2, 1]$ and $I_P(f_1 \cdot f_1) = P$.

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