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ON A ZAREMBA'S CONJECTURE FOR POWERS

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ABSTRACT. A conjecture of Zaremba says that for every $m \geq 2$ there exists a reduced fraction a/m such that its simple continued fraction has all its partial quotients bounded by 5. This conjecture is proved with all partial quotients bounded by 3 for m being $c \cdot 2^n$-th power of 7 where $n \geq 0$ and $c = 1, 3, 5, 7, 9, 11$. A more general case is considered.

1. Introduction

Let $m \geq 2$ be an integer and let a be an integer with $1 \leq a \leq m-1$ and $\gcd(a, m) = 1$. Suppose that the simple continued fraction expansion of the rational number a/m is given by

$$\frac{a}{m} = [0; a_1, a_2, \ldots, a_h]\,,$$

where a_1, a_2, $\ldots$, a_n are the partial quotients and $a_i \geq 1$ $(1 \leq i \leq h)$. We define

$$K\left(\frac{a}{m}\right) = \max(a_1, a_2, \ldots, a_h)\,.$$

For the purpose of this definition we take $a_h > 1$ to guarantee the uniqueness of the continued fraction expansion, but later on we will also allow $a_h = 1$.

A conjecture of Zaremba [5, pp. 69 and 76] states that for every $m \geq 2$ there exists a reduced fraction a/m such that $K(a/m) \leq 5$. In 1986, Niederreiter [2] proved that for m being powers of 2, 3 and 5, there exists an integer a with $1 \leq a \leq m-1$ and $\gcd(a, m) = 1$ such that $K(a/m) \leq 3$. In 2002, Yodphotong and Laohakosol [4] proved that for m being powers of 6, there exists an integer a with $1 \leq a \leq m-1$ and $\gcd(a, m) = 1$ such that $K(a/m) \leq 5$. Nevertheless, $K(a/m) \leq 3$ is conjectured except for $m = 6$.

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In this paper we shall prove that for $m=7^{c\cdot 2^n}$ ($n\ge 0$; $c=1,3,5,7,9,11$) there exists a reduced fraction a/m such that $K(a/m)\le 3$. A more general case is considered.

2. Folding's Lemma and its application

The following famous result is called Folding's Lemma. See [1] or [3, Proposition 2]. This is also useful in this paper.

Lemma 1 (**Folding's Lemma**). *If $p_n/q_n=[a_0;a_1,a_2,\dots,a_n]$ and b is a nonnegative integer, then*

$$\begin{aligned}\frac{p_n}{q_n}+\frac{(-1)^n}{(b+1)q_n^2} &= [a_0;a_1,a_2,\dots,a_n,b+1,-a_n,-a_{n-1},\dots,-a_2,-a_1]\\ &= [a_0;a_1,a_2,\dots,a_n,b,1,a_n-1,a_{n-1},\dots,a_2,a_1]\,.\end{aligned}$$

Remark. If $b=0$ or $a_n=1$, then the convenient rule $[\dots,c,0,d,\dots]=[\dots,c+d,\dots]$ ([3, Proposition 3]) is applied.

Theorem 1. *If $a/m=[0;1,1,a_3,\dots,a_{h-1},2]$, then*

$$\frac{ma+(-1)^h}{m^2}=[0;1,1,a_3,\dots,a_{h-1},3,1,a_{h-1},\dots,a_3,2]$$

and

$$\frac{ma-(-1)^h}{m^2}=[0;1,1,a_3,\dots,a_{h-1},1,3,a_{h-1},\dots,a_3,2]\,.$$

Corollary 1. *If $a/m=[0;1,1,a_3,\dots,a_{h-1},2]$, then we have*

$$\begin{aligned}2\le K\left(\frac{a}{m}\right)\le K\left(\frac{ma\pm 1}{m^2}\right)&=K\left(\frac{m^2(ma\pm 1)\pm 1}{m^4}\right)\\ &=K\left(\frac{m^4(m^2(ma\pm 1)\pm 1)\pm 1}{m^8}\right)=\cdots\\ =K&\left(\frac{m^{2^n-1}a\pm m^{2^n-2}\pm m^{2^n-2^2}\pm\cdots\pm m^{2^n-2^{n-2}}\pm m^{2^{n-1}}\pm 1}{m^{2^n}}\right)\,.\end{aligned}$$

Proof of Theorem 1. Put $n=h$, $a_0=0$, $a_1=a_2=1$, $a_h=2$ and $b=0$ in Folding's Lemma. Then we have

$$\begin{aligned}\frac{a}{m}+\frac{(-1)^h}{m^2} &= [0;1,1,a_3,\dots,a_{h-1},2,0,1,1,a_{h-1},\dots,a_3,1,1]\\ &= [0;1,1,a_3,\dots,a_{h-1},3,1,a_{h-1},\dots,a_3,2]\,.\end{aligned}$$

Another expression can be obtained by considering the different form,

$a/m = [0; 1, 1, a_3, \ldots, a_{h-1}, 1, 1]$. Put $n = h + 1$, $a_0 = 0$, $a_1 = a_2 = 1$, $a_h = a_{h+1} = 1$ and $b = 0$ in Folding Lemma. Then we have

$$\begin{aligned} \frac{a}{m} + \frac{(-1)^{h+1}}{m^2} &= [0; 1, 1, a_3, \ldots, a_{h-1}, 1, 1, 0, 1, 0, 1, a_{h-1}, \ldots, a_3, 1, 1] \\ &= [0; 1, 1, a_3, \ldots, a_{h-1}, 1, 3, a_{h-1}, \ldots, a_3, 2]\,. \end{aligned}$$

It is clear that $\gcd(ma \pm 1, m^2) = 1$ and $1 \le ma \pm 1 \le m^2 - 1$. □

3. Main results and examples

Theorem 2. *For any nonnegative integer n there exists a positive integer a with $1 \le a < 7^{2^n}$ and $\gcd(a, 7^{2^n}) = 1$ such that $K(a/7^{2^n}) \le 3$.*

Proof. We start from the fractions

$$\begin{aligned} \frac{5}{7} &= [0; 1, 2, 2]\,, \\ \frac{30}{7^2} &= [0; \underbrace{1, 1, 1, 1, 2, 1, 2}_{7}]\,. \end{aligned}$$

Since $30/7^2 = [0; 1, 1, a_3, \ldots, a_{h-1}, 2]$ with $a_i \le 3$ $(3 \le i \le h - 1)$, by Corollary 1 we have the desired result. □

One can see the further details without difficulty. We apply Theorem 1 to $30/7^2$ to obtain that

$$\frac{7^2 \cdot 30 + (-1)^7}{7^4} = \frac{1469}{7^4} = [0; 1, 1, 1, 1, 2, 1, 3, 1, 1, 2, 1, 1, 2]$$

and

$$\frac{7^2 \cdot 30 - (-1)^7}{7^4} = \frac{1471}{7^4} = [0; \underbrace{1, 1, 1, 1, 2, 1, 1, 3, 1, 2, 1, 1, 2}_{13}]\,.$$

We apply Theorem 1 again to the latter fraction $1471/7^4$. Then we have

$$\begin{aligned} &\frac{7^4 \cdot 1471 + (-1)^{13}}{7^8} = \frac{3531870}{7^8} \\ &\qquad = [0; 1, 1, 1, 1, 2, 1, 1, 3, 1, 2, 1, 1, 3, 1, 1, 1, 2, 1, 3, 1, 1, 2, 1, 1, 2] \end{aligned}$$

and

$$\begin{aligned} &\frac{7^4 \cdot 1471 - (-1)^{13}}{7^8} = \frac{3531872}{7^8} \\ &\qquad = [0; 1, 1, 1, 1, 2, 1, 1, 3, 1, 2, 1, 1, 1, 3, 1, 1, 2, 1, 3, 1, 1, 2, 1, 1, 2]\,. \end{aligned}$$

In a similar manner, we apply Theorem 1 to one of two expressions to obtain the reduced fractions $A/7^{16}$ such that $K(A/7^{16}) \le 3$. By repeating the process we have the result.

Now, we start from e.g.

$$\frac{199}{7^3}=[0;1,1,2,1,1,1,1,1,1,1,1,2]\,,$$

$$\frac{9861}{7^5}=[0;1,1,2,2,1,1,1,1,2,1,2,1,2,1,1,2]\,,$$

$$\frac{475635}{7^7}=[0;1,1,2,1,2,1,1,1,1,1,3,2,1,1,1,2,1,2,1,2,1,2]\,,$$

$$\begin{aligned}\frac{23670145}{7^9}&=[0;1,1,2,2,1,1,2,1,\\&\qquad 2,2,1,1,1,1,1,2,1,1,1,1,2,1,2,1,1,2,1,1,2],\\ \frac{1141612802}{7^{11}}&=[0;1,1,2,1,2,1,2,1,2,1,1,\\&\qquad 2,2,2,2,2,1,1,1,2,1,1,1,2,1,2,1,1,1,1,1,2,1,2]\end{aligned}$$

and so forth, then apply to each of them Theorem 1, yielding $K(a/7^{c\cdot 2^n})\le 3$ for any nonnegative integer n and $c=3,5,7,9,11$, respectively. It is not difficult to continue to check the validity for concrete odd numbers c.

Notice that there exists no reduced fraction $a/7^7$ satisfying $K(a/7^7)\le 2$.

4. A more general case

Since Theorem 1 with Corollary 1 does not restrict the denominator to only 7, we can choose the denominator as powers of any integer greater than 1. If we can find an initial fraction of the form

$$\frac{a}{N^c}=[0;1,1,a_3,\dots,a_{h-1},2]\,,$$

where $a_i\le 3$ $(3\le i\le h-1)$, then for any nonnegative integer n there exists a positive integer a with $1\le a<N^{c\cdot 2^n}$ and $\gcd(a,N^{c\cdot 2^n})=1$ such that $K(a/N^{c\cdot 2^n})\le 3$. If $a_i\le 4$ $(3\le i\le h-1)$, then $K(a/N^{c\cdot 2^n})\le 4$. If $a_i\le 5$ $(3\le i\le h-1)$, then $K(a/N^{c\cdot 2^n})\le 5$.

Conjecture. Let N be an integer with $N\ge 2$. For a positive integer n there exists a positive integer a with $1\le a<N^{2^n}$ and $\gcd(a,N^{2^n})=1$ such that $K(a/N^{2^n})\le 3$.

It is easy to see that this holds for $N=2,3,\dots,2052$ and so forth. Unless $N=6,20,28,38,42,54,90,96,\dots$, this holds for any nonnegative integer n. In fact, $K(a/m)\ge 4$ $(m=20,28,38,42,90,96,156,164,216,228,252,318,336,350,384,386,442,508,558,770,876,922,978,1014,1155,1170,1410,1450,1692,1870,2052,\dots)$ and $K(a/m)\ge 5$ $(m=6,54,150)$ for any positive integer a such that each fraction is reduced. It is unknown if we can say something more about such a sequence of numbers m.

It has been unknown whether there are infinitely many m's such that $K(a/m) = 4$ and/or what the largest m satisfying $K(a/m) = 5$ is if it exists.

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