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ON A QUESTION RAISED BY BROWN, GRAHAM AND LANDMAN

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Abstract. We construct non-periodic 2-colorings that avoid long monochromatic progressions having odd common differences. Also we prove that the set of all arithmetic progressions with common differences in $(\mathbb{N}!-1)\cup\mathbb{N}!\cup(\mathbb{N}!+1)-\{0\}$ does not have the 2-Ramsey property.

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and for integers $a<b$, denote by $[a,b]$ the set $\{a,a+1,\dots,b\}$. For $r\in\mathbb{N}$, an r-coloring of $\mathbb{N}$ is a map $f:\mathbb{N}\to A$, with $|A|=r$. A coloring is an r-coloring for some r. If f is a coloring and if $B\subseteq\mathbb{N}$ satisfies $|f(B)|=1$, we say that B is f-monochromatic. An arithmetic progression of length k and common difference d, $k,d\in\mathbb{N}$, is a set of the form $\{a+(i-1)\,d:i\in[1,k]\}$, for some $a\in\mathbb{N}$.

Van der Waerden's theorem [4] on arithmetic progressions says that for any coloring f and any $k\in\mathbb{N}$ there is an f-monochromatic arithmetic progression of length k. Brown, Graham, and Landman in [1] study subsets L of $\mathbb{N}$ such that van der Waerden's theorem can be strengthened to guarantee the existence of arbitrarily long f-monochromatic progressions having common differences in L.

Another, more general, approach in which the set of arithmetic progressions having common difference in L is replaced with the set of sequences with special gaps in L, is studied in [3].

For $r\in\mathbb{N}\backslash\{1\}$ we say that L, $L\subseteq\mathbb{N}$, is r-large if every r-coloring yields arbitrarily long monochromatic progressions having common differences in L. We say that L is large if it is r-large for every r. In general, if T is a family of integer sequences such that for any r-coloring of $\mathbb{N}$ there are arbitrarily long monochromatic members of T, then we say that T has the

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r-Ramsey property. Thus a set L is r-large if and only if the set of all arithmetic progressions having common differences in L has the r-Ramsey property. Perhaps surprisingly, there are many large sets; for example, for any $m \in \mathbb{N}$, $m\mathbb{N}$ is large.

On the other hand, it is known that $2\mathbb{N}-1$ is not 2-large. One can see that, for $n \in \mathbb{N}\backslash\{1\}$, the coloring $f_n : \mathbb{N} \to \{0,1\}$ defined by

$$f_n(i) = 0 \Leftrightarrow ((\exists t \in \mathbb{N}_0)\ i \in [2(n-1)t+1, (2t+1)(n-1)])$$

avoids n-term f_n-monochromatic arithmetic progressions having odd common differences. Clearly, f_n is periodic with a period $2(n-1)$.

In [1] it was shown that a necessary condition for 2-largeness is that the set contains an infinite number of multiples of any integer. Obviously, the set of odd positive integers fails to satisfy this condition.

Another example of a set that is not 2-large is $\mathbb{N}! = \{n! : n \in \mathbb{N}\}$. This set is not 2-large because it is too sparse. It is known [1] that for a set $L = \{a_n\}_{n\in\mathbb{N}}$ to be 2-large it is necessary that for any $N \in \mathbb{N}$ there is $n > N$ such that $\frac{a_{n+1}}{a_n} < 3$. This fact implies that for a set $L = \{a_n\}_{n\in\mathbb{N}}$ to be r-large it is necessary that for any $N \in \mathbb{N}$ there is $n > N$ such that $\frac{a_{n+1}}{a_n} < 3^{\frac{1}{\lfloor \log_2 r \rfloor}}$. Ultimately, for any large set $L = \{a_n\}_{n\in\mathbb{N}}$, $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$.

Brown, Graham and Landman [1] conjectured that any 2-large set is large.

One notable distinction between the known properties of the family of large sets and the family of 2-large sets is as follows. It is known that if $L_1 \cup L_2$ is large then at least one of L_1 and L_2 is large. Whether or not the same is true for 2-large sets is an open question.

Thus, one way to approach the conjecture is to find two sets that are not 2-large and to show that their union is 2-large. Brown, Graham, and Landman suggest that $2\mathbb{N}-1$ and $\mathbb{N}!$ could be such sets. Clearly, $(2\mathbb{N}-1)\cup\mathbb{N}!$ contains an infinite number of multiplies of any integers and it is not sparse, therefore it satisfies both of the necessary conditions mentioned above.

In this note we clarify two issues related to the question whether $(2\mathbb{N}-1)\cup \mathbb{N}!$ is 2-large.

If χ is a periodic 2-coloring of $\mathbb{N}$ with a period T, then the set $\{1 + i \cdot T! : i \in \mathbb{N}\}$ is χ-monochromatic. Thus if there is a 2-coloring that avoids long monochromatic progressions having their common differences in $(2\mathbb{N}-1)\cup\mathbb{N}!$ then it must be non-periodic.

We prove that the family of non-periodic 2-colorings of $\mathbb{N}$ that avoid long monochromatic arithmetic progressions having their common differences odd is non-empty. As part of our construction we show that, for any $p \geq 2$, the coloring f_{2p} could be modified by changing colors of an infinite number of integers, but still obtaining a 2-coloring that avoids long monochromatic arithmetic progressions having their common differences odd. We prove that this modification can be done so that the new coloring is not periodic. These colorings are very different from the counterexamples considered in [1].

Our construction does not avoid periodicity on all of the classes modulo $2p$ and therefore we are not able to avoid long arithmetic progressions having their common differences in $\mathbb{N}!$.

Secondly, we prove that $(\mathbb{N}!{-}1)\cup\mathbb{N}!\cup(\mathbb{N}!{+}1)-\{0\}$ is not 2-large. Hence we give an example of a subset of $(2\mathbb{N}-1)\cup\mathbb{N}!$ that contains $\mathbb{N}!$ and an infinite subset of $2\mathbb{N}-1$ and that is not 2-large. An interesting possibility suggested by this example is that the gaps between consecutive elements of a 2-large set cannot be too big. We note that there are large sets with unbounded gaps. One example of such a set is $\mathbb{N}^2$. A result from [1] implies that $(\mathbb{N}!{-}1)\cup\mathbb{N}!\cup(\mathbb{N}!{+}1)-\{0\}$ is not 3-large. Here we prove that, in fact, this set is not 2-large.

2. A set of 2-colorings

Let $\mathcal{K}_{2\mathbb{N}-1}$ be the set of all finite colorings of $\mathbb{N}$ that do not yield long monochromatic arithmetic progressions having odd common differences.

We start with a lemma.

Let $p \in \mathbb{N}\setminus\{1\}$ be given.

Lemma 1. *Let $a \in [1, 2\,(2p-1)]$ and let $l \in [0, 2\,(p-1)]\setminus\{p-1\}$. There are $i \in [0, 2p-1]$, $j \in [1, p]$, and $k \in [2p, 2\,(2p-1)]$ so that $a+i\,(2l+1) \equiv 2j \pmod{2p}$ and $a+i\,(2l+1) \equiv k \pmod{2\,(2p-1)}$.*

Proof. Let $\{a, a+(2l+1)\}\cap 2\mathbb{N} = \{\alpha\}$. Suppose that there is $q \in [1, p-1]$ so that

$$(\forall i \in [0,q])\,(\exists k_i \in [1, 2p-1])\;\; \alpha + 2i\,(2l+1) \equiv k_i \pmod{2\,(2p-1)}.$$

We note that $2(2l+1) \not\equiv 0 \pmod{2(2p-1)}$ implies $k_1 \neq k_0$. Since, for all $i, i' \in [0,q]$, $i \neq i'$ implies $k_i \neq k_{i'}$, from the fact that k_i is even it follows that $q < p-1$.

Therefore, there are $i \in [0, p-1]$ and $k \in [2p, 2\,(2p-1)]$ so that $\alpha + 2i\,(2l+1) \equiv k \pmod{2\,(2p-1)}$. $\square$

Let $p \in \mathbb{N}$ and let f_{2p} be as above. Let M'_p be the set of 2-colorings such that $f \in M'_p$ if and only if the following two conditions are satisfied:

1. For any $n \in 2\mathbb{N}$, $f(n) = f_{2p}(n)$.
2. There is an odd number $N = N(f) \in [1, 2p]$ so that for $K'_N = \{i \in \mathbb{N} : i \equiv N \pmod{2p}\}$ and $K''_N = \{i \in (2\mathbb{N}-1)\setminus K'_N : (\exists j \in [1, 2p-1])\; i \equiv j \pmod{2\,(2p-1)}\}$ we have that the restrictions of f and f_{2p} on $K'_N \cup K''_N$ coincide.

Theorem 2. $M'_p \subseteq \mathcal{K}_{2\mathbb{N}-1}$.

Proof. Let $f \in M'_p$. Let $a, l \in \mathbb{N}$, let $a' \in [1, 2(2p-1)]$, and let $l' \in [0, 2(p-1)]$ be so that $a \equiv a' \pmod{2(2p-1)}$ and $l \equiv l' \pmod{(2p-1)}$.

If $l' \neq p-1$, by Lemma 1 there is $i \in [0, 2p-1]$ so that $a' + i\,(2l'+1)$ is even and $f_{2p}\,(a' + i\,(2l'+1)) = f\,(a + i\,(2l+1)) = 1$. On the other hand, $1 \leq |\{i \in [0, 2p-1] : f_{2p}\ (a + i(2l+1)) = 0\}| \leq |\{i \in [0, 2p-1] : f(a + i(2l+1)) = 0\}|$. Therefore, $\{a + i\,(2l+1) : i \in [0, 2p-1]\}$ is not f-monochromatic.

If $l' = p-1$ then $2l'+1 = 2p-1$ and, for all $i \in [0, 2p-2]$, $f_{2p}(a'+i(2p-1)) \neq f_{2p}\,(a'+(i+1)(2p-1))$. Note that if $\beta \in \{a'+i(2p-1) : i \in [0, 2p-1]\} \cap K'_N$ then $\{\beta - (2p-1), \beta + (2p-1)\} \cap \{a' + i(2p-1) : i \in [0, 2p-1]\} \neq \phi$.

Thus, $\{a' + i(2p-1) : i \in [0, 2p-1]\}$ is not f-monochromatic. $\square$

The following corollary gives a way to construct non-periodic elements of $\mathcal{K}_{2\mathbb{N}-1}$.

Corollary 3. *Let g be a non-periodic 2-coloring of $\mathbb{N}_0$ and let $f : \mathbb{N} \to \{0, 1\}$ be defined by*

$$f\,(n) = \begin{cases} g\left(\frac{n-2p-1}{2p(2p-1)}\right) & \text{if } n \equiv (2p+1) \pmod{2p\,(2p-1)} \\ f_{2p}\,(n) & \text{otherwise.} \end{cases}$$

Then f is a non-periodic element of M'_p.

An example of a non-periodic 2-coloring of $\mathbb{N}$ is the Morse sequence. See, for example, [2].

Note that for any f obtained in the way described in Corollary 3, there are arbitrarily long monochromatic progressions having their common differences in $\mathbb{N}!$.

Let M''_p be the set of all 2-colorings so that $f \in M''_p$ if and only if the following two conditions are satisfied:

1. For any odd integer n, $f(n) = f_{2p}(n)$.
2. There is an even $N = N\,(f) \in [1, 2p]$ so that for $L'_N = \{i \in \mathbb{N} : i \equiv N \pmod{2p}\}$ and $L''_N = \{i \in 2\mathbb{N} \backslash L'_N : (\exists j \in [2p, 2(2p-1)])\ i \equiv j \pmod{2(2p-1)}\}$ we have that the restrictions of f and f_{2p} on $L'_N \cup L''_N$ coincide.

Then $M''_p \subseteq \mathcal{K}_{2\mathbb{N}-1}$. Therefore, $M_p = \left\{\chi : \{\chi, 1-\chi\} \cap \left(M'_p \cup M''_p\right) \neq \phi\right\} \subseteq \mathcal{K}_{2\mathbb{N}-1}$.

We note that the elements of M_p permit monochromatic $(2p-1)$-term arithmetic progressions having common difference 1.

3. An interesting example

By [1], Corollary 2.2, $(\mathbb{N}! - 1) \cup \mathbb{N}! \cup (\mathbb{N}! + 1) - \{0\}$ is not 3-large. We prove that $(\mathbb{N}! - 1) \cup \mathbb{N}! \cup (\mathbb{N}! + 1) - \{0\}$ is not 2-large.

Let $\mathbb{N}! = \{n!\}_{n\in\mathbb{N}}$ and let $\chi : \mathbb{N} \to \{0, 1\}$ be defined by $\chi(1) = 1$ and, inductively, if χ is defined on $[1, n!]$ then

$$(\forall x \in [n! + 1, (n + 1)!])\ \chi(x) \neq \chi(x - n!).$$

It is not difficult to check that there is no χ-monochromatic 3-term arithmetic progression having its common difference in $\mathbb{N}!$. Also, let $q, n \in \mathbb{N}$ and let $x, k \in [1, n+1]$ be such that $\{x, x+k\cdot n!\} \subseteq [q(n+1)!+1, (q+1)(n+1)!]$. If k is even then $\chi(x) = \chi(x + k \cdot n!)$.

Theorem 4. $(\mathbb{N}! - 1) \cup \mathbb{N}! \cup (\mathbb{N}! + 1) - \{0\}$ *is not 2-large.*

Proof. Let $q \in \mathbb{N}_0$ and let $n \geq 9$. For $j \in [0, 8]$, let $I_j = [(q + j)n! + 1, (q + j + 1)n!]$. Let $r \in \mathbb{N}$ and $l \in \{0, 4\}$ be such that $\cup_{j=0}^{4} I_{l+j} \subseteq [r(n + 1)! + 1, (r + 1)(n + 1)!]$.

For $x \in [qn! + 1, (q + 1)n! - 8]$ let $X = \{x + j(n! + 1) : j \in [0, 8]\}$ and $X' = \{x+(l+j)(n!+1) : j \in \{0, 2, 4\}\}$. Since j is even, $\chi(x+(l+j)(n!+1)) = \chi(x + l(n! + 1) + j)$ and it follows that X' is not χ-monochromatic.

Therefore, there is no χ-monochromatic 9-term arithmetic progression having its least element in $[qn! + 1, (q + 1)n! - 8]$, and having common difference $n! + 1$.

Let $x \in [(q+1)n!-7, (q+1)n!]$ and let us consider the 17-term arithmetic progression $\{x+j(n!+1) : j \in [0, 16]\}$. For $s \in [1, 8]$ such that $x+s(n!+1) = (q+s+1)n!+1$, the 9-term arithmetic progression $\{x+s(n!+1)+j(n!+1) : j \in [0, 8]\} = \{x + j(n! + 1) : j \in [s, s + 8]\}$ is not monochromatic.

Therefore, χ avoids monochromatic 17-term arithmetic progressions having common difference in $\mathbb{N}! + 1$.

Similarly, we can see that χ avoids monochromatic 17-term arithmetic progressions having common difference in $\mathbb{N}! - 1$. $\square$

With the results presented in this note we would like to draw the reader's attention to the Brown-Graham-Landman conjecture that every 2-large set is large. It seems that this simply stated conjecture is difficult to prove or disprove and that the answer could go either way.

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