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REMARK ON THE SECOND BOUNDED COHOMOLOGY OF AMALGAMATED PRODUCT OF GROUPS

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ABSTRACT. For any cardinal number $\mathcal{M}$ we construct examples of amalgamated products and HNN extensions of groups such that the dimension of the space of second bounded cohomologies is at least $\mathcal{M}$. Also we describe the space of pseudocharacters of the group $GL(2, F_2[z])$.

1. INTRODUCTION

Bounded cohomology was defined first for discrete groups by F. Trauber and then for topological spaces by Gromov [29]. Moreover, Gromov developed the theory of bounded cohomology and applied it to Riemannian geometry, thus demonstrating the importance of this theory. The second bounded cohomology group is related to some topics of the theory of right orderable groups and has application in the theory of groups acting on a circle [25, 47, 48]. In [4], Brooks made a first step in understanding the theory of bounded cohomology from the point of view of relative homological algebra. The papers of Gromov, Brooks, Ghys, Mitsumatsu, Matsumoto, Morita and others give excellent examples of applications of abstract theory of cohomology in Banach algebras, Riemannian geometry, topology, dynamics and other branches of mathematics. An important feature of the theory is that the bounded cohomology of a topological space and its fundamental group coincide [29, 4, 47, 48, 49]. This makes it possible to study them simultaneously from two basic view points: group theory and topology.

The bounded cohomology, $H_b^*(G, \mathbb{R})$, of an amenable group G is zero (Trauber's theorem). In [4] some examples are given showing that for nonamenable groups bounded cohomology may be nonzero and even infinite dimensional. The first dimension in which bounded cohomology should be investigated is dimension 2 because $H_b^{(0)}(G, \mathbb{R}) = \mathbb{R}$ and $H_b^{(1)}(G, \mathbb{R}) = 0$ for any group G. In Faiziev's papers [8, 12], the space of pseudocharacters of free group were described. Using the space of pseudocharacters and results

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and methods of the papers [8, 12] Grigorchuk in [27] reformulated these results of Faiziev in terms of second bounded cohomology. Then he, using the methods and results of the papers [5, 12], gave calculation of $H_b^{(2)}(G,\mathbb{R})$ for surface groups.

The ordinary cohomology group $H^*(G)$ is given by the cohomology of the cochain complex $C^*(G)$:

$$\stackrel{\delta^{(n)}}{\longleftarrow} C^{(n)}(G) \stackrel{\delta^{(n-1)}}{\longleftarrow} C^{(n-1)}(G) \longleftarrow \cdots$$
$$\cdots \longleftarrow C^{(2)}(G) \stackrel{\delta^{(1)}}{\longleftarrow} C^{(1)}(G) \stackrel{\delta^{(0)}=0}{\longleftarrow} \mathbb{R} \stackrel{\delta^{(-1)}=0}{\longleftarrow} 0,$$

where $C^{(n)}(G), n \geq 0$ consists of mappings

$$G \times \cdots \times G \to \mathbb{R},$$

and the differential $\delta = (\delta^{(n)}), n \geq 0$:

$$\delta^{(n)} : C^{(n)}(G) \to C^{(n+1)}(G)$$

is given by the formula

$$\begin{aligned}(\delta^{(n)} f)(g_1, \cdots, g_{n+1}) =& f(g_2, \cdots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(g_1, \cdots, g_{i-1}, g_i g_{i+1}, g_{i+2} \cdots, g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, \cdots, g_n),\end{aligned}$$

where $f \in C^n(G)$. Now let us consider bounded cochains $f \in C^{(n)}(G)$, that is, cochains for which there exists $M_f > 0$ such that

$$|f(g_1, \cdots, g_n)| \leq M_f$$

for all $g_1, \ldots, g_n \in G$. We have the cochain complex $C_b^*(G)$:

$$\stackrel{\delta_b^{(n)}}{\longleftarrow} C_b^{(n)}(G) \stackrel{\delta_b^{(n-1)}}{\longleftarrow} C_b^{(n-1)}(G) \longleftarrow \cdots$$
$$\cdots \longleftarrow C_b^{(2)}(G) \stackrel{\delta_b^{(1)}}{\longleftarrow} C_b^{(1)}(G) \stackrel{\delta_b^{(0)}=0}{\longleftarrow} \mathbb{R} \stackrel{\delta_b^{(-1)}=0}{\longleftarrow} 0,$$

of bounded cochains with values in $\mathbb{R}$ and can define ℓ_∞ (or bounded) - cohomology

$$H_b^*(G, \mathbb{R}) = H_b^*(C_b^*(G));$$

that is

$$H_b^{(n)}(G) = \ker \delta_b^{(n)} / \Im \delta_b^{(n-1)}, \quad n \geq 0$$

where

$$\delta_b^{(n)} = \delta_b^{(n)} \Big|_{C_b^{(n)}(G)}$$

is the bounded differential operator (the restriction of $\delta^{(n)}$ to the bounded cochain complex). It is easy to see that $H_b^{(1)}(G) = 0$ for all G. The fact is that there do not exist nontrivial bounded homomorphisms $G \to \mathbb{R}$.

The inclusion homomorphism $C_b^*(G) \to C^*(G)$ induces a homomorphism $\xi : H_b^*(G) \to B(G)$ which in general is neither injective nor surjective. The image of this homomorphism is called the bounded part of $H^*(G)$ and will be denoted by $H_{b,1}^*(G)$ (see [27]). Denote by $H_{b,2}^{(n)}(G)$ the subspace $\operatorname{Im} \delta^{(n-1)} \cap \ker \delta_b^{(n)} / \operatorname{Im} \delta_b^{(n-1)}$ of $H_b^{(n)}(G)$. The space $H_{b,2}^{(n)}(G)$ is called the singular part of the bounded cohomology group.

In [27], Grigorchuk obtained the following result.

Theorem 1.1. *An isomorphism of vector spaces*

$$H_{b,2}^*(G) \cong H_{b,1}^*(G) \oplus H_{b,2}^*(G)$$

holds.

Let l^1 denote the Banach space of summable sequences of real numbers with the norm $||x_i|| = \sum_{i=1}^{\infty} |x_i|$. Let $|A :: C|$ denote the number of double cosets of A by C, and $A *_C B$ denote an amalgamated free product of groups A and B (for definition of $A *_C B$ see [44]). For amalgamated free product of groups, Fujiwara proved the following results in [23].

Theorem 1.2. *Let $G = A *_C B$. If $|A :: C| \geq 3$ and $|B : C| \geq 2$, then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G, \mathbb{R})$. In particular, the dimension of $H_b^2(G, \mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinality of the continuum.*

Corollary 1.3. *Let $G = A * B$ with $A \neq \{1\}$, $B \neq \{1\}$. If $G \neq \mathbb{Z}_2 * \mathbb{Z}_2$, then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G, \mathbb{R})$. In particular, the dimension of $H_b^2(G, \mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinality of the continuum.*

Corollary 1.4. *Let $G = A *_C B$. If $|A| = \infty$, $|C| < \infty$ and $|B/C| \geq 2$ then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G, \mathbb{R})$. In particular, the dimension of $H_b^2(G, \mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinality of the continuum.*

Corollary 1.5. *Let $G = A *_C B$. If A is abelian, $|A/C| \geq 3$ and $|B/C| \geq 2$ then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G, \mathbb{R})$. In particular, the dimension of $H_b^2(G, \mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinality of the continuum.*

In the case of HNN extensions of groups, Fujiwara proved the following results in [23].

Theorem 1.6. *Let $G = A *_{C,\varphi}$. If $|A/C| \geq 2$ and $|A/\varphi(C)| \geq 2$, then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G, \mathbb{R})$. In particular, the dimension of $H_b^2(G, \mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinality of the continuum.*

Theorem 1.7. *If G is a finitely generated group with infinitely many ends, then there is an injective $\mathbb{R}$-linear map $\omega : l^1 \to H_b^2(G, \mathbb{R})$. In particular, the dimension of $H_b^2(G, \mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinality of the continuum.*

In 1940, Ulam [55] posed the following problem. Given a group G_1, a metric group (G_2, d) and a positive number ε, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $T : G_1 \to G_2$ exists with $d(f(x), T(x)) < \varepsilon$ for all $x, y \in G_1$? The first affirmative answer was given by Hyers [31] in 1941.

Theorem 1.8. (Hyers [31]**).** *Let E_1 and E_2 be Banach spaces. If $f : E_1 \to E_2$ satisfies the inequality*

$$\| f(x+y) - f(x) - f(y) \| < \varepsilon \tag{1.1}$$

for some $\varepsilon > 0$ and for all $x, y \in E_1$, then there exists a unique map $T : E_1 \to E_2$ such that

$$T(x+y) - T(x) - T(y) = 0 \qquad \text{for all} \quad x, y \in E_1 \tag{1.2}$$

and

$$\| f(x) - T(x) \| < \varepsilon \qquad \text{for all} \quad x \in E_1. \tag{1.3}$$

The subject rested there until Rassias [50] considered a generalized version of the previous result which permitted the Cauchy difference to become unbounded. That is, he assumed that

$$\| f(x+y) - f(x) - f(y) \| \leq \varepsilon \, (\|x\|^p + \ |y\|^p) \qquad \text{for all} \quad x, y \in E_1,$$

where ε and p are constants with $\varepsilon > 0$ and $0 \leq p < 1$.

Rassias proved in this case too, that there is an additive function T from E_1 into E_2 such that

$$||T(x) - f(x)|| \leq k \cdot \varepsilon \cdot ||x||^p,$$

where k depends on p as well as ε.

In 1990, during the 27^{th} International Symposium on Functional Equations, Rassias [51] asked whether such a theorem can also be proved for $p \geq 1$. Gajda [24], following the same approach as in [50], gave an affirmative solution to this question for $p > 1$. Several generalizations of these results can be found in [35]–[39] and [50, 51].

In connection with these results the following question arises. Let S be an arbitrary semigroup or group and let a mapping $f : S \to \mathbb{R}$ (the set of reals) be such that the set $\{f(xy) - f(x) - f(y) \mid x, y \in S\}$ is bounded. Is it true that there is a mapping $T : S \to \mathbb{R}$ that satisfies

$$T(xy) - T(x) - T(y) = 0 \quad \text{for all } x, y \in S,$$

and the set $\{T(x) - f(x) \mid x \in S\}$ is bounded. A negative answer was given by Forti [21]. It turns out that the existence of mappings that are "almost homomorphisms" but are not small perturbations of homomorphisms has an algebraic nature.

Definition 1.9. A *quasicharacter* of a semigroup S is a real-valued function f on S such that the set $\{f(xy) - f(x) - f(y)|x, y \in S\}$ is bounded.

Definition 1.10. By a *pseudocharacter* of a semigroup S (group S) we mean its quasicharacter f that satisfies $f(x^n) = nf(x)$ for all $x \in S$ and all $n \in \mathbb{N}$ (and all $n \in \mathbb{Z}$, if S is a group).

The set of quasicharacters of a semigroup S is a vector space (with respect to the usual operations of addition of functions and their multiplication by numbers), which will be denoted by $KX(S)$. The subspace of $KX(S)$ consisting of pseudocharacters will be denoted by $PX(S)$ and the subspace consisting of real additive characters of the semigroup S, will be denoted by $X(S)$. We say that a pseudocharacter φ of the group G is *nontrivial* if $\varphi \notin X(G)$. In the papers [7, 8, 9, 12, 13, 15] a description the set of pseudocharacters of free groups and semigroups, on the free and semidirect products of groups and semigroups were given.

For a real constant c and a mapping f of the group G into a semigroup of linear transformations of a vector space, sufficient conditions of the coincidence of the solution of a functional inequality $\|f(xy) - f(x) \cdot f(y)\| < c$ with the solution of the corresponding functional equation $f(xy) - f(x) \cdot f(y) = 0$ was studied in [2, 30, 43]. In the papers [30, 43], it was independently shown that if a continuous mapping f of a compact group G into the algebra of endomorphisms of a Banach space satisfies the relation $\|f(xy) - f(x) \cdot f(y)\| \leq \delta$ for all $x, y \in G$ with a sufficiently small $\delta > 0$, then it is ε-close to a continuous representation g of the same group in the same Banach space (that is, we have $\|f(x) - g(x)\| < \varepsilon$ for all $x \in G$).

The study of pseudocharacters and quasicharacters as independent objects began in the papers [7]–[15]. However earlier in the paper [53] quasicharacters were constructed to investigate the problem of expressibility in the theory of groups and in [42] a quasicharacter was constructed in a free group for studying the groups of cohomology of a Banach algebra.

In [17] it was shown that for any group G the following decomposition holds

$$KX(G) = PX(G) \oplus B(G),$$

where $B(G)$ denotes the set of real valued functions on G.

From this result, the following theorem follows (see [27])

Theorem 1.11. *An isomorphism of vector spaces*

$$H_{b,2}^{(2)}(G) \cong PX(G)/X(G)$$

holds.

In the papers [19, 20] an application of pseudocharacters to the problem of expressibility in groups was given.

Let G be an arbitrary group and let S be its subset such that $S^{-1} = S$. Denote by $gr(S)$ the subgroup of G generated by S. We say that the width of the set S is finite if there is a number $k \in \mathbb{N}$ such that any element g of $gr(S)$ is representable in the form

$$g = s_1 s_2 \cdots s_n, \quad \text{where} \quad s_i \in S \cup S^{-1}, \quad n \leq k. \tag{1.4}$$

The minimal k with this property is called *the width* of the set S in G and will be denoted by $wid(S, G)$. We say that the width of the set S in the group G is infinite if for any $k \in \mathbb{N}$ there is an element $g_k \in gr(S)$ which does not have a presentation of the form (1.4). Many papers have been devoted to the problem of the width of different subsets (see for example [1, 3, 6, 26, 46, 53, 54]).

Let V be a finite subset of the free group F of the countable rank. We say that V is proper if the verbal subgroup $V(F)$ is a proper subgroup of F. Let G be an arbitrary group. Denote by $\overline{V}(G)$ the set of values in the group G of all the words from the set V. By the width of verbal subgroup $V(G)$ we mean the width of the set $\overline{V}(G) \cup \overline{V}(G)^{-1}$ in the group G. Many papers have also been devoted to the problem of the width of verbal subgroups (see [3, 26, 53] and references therein).

If the set V contains only one word $[x, y] = x^{-1}y^{-1}xy$ we will say about commutator width.

In the paper [28], Grigorchuk made assumption that if $G = A *_H B$ is an amalgamated free product such that

$$|A :: H| = 2 \quad \text{and} \quad |B : H| = 2 \tag{1.5}$$

then the width of commutator subgroup G' is finite.

The goals of this paper are:

1) To show that Theorems 1.2, 1.6, 1.7 and Corollaries 1.3, 1.4, 1.5 of Fujiwara in [23] are not quite true. Namely, for any Fujiwara's Theorem or Corollary mentioned above and for any cardinal number $\mathcal{M}$, we construct a group $G = A *_H B$ satisfying the assumptions of the corresponding theorem or corollary such that the dimension of the linear space $H_{b,2}^2(G)$ is at least $\mathcal{M}$. Moreover in the paper [23] no information about the group $G = A *_H B$ was given when $|A :: H| = 2$ and $|B : H| = 2$. Using results of this paper,

it can be shown that, in this case too, for any cardinal number $\mathcal{M}$ one can construct a group $G = A *_H B$ such that the dimension of the linear space $H_b^2(G)$ is at least $\mathcal{M}$. Also, we construct a group $G = A*_{C,\varphi}$ such that the dimension of the linear space $H_b^2(G)$ is at least $\mathcal{M}$. Moreover in the paper [23] no information about the group $G = A*_{C,\varphi}$ was given when $|A/H| \leq 2$ or $|A/\varphi(H)| \leq 2$. Using results of this paper, it can be shown that, in this case too, for any cardinal number $\mathcal{M}$ one can construct a group $G = A*_{C,\varphi}$ such that the dimension of the linear space $H_b^2(G)$ is at least $\mathcal{M}$.

2) To show that the assumption of Grigorchuk in [28] is not true. Moreover, from our construction, it will follow that in the case $|A :: H| = 2$ and $|B : H| = 2$, one can construct groups such that the width of every proper verbal subgroups will be infinite.

3) To show the space of pseudocharacters $GL(2, F_2[z])$.

2. Some auxiliary facts

Let G be an arbitrary group and $\tau : G \to C$ be an epimorphism from G onto a group C. Denote by τ^* the mapping that takes each element $\varphi \in PX(C)$ to $\varphi \circ \tau \in PX(G)$. It is evident that τ^* is an embedding of $PX(C)$ into $PX(G)$.

Let $H = A * B$ be the free product of nontrivial groups A and B. There are natural epimorphisms $\tau_A : H \to A$ and $\tau_B : H \to B$. Let τ_A^* and τ_B^* be embedding of the spaces $PX(A)$ and $PX(B)$ into $PX(G)$, respectively. Below we shall identify the spaces $PX(A)$ and $PX(B)$ with their τ_A^* and τ_B^* isomorphic images, respectively. Set $A_0 = A \setminus \{1\}, B_0 = B \setminus \{1\}$ and $M = \{a \cdot b \,|\, a \in A_0, b \in B_0\}$. It is clear that subsemigroup $\widetilde{D}$ of group H generated by the set M is free and M is the system of free generators for $\widetilde{D}$. By D we denote a semigroup generated by $\widetilde{D}$ and 1. Let $v \in D$. By $|v|$ we denote the length of the word v in alphabet M. If $v = 1$ we set $|v| = 0$.

Let $v = a_1 b_1 \cdots a_n b_n \in \widetilde{D}$. By $\overline{v}$ we denote the element $b_1 a_2 b_2 \cdots a_n b_n a_1$. Let $PX(D, -1)$ be the subspace of $PX(D)$ consisting of the pseudocharacters φ of D satisfying the following conditions:

1) the set $\varphi(M)$ is bounded,
2) $\varphi((\overline{v})^{-1}) = -\varphi(v)$, $\forall v \in D$.

Remark 2.1. Recall that by the Proposition 3 from [9] for any pseudocharacter φ of arbitrary semigroup S the relation $\varphi(xy) = \varphi(yx)$ holds for all $x, y \in S$.

Hence a pseudocharacter is constant in a class of conjugate elements in a group because $\varphi(x^{-1}yx) = \varphi(yxx^{-1}) = \varphi(y)$.

Let $\varphi \in PX(D, -1)$. Denote by $\overline{\varphi}$ the function on the group G defining as follows. If element v from G is conjugate to some element $a \in A$ or some element $b \in B$, then we set $\overline{\varphi}(v) = 0$. Otherwise we set $\overline{\varphi}(v) = \varphi(t)$, where $t \in D$ and elements v and t are conjugate in G. Remark 2.1 implies that the function $\overline{\varphi}$ is well defined. It is clear that the function $\overline{\varphi}$ is constant on the classes of conjugacy in H. Denote by $\sim$ the relation of conjugacy in the group H.

In [16] the following two theorems were established.

Theorem 2.2. *Let $\varphi \in PX(D, -1)$ and $c > 0$ such that $|\varphi(xy) - \varphi(x) - \varphi(y)| \le c$ for all $x, y \in D$. Then the function $\overline{\varphi}$ is a pseudocharacter of group G such that $\overline{\varphi}\big|_{A\cup B} \equiv 0$ and for any u, v from G the inequality*

$$|\overline{\varphi}(uv) - \overline{\varphi}(u) - \overline{\varphi}(v)| \le 261\, c$$

holds.

Theorem 2.3. *The mapping $\lambda : \varphi \to \overline{\varphi}$ is an embedding of $PX(D, -1)$ into $PX(G)$, and $PX(G) = PX(A) \oplus PX(B) \oplus PX(D, -1)$.*

Since $X(G) \cap PX(D, -1) = \{0\}$, we have the following corollary.

Corollary 2.4. $H^2_{b,2}(G) = PX(A)/X(A) \oplus PX(B)/X(B) \oplus PX(D, -1)$.

3. Some auxiliary facts about free products of groups

Let D^* be a free subsemigroup of the group H generated by the set $M^* = \{ba \mid b \in B_0, a \in A_0\}$. For any word v in alphabet M we introduce the set of "beginnings" $B(v)$ and the set of "ends" $E(v)$ as follows: $B(v) = E(v) = \emptyset$, if $|v| \le 1$, and

$$B(v) = \{x_{i_1}, x_{i_1}x_{i_2}, \dots, x_{i_1}x_{i_2}\dots x_{i_{n-1}}\},$$
$$E(v) = \{x_{i_2}, \dots, x_{i_n}, x_{i_3}\dots x_{i_n}, \dots, x_{i_{n-1}}x_{i_n}, x_{i_n}\},$$

if $v = x_{i_1} \cdot \dots \cdot x_{i_n}, n > 1$.

For any element w in D such that $B(w) \cap E(w) = \emptyset$, the functions $\eta_w(v)$ and $e_w(v)$ were defined in [11] as follows: If $v \in D$, then $\eta_w(v)$ is equal to the number of occurrences of w in the word v, and

$$e_w(v) = \max\{\eta_w(v') \mid v' \sim_D v\}.$$

An element v from the free semigroup D is called *simple* if it is not a nontrivial power of another element $u \in D$. The set of simple elements of semigroup D will be denoted by $\mathcal{P}$. Obviously, if $u \sim_D v$, then $u \in \mathcal{P}$ if and only if $v \in \mathcal{P}$.

By Lemma 8 from [11] we have that in any class of $\sim_D$ conjugate elements belonging to the set $\mathcal{P}$ there is a representative w that satisfies to the condition

$$B(w) \cap E(w) = \emptyset. \tag{3.1}$$

Denote by P the set of representatives w of classes of conjugate elements belonging to $\mathcal{P}$ and satisfying relation (3.1).

It is clear that if w is a word in alphabet M such that $B(w) \cap E(w) = \emptyset$, then the word w^{-1} in alphabet M^* satisfies the condition $B(w^{-1}) \cap E(w^{-1}) = \emptyset$. By Lemma 13 from [11] we have that for any $w \in P$ the function e_w is the pseudocharacter of the semigroup D such that for any u, v in D the relation

$$|e_w(uv) - e_w(u) - e_w(v)| \leq 2$$

holds. A similar pseudocharacter of semigroup group D^* which corresponds to the word w^{-1} will be denoted by $e_{w^{-1}}$. Denote by P_0 a subset of P consisting of elements w such that $w \sim w^{-1}$ in the group H. Let $Q = P \setminus P_0$. The set Q is nonempty (see [16]).

Define a relation $\sim_1$ on the set Q as follows. Set $w_1 \sim_1 w_2$ if and only if either $w_1 = w_2$ or $w_1^{-1} \sim w_2$. It is clear that $\sim_1$ is an equivalence relation such that there are only two elements in each class of $\sim_1$ equivalency.

Let us choose, in each of these classes, a representative. Denote by Q^+ the set of these representatives. By Q_n^+ denote subset of Q^+ consisting of elements of length n in alphabet M. Obviously, if $\varphi \in PX(D, -1)$, then φ is fully defined by its restriction to Q^+.

Now define a function $\pi_w : D \to \mathbb{R}$ by the formula

$$\pi_w(v) = e_w(v) - e_{w^{-1}}(\overline{v}), \quad \forall v \in D.$$

Lemma 3.1. (see [16]) *Let $w \in Q^+$. Then the function π_w is an element of the space $PX(D, -1)$ and the following relation holds*

$$|\pi_w(uv) - \pi_w(u) - \pi_w(v)| \leq 10. \tag{3.2}$$

Lemma 3.2. *Let $w \in Q^+$, $u \in D$. Then*

1) *if $|u| < |w|$, then $\pi_w(u) = 0$;*
2) *if $|u| = |w|$ and u is not conjugate neither w nor to w^{-1}, then $\pi_w(u) = 0$; if $u \sim w^{\varepsilon}$ where $\varepsilon \in \{+1, -1\}$, then $\pi_w(u) = \varepsilon$.*

Lemma 3.3. (See [16]) *Let $n \in \mathbb{N}$ and λ is a bounded function on Q_n^+. Then the function*

$$\psi_\lambda = \sum_{w \in Q_n^+} \lambda(w) \pi_w$$

is an element of the space $BPX(D, -1)$, and for any $u, v \in D$ the following inequality holds:

$$|\, \psi_\lambda(uv) - \psi_\lambda(u) - \psi_\lambda(v) \,| \leq 240\, \lambda_0\, (n - 1), \tag{3.3}$$

where $\lambda_0 = \sup\{\lambda(w) \,|\, w \in Q_n^+\}$. Moreover, for any $w_0 \in Q_n^+$ we have $\psi_\lambda(w_0) = \lambda(w_0)$.

4. Semidirect product

Let G be a group and α be its automorphism. For any $\varphi \in PX(G)$ we set $\varphi^\alpha(x) = \varphi(x^\alpha)$ for all $x \in G$. It is clear that φ^α is a pseudocharacter of G.

Definition 4.1. Let $\varphi \in PX(G)$. The map φ is said to be *invariant* relative to α if $\varphi^\alpha = \varphi$. If this relation holds for each a in $A \subseteq Aut\, G$, we will say that φ is invariant relative to A.

The subspace consisting of pseudocharacters of G invariant relative to A will be denoted by $PX(G, A)$.

Let $G = A \cdot B$ be a semidirect product of its subgroups A and B such that B is an invariant subgroup of G. In [12] it was shown that any element from $PX(B, A)$ can be extended to G as a pseudocharacter that is equal to zero on subgroup A. The following theorem was established in [12].

Theorem 4.2. *Let the group $G = A \cdot B$ be a semidirect product of its subgroups A and B such that B is an invariant subgroup of G. Then*

$$PX(A \cdot B) = PX(A) \oplus PX(B, A). \tag{4.1}$$

Corollary 4.3. $H^{(2)}_{b,2}(A \cdot B) = PX(A)/X(A) \oplus PX(B, A)/X(B, A)$.

By the last theorem, the problem of describing $PX(G)$ is reduced to that of $PX(A)$ and $PX(B, A)$.

We will use the following notations for the rest of this paper. Let A and B be groups, and $H = A * B$ be their free product. Further, let T_1 be a subgroup of $Aut\, A$, and T_2 be a subgroup of $Aut\, B$, and $T = T_1 \times T_2$.

Let $G = T \cdot H$ be the semidirect product such that T acts on H by the rule that

$$a^t = a^{t_1}, \quad b^t = b^{t_2}, \quad a^{t_2} = a, \quad b^{t_1} = b$$

for any $a \in A$, $b \in B$, $t_1 \in T_1$, $t_2 \in T_2$ and $t = t_1 t_2$. The relation of conjugacy in the group H will be denoted by $\sim$ and by $\sim$ we will also denote the relation of conjugacy in the semigroup D.

Definition 4.4. We will say that elements u and v from H are T-conjugate if there is $t \in T$ such that $u^t \sim v$.

The subset of $PX(D, -1)$ consisting of pseudocharacters invariant relative to the group T we denote by $PX(D, -1, T)$.

Lemma 4.5. $PX(H, T) = PX(A, T_1) \oplus PX(B, T_2) \oplus PX(D, -1, T)$.

Corollary 4.6.

$$PX(G) = PX(T) \oplus PX(A, T_1) \oplus PX(B, T_2) \oplus PX(D, -1, T). \tag{4.2}$$

Thus the problem of describing $PX(G)$ is reduced to $PX(D,-1,T)$.

Corollary 4.7.

$$\begin{aligned} H^{(2)}_{b,2}(G) &= PX(T)/X(T) \oplus PX(A,T_1)/X(A,T_1) \\ &\quad \oplus PX(B,T_2)/PX(B,T_2) \oplus PX(D,-1,T). \end{aligned} \tag{4.3}$$

5. DEFINITION OF δ_w

In this section, we recall some facts from the paper [16]. Let $\mathcal{P}$ be a set of simple elements of the semigroup D, and the sets P, P_0 are the same as before. Denote by E_0 a subset of P consisting of elements w such that $w \sim w^{-1}$ in the group G. That is, there is a t in T such that elements w^t and w^{-1} conjugate in the group H. It is easy to verify that $P \setminus E_0 \neq \emptyset$.

Denote by E a system of representatives of classes of T–conjugate elements belonging to $\mathcal{P}$ such that if $w \in E$, then w and w^{-1} are not T–conjugate. As it was shown in [20], the set E is nonempty. The subset of E consisting of elements of the length n in alphabet M will be denoted by E_n. It is clear that we can assume $E \subseteq Q$ and $E_n \subseteq Q_n$. If we let $E^+ = E \cap Q^+$, then $E_n^+ = E \cap Q_n^+$.

Let $M(w)$ be the set of values of the function $t \to w^t$ for $t \in T$. For any $w \in E$ define the function $\delta_w : D \to \mathbb{R}$ by letting

$$\delta_w(v) = \sum_{u \in M(w)} \pi_u(v).$$

From Lemma 3.3 it follows that $\delta_w \in PX(D,-1)$, and if $|w| \geq 2$ then

$$|\,\delta_w(uv) - \delta_w(u) - \delta_w(v)\,| \leq 240\,(|w|-1). \tag{5.1}$$

Proposition 5.1. *For any $w \in E^+$, the function δ_w belongs to the space $PX(D,-1,T)$.*

Proposition 5.2. *Let $w \in E^+$. Then*

1) *For any $u \in M(w)$, we have $e_{u^{-1}}(\overline{w}) = 0$, and*
2) *$\delta_w(w) =$ number of elements in the set $M(w)$ conjugate to w.*

Lemma 5.3. *The set $\{\delta_w \mid w \in E^+\}$ is a system of linearly independent elements of $PX(D,-1,T)$.*

Proof. Suppose that there are different elements $w_1, w_2, \dots, w_k \in E^+$ (we may assume $|w_1| \leq |w_2| \leq \dots \leq |w_k|$) and $r_1, r_2, \dots, r_k \in \mathbb{R} \setminus \{0\}$ such that

$$\sum_{i=1}^{k} r_i \delta_{w_i} \equiv 0.$$

By Collorary 3.2 we have

$$\sum_{i=1}^{k} r_i\delta_{w_i}(w_1) = r_1\delta_{w_1}(w_1) = 0$$

and we have contradiction with Proposition 5.2. □

6. On Fujiwara Theorem 1.2

In this section we will show, using the decomposition (4.2), how to construct examples that will show that the statement of Theorem 1.2 of Fujiwara in [23] is incorrect.

Let us consider a particular case of the amalgamated products of groups $G = A *_T B$. Namely, we will consider below the amalgamated products of two groups that are semidirect products $T \cdot A$ and $T \cdot B$. In this case we have $G = T \cdot A *_T T \cdot B$. It easy to see that in this case the group $G = T \cdot A *_T T \cdot B$ is a semidirect product $G = T \cdot (A * B)$. Hence from (4.2) it follows

$$PX(G) = PX(T) \oplus PX(A,T) \oplus PX(B,T) \oplus PX(D,-1,T), \tag{6.1}$$

$$\begin{aligned} H^{(2)}_{b,2}(G) &= PX(T)/X(T) \oplus PX(A,T)/X(A,T) \\ &\quad \oplus PX(B,T)/X(B,T) \oplus PX(D,-1,T). \end{aligned} \tag{6.2}$$

6.1. DIMENSION OF $PX(T)/X(T)$ **.** Let us construct a class of amalgamated products of groups $\mathcal{K}$ of the form $G = T \cdot A *_T T \cdot B$ such that for any cardinal number $\mathcal{M}$ there is a $G \in \mathcal{K}$ such that the cardinality of the the basis of the space $PX(G)$ is at least $\mathcal{M}$.

Indeed, let T be some group such that the linear dimension of the factor space $PX(T)/X(T)$ is at least $\mathcal{M}$. For example, for such a group we can take a free group F with free generators X such that the cardinality of the set X is at least $\mathcal{M}$. We can construct similar groups using free product of groups.

Now let A and B be an arbitrary non unit groups. Consider semidirect products $T \cdot A$ and $T \cdot B$. For example if the group T acts trivially on A or on B respectively, then we have $T \cdot A = T \times A$ or $T \cdot B = T \times B$ respectively. In this case we have $G = T \cdot A *_T T \cdot B = T \cdot (A * B)$. Hence $PX(G) = PX(T \cdot (A * B)) = PX(T) \oplus PX(A * B, T)$ and we see that the subspace $PX(T)/X(T)$ of $H^{(2)}_{b,2}(G)$ has linear dimension at least $\mathcal{M}$.

If $A = B = \mathbb{Z}_2$ the group of order 2, then we get $|T \cdot A :: T| = |T \cdot A : T| = |T \cdot B :: T| = |T \cdot B : T| = 2$. It is well known that $PX(A * B) = 0$. Hence, we have $PX(G) = PX(T \times (A * B)) = PX(T) \oplus PX((A * B), T) = PX(T)$, and we see that dimension of $H^{(2)}_{b,2}(G)$ can be arbitrarily large.

6.2. DIMENSION OF $PX(A,T)/X(A,T)$. Now using the space $PX(A)$, we consider how to construct the group $G = T \cdot A *_T T \cdot B$ with the required property. Let A and T be some groups and $T \cdot A$ be their semidirect product. Let the linear dimension of $PX(A,T)/X(A,T)$ be at least $\mathcal{M}$.

For example, if the group T acts trivially on A, then we have $T \cdot A = T \times A$. Hence, $PX(A,T)/X(A,T) = PX(A)/X(A)$, and we can choose the group A to be any group such that the dimension of the space $PX(A)/X(A)$ is at least $\mathcal{M}$.

6.3. DIMENSION OF $PX(D,-1,T)$. Let J be a set such that $|J| \geq \mathcal{M}$. Further, let A_i, $i \in J$ be nontrivial groups, and $T_i \subseteq Aut\, A_i$, $A = \prod^{\times}_{i \in J} A_i$, $T = \prod^{\times}_{i \in J} T_i$. Let us continue the action of T_i onto A as follows: If $t_i \in T_i$, $a_j \in A_j$, $i \neq j$, then $a_j^{t_i} \in a_j$. Hence T becomes a subgroup of $Aut\, A$ and we can construct semidirect product $T \cdot A$.

Now let B be an arbitrary group and $T \times B$ be direct product of T and B. Now we can construct the amalgamated product $G = (T \cdot A) *_T (T \times B)$. We may assume that J is an ordered set. Let us denote by J_3 the set of all subset of J consisting three different elements, that is $J_3 = \{(i,j,k) \,|\, i < j < k\}$. For every $i \in J$, let us fix some nonunit element $a_i \in A_i$, and let b be nonunit element from B. Let $p = (i,j,k) \in J_3$. Then $w_p = a_i b a_j b a_k b \in D$, and we can construct elements $e_{w_p} \in PX(D,-1)$ and $\delta_{w_p} \in PX(D,-1,T)$ as above.

Lemma 6.1. *Let p and q be different elements from J_3, then $e_{w_p} \not\sim_T e_{w_q}$ and $e_{w_p} \not\sim_T e_{w_q^{-1}}$. Hence, $\pi_{w_p}(w_p) = 1$ and $\pi_{w_p}(w_q) = 0$.*

Lemma 6.2. *Let p and q be different elements from J_3, then $\delta_{w_p} \not\sim_T \delta_{w_q^{-1}}$. Further $\delta_{w_p}(w_p) = 1$ and $\delta_{w_p}(w_q) = 0$.*

Proof. $\delta_{w_p}(v) = \sum_{u \in M(w_p)} \pi_{w_p}(v)$. Let $u \in M(w_p)$ and $u \neq w_p$. Then for some $t \in T$ we have $u = (w_p)^t = a_i^t b a_j^t b a_i^t b$ and either $a_i^t \neq a_i$ or $a_j^t \neq a_j$ or $a_k^t \neq a_k$. In this case we have $\pi_u(w_p) = 0$. Hence $\delta_{w_p}(w_p) = 1$. □

Corollary 6.3. *The set δ_{w_p}, $p \in J_3$ is linearly independent.*

Proof. Suppose that for some $p_1, \ldots, p_m \in J_3$ and nonzero reals $\lambda_1, \ldots, \lambda_m$ we have $\varphi = \sum_{l=1}^m \lambda_l \delta_{p_l} \equiv 0$. Then by previous lemma we have $\varphi(w_{p_1}) = \lambda_1 = 0, \ldots, \varphi(w_{p_m}) = \lambda_m = 0$, and we come to contradiction with the assumption about λ_l, $l = 1, \ldots, m$. □

From this corollary it follows that the dimension of the space $PX(D,-1,T)$ is at least $\mathcal{M}$.

Remark 6.4. The statements of Corollaries 1.3– 1.5 are not accurate.

Indeed, let A_i, $i \in I$ be abelian groups and all groups T_i are trivial. Then in this case we have $G = (T \cdot A) *_T (T \times B) = A * B$, and as was shown above the cardinality of the space $PX(D, -1)$ is at least $\mathcal{M}$.

7. On Fujiwara Theorem 1.6

In this section, we will show how to construct examples that will show that the statement of Theorem 1.6 of Fujiwara in [23] is incorrect. Let a group G be an HNN extension $G = A *_{C,\varphi}$.

We recall the notion of HNN extension. Let G be an arbitrary group, A and B its subgroups and $\varphi : A \to B$ an isomorphism. Let T be an infinite cyclic group with generator t.

The group K is denoted by $K = G*_{A,\varphi} = \langle G, t;\ t^{-1}at = \varphi(a),\ \forall\, a \in A\rangle$ is an HNN extension of G with connected subgroups A and B. In other words K is a factor group of $G * T$ by its invariant subgroup generated by the set $\{t^{-1}at\varphi(a)^{-1},\ a \in A\}$.

For more on HNN extension, the interested reader is referred to [44].

7.1. FOR THE CASE WHEN $A = B = G$**.** It is clear that in this case we have $K = T \cdot G$, that is, K is a semidirect product of its subgroups T, G and G is invariant in K. By Theorem 4.2 we have $PX(K) = PX(T) \oplus PX(G, T)$. In this case we have $PX(K)/X(K) = PX(G, T)/X(G, T)$.

Now we need to construct a group G such that the dimension of the space $PX(G, T)/X(G, T)$ is at least $\mathcal{M}$. It is not difficult to construct such groups. When the group T acts trivially on G, we obtain $PX(K)/X(K) = PX(G, T)/X(G, T) = PX(G)/X(G)$, and we see that we can construct groups such that the dimension of the space $PX(K)/X(K)$ is at least $\mathcal{M}$.

7.2. FOR THE GENERAL CASE WHEN $A \subseteq G$, $B \subseteq G$**.** Let

$$K = G*_{A,\varphi} = \langle G, t;\ t^{-1}at = \varphi(a),\ \forall\, a \in A\rangle$$

is HNN extension. Let $\varphi(A) = B$ Consider the following HNN extension. Let C be an arbitrary group and G acts on C by automorphisms. Further, let $H = G \cdot C$ their semidirect product. Then subgroup of H generated by A and C is a semidirect product $A_1 = A \cdot C$. More over subgroup of H generated by B and C is a semidirect product $B_1 = B \cdot C$.

Let α be an automorphism of C. Define $\varphi' : A \cdot C \to B \cdot C$ as follows:

$$\varphi'(ac) = \varphi(a)\alpha(c) \quad \forall\, a \in A, \quad \forall c \in C.$$

Lemma 7.1. *The map φ' is an isomorphism if and only if for any $a \in A$ and any $c \in C$ the relation*

$$\alpha(c)^{\varphi(a)} = \alpha(c^a) \tag{7.1}$$

holds.

Suppose that the relation (7.1) holds. Then we can define the following HNN extension

$$Q = (G \cdot C) *_{A \cdot C, \varphi'} = \langle G \cdot C, t;\ t^{-1}xt = \varphi'(x),\ \forall\, x \in A \cdot C \rangle.$$

If the group G acts trivially on C, then we have $G \cdot C = G \times C$, $A \cdot C = A \times C$, and $B \cdot C = B \times C$. Hence, the relation (7.1) is fulfilled and in this case we have

$$Q = (G \times C) *_{A \times C, \varphi'} = \langle G \times C, t;\ t^{-1}xt = \varphi'(x),\ \forall\, x \in A \times C \rangle.$$

Here C is an arbitrary group, $\varphi' : A \times C \to B \times C$ and $\varphi'(a) = \varphi(a)$ for all $a \in A$, $\varphi'(c) = c$ for all $c \in C$.

Let L be a subgroup of Q generated by G and T. It is easy to see that C and L are normal subgroups of Q such that $Q = C \times L$. Hence, $PX(Q) = PX(C) \oplus PX(L)$. Because C is an arbitrary group we see that for any cardinal number $\mathcal{M}$ we can construct a group Q which is HNN extension and the dimension of the space $PX(Q)/X(Q)$ is at least $\mathcal{M}$.

8. On Grigorchuk's assumption

In this section we will show that the assumption made by Grigorchuk is not true. Let A and B be a cyclic group of order two. Then we have $T \cdot A = T \times A$, $T \cdot B = T \times B$ and $G = T \cdot A *_T T \cdot B = T \cdot (A * B) = T \times (A * B)$. The group $A * B = \mathbb{Z}_2 * \mathbb{Z}_2$ is amenable. Hence $PX(A * B) = X(A * B) = 0$, and we see that $PX(G) = PX(T) \oplus PX(A * B) = PX(T)$.

Thus if we take T to be a group with nontrivial pseudocharacters φ, we obtain that the width of commutator subgroup T' of the group T is infinite. Indeed, if we suppose that there is $k \in \mathbb{N}$ such that every element t of T' can be represented as a product of no more than k commutators, then we obtain that φ is bounded on T. Indeed, suppose for some $c > 0$ we have $|\varphi(xy) - \varphi(x) - \varphi(y)| \le c$ for all $x, y \in T$. Then we get $|\varphi([x, y]) - \varphi(x^{-1}) - \varphi(y^{-1}xy)| = |\varphi([x, y]) + \varphi(x) - \varphi(y^{-1}xy)| = |\varphi([x, y])| \le c$. Hence for any $t \in T'$ we get $|\varphi(t)| \le (k - 1)c$, and we see that φ is bounded on T. Thus $\varphi \equiv 0$, and we came to contradiction with the assumption regarding φ.

The group T is an epimorphic image of G, hence if the group T has the property that for a word W the verbal subgroup $W(T)$ has infinite width, then the verbal subgroup $W(G)$ of G also has infinite width.

9. On the group $G = GL(2, F_2[z])$

Let F_2 be a field consisting of two elements $\{0, 1\}$, and let $F_2[z]$ be the ring of polynomials over F_2. Further, let T be a subgroup of the group $A =$

$GL(2,F_2)$ consisting of matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Denote by t the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, hence $t^2 = 1$. Let

$$Q = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

If $a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, then $a^2 = a^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then Q is a subgroup of order three.

Let B be the subgroup of $G = GL(2,F_2[z])$ consisting of matrices

$$\begin{bmatrix} 1 & f(z) \\ 0 & 1 \end{bmatrix}; \quad \text{where} \quad f(z) \in F_2[z].$$

It is clear that $T \subset B$. It is well known that the group $G = GL(2,F_2[z])$ is an amalgamated product $G = A *_T B$ (see [44]). It is clear that B is an abelian group such that for any $b \in B$ we have $b^2 = 1$. Let B_n be subgroup of B generated by $b_n = \begin{bmatrix} 1 & z^n \\ 0 & 1 \end{bmatrix}$.

Lemma 9.1. 1) *Q is normal subgroup in A, A is semidirect products $A = T \cdot Q$ and $t^{-1}at = a^{-1}$.*

2) *Elements*

$$P = \left\{ \begin{bmatrix} 1 & \varphi(z) \\ 0 & 1 \end{bmatrix}; \quad \varphi(0) = 0 \right\}.$$

form a subgroup of B and $P = \prod^{\times}_{n \in \mathbb{N}} B_n$.

3) *B is direct product $B = T \times P$.*

Proof. The proof is obtained by direct calculations. □

Corollary 9.2. 1) *Subgroup H of G generated by Q and P is their free product.* 2) *H is invariant in G, and G is semidirect product $G = T \cdot H$.*

Hence to describe of the space of $H^{(2)}_{2,b}(G)$ we can use the results of the Section 4. In our situation we have $PX(T) = PX(Q) = PX(P) = 0$.

Corollary 9.3. $H^{(2)}_{b,2}(G) = PX(D, -1, T)$.

10. Description of the space $PX(D, -1, T)$

Now let us describe the space $PX(D, -1, T)$. In what follows we will use results of Sections $2-5$.

Let $\overline{B}(w) = B(w) \cup \{w\}$ and $\overline{E}(w) = E(w) \cup \{w\}$. We set $\mu_{u,v}(w) = 1$ if there exist x and y such that $x \in \overline{E}(w)$, $y \in \overline{B}(w)$, and $w = xy$; otherwise we set $\mu_{u,v}(w) = 0$. Similarly the measures $\mu_{u,v}$ for all $u, v \in D^*$ on Q^{-1} are defined. Here Q is the same as in Sections $3-5$. Let

$$\nu_{u,v}(w) = \mu_{u,v}(w) + \mu_{uv,uv}(w) - \mu_{u,u}(w) - \mu_{v,v}(w).$$

Hence, the measure $\nu_{u,v}$ takes values from the set $\{-2, -1, 0, 1, 2\}$. In [11] it was shown that for any $u, v \in D$ and for any $w \in Q^+$, the following equality

$$e_w(uv) - e_w(u) - e_w(v) = \nu_{u,v}(w) \tag{10.1}$$

holds.

Let the group $C = A * B$ be a free product of A and B.

Definition 10.1. By *canonical* form of nonunit element g from $G = A * B$ we mean its presentation in the form $g = c_1 c_2 \cdots c_n$, where $c_i \in A_0 \cup B_0$, and $c_i c_{i+1} \notin A \cup B$.

Let $v = c_1 c_2 \cdots c_k c_{k+1}$ be a canonical form. Then we set $\dot{v} = c_1$, $\ddot{v} = c_{k+1}$, $\widetilde{v} = 1$, if $k = 1$ and $\widetilde{v} = c_2 \cdots c_k$, if $k > 1$. For any $t \in M^*$ and any $w \in Q^+$ we set $r_t(w) = 1$, if $t = w^{-1}$ and $r_t(w) = 0$, if $t \neq w^{-1}$. For any u, v in D that differ from unit element, let us define a measure $\zeta_{u,v}(w)$ on Q^+ as follows:

$$\begin{aligned}\zeta_{u,v}(w) = r_{\ddot{u}\dot{v}}(w) + r_{\ddot{v}\dot{u}}(w) - r_{\ddot{u}\dot{u}}(w) - r_{\ddot{v}\dot{v}}(w) \\ +\nu_{\widetilde{u}\ddot{u}\dot{v},\widetilde{v}\ddot{v}\dot{u}}(w^{-1}) + \nu_{\widetilde{u},\ddot{u}\dot{v}}(w^{-1}) + \nu_{\widetilde{v},\ddot{v}\dot{u}}(w^{-1}) \\ -\nu_{\widetilde{u},\ddot{u}\dot{u}}(w^{-1}) - \nu_{\widetilde{v},\ddot{v}\dot{v}}(w^{-1}).\end{aligned} \tag{10.2}$$

Lemma 10.2. (see [16]) *For any $w \in Q^+$ and $u, v \in D$ the following equality holds*

$$e_{w^{-1}}(\overline{uv}) - e_{w^{-1}}(\overline{u}) - e_{w^{-1}}(\overline{v}) = \zeta_{u,v}(w).$$

If $w \in M$, that is, the length of the word w in alphabet M equal to one, then it is clear that

$$\nu_{\widetilde{u}\ddot{u}\dot{v},\widetilde{v}\ddot{v}\dot{u}}(w^{-1}) = \nu_{\widetilde{u},\ddot{u}\dot{v}}(w^{-1}) = \nu_{\widetilde{v},\ddot{v}\dot{u}}(w^{-1}) = \nu_{\widetilde{u},\ddot{u}\dot{u}}(w^{-1}) = \nu_{\widetilde{v},\ddot{v}\dot{v}}(w^{-1}) = 0.$$

Hence

$$\zeta_{u,v}(w) = r_{\ddot{u}\dot{v}}(w) + r_{\ddot{v}\dot{u}}(w) - r_{\ddot{u}\dot{u}}(w) - r_{\ddot{v}\dot{v}}(w), \qquad w \in M. \tag{10.3}$$

If $w \notin M$, then

$$r_{\ddot{u}\dot{v}}(w) = r_{\ddot{v}\dot{u}}(w) = r_{\ddot{u}\dot{u}}(w) = r_{\ddot{v}\dot{v}}(w) = 0,$$

and by (10.2) we get

$$\zeta_{u,v}(w) = \nu_{\widetilde{u}\ddot{u}\dot{v},\widetilde{v}\ddot{v}\dot{v}}(w^{-1}) + \nu_{\widetilde{u},\ddot{u}\dot{v}}(w^{-1}) \\ +\nu_{\widetilde{v},\ddot{v}\dot{u}}(w^{-1}) - \nu_{\widetilde{u},\ddot{u}\dot{u}}(w^{-1}) - \nu_{\widetilde{v},\ddot{v}\dot{v}}(w^{-1}). \quad (10.4)$$

For any $w \in Q^+$ and $u, v \in D$ define $\Theta_{u,v}(w)$ by setting

$$\Theta_{u,v}(w) = \nu_{u,v}(w) - \zeta_{u,v}(w). \quad (10.5)$$

Remark 10.3. Let us note that if either $\dot{u} = \dot{v}$ or $\ddot{u} = \ddot{v}$, then $\Theta_{u,v}(w) = 0$ for all $w \in M$. Indeed, for example if $\dot{u} = \dot{v}$, then we obtain $\ddot{u}\dot{v} = \ddot{u}\dot{u}$ and $\ddot{v}\dot{u} = \ddot{v}\dot{v}$.

Hence, (10.5) implies $\Theta_{u,v}(w) = 0$. Thus, if $\Theta_{u,v}(w) \neq 0$, then $\dot{u} \neq \dot{v}$ and $\ddot{u} \neq \ddot{v}$. From the latter we obtain that the words $\ddot{u}\dot{v}$, $\ddot{v}\dot{u}$, $\ddot{u}\dot{u}$, $\ddot{v}\dot{v}$ are pair wise different. Hence, the measure $\Theta_{u,v}$ on Q_1^+ takes values from the set { -1, 0, 1 }.

The measure $\nu_{u,v}$ is a sum of four measures of the form $\mu_{u,v}$, and for $n > 1$ the measure $\Theta_{u,v}$ is the sum of 24 measures of the form $\mu_{u,v}$. Hence for any $n > 1$ the following relations hold:

$$|\operatorname{supp} \nu_{u,v} \cap Q_n^+| \leq 4\,(n-1) \quad (10.6)$$

$$|\operatorname{supp} \Theta_{u,v} \cap Q_n^+| \leq 24\,(n-1). \quad (10.7)$$

Note that if $w \in M$, then $|\Theta_{u,v}(w)| \leq 4$, and if $w \notin M$, then $|\Theta_{u,v}(w)| \leq 10$.

Lemma 10.4. (see [16]) *Let $w \in Q^+$. then the following relations hold*

$$\pi_w(uv) - \pi_w(u) - \pi_w(v) = \Theta_{u,v}(w), \quad \text{and} \quad |\Theta_{u,v}(w)| \leq 10. \quad (10.8)$$

It is clear that for any $w \in D$ we have the estimate $1 \leq |\delta_w(w)| \leq |w|$. Set $\widehat{\delta}_w = \frac{1}{\delta_w(w)}\delta_w$, then $\widehat{\delta}_w(w) = 1$, $\delta_w(v) = \delta_w(w)\widehat{\delta}_w(v)$. Set

$$\Theta_{u,v}^T(w) = \sum_{g \in M(w)} \Theta_{u,v}(g).$$

Proposition 10.5. *Let $|w| = n \geq 2$. Then*

$$|\Theta_{u,v}^T(w)| \leq 240\,(|w|-1)|\operatorname{supp} \Theta_{u,v}^T \cap E_n^+| \leq 240(n-1)^2.$$

Proposition 10.6. *Let λ be a bounded function on E_n^+ for all $n \in N$. Then the functions*

$$\psi_\lambda = \sum_{w \in E_n^+} \lambda(w)\delta_w \quad \text{and} \quad \overline{\psi}_\lambda = \sum_{w \in E_n^+} \lambda(w)\widehat{\delta}_w$$

belong to $PX(D, -1, T)$, and $\overline{\psi}_\lambda(w) = \lambda(w)$ for all $w \in E_n^+$.

Denote by $E(D)$ the set of functions φ on the semigroup D satisfying the relations:

1) $\varphi(x^n) = n\varphi(x)$ for all $n \in \mathbb{N}$ and for all $x \in D$;
2) $\varphi(xy) = \varphi(yx)$ for all $x, y \in D$;
3) $\varphi((\overline{v})^{-1}) = -\varphi(v)$ for all $v \in D$;
4) $\varphi\big|_{E_i^+}$ is a bounded function for all $i \in \mathbb{N}$;
5) $\varphi(x^t) = \varphi(x)$ for all $x \in D$ and for all $t \in T$.

It is clear that $E(D)$ is a linear space (with respect to ordinary operations).

Lemma 10.7. *Let $\varphi \in PX(D, -1)$. Then φ is bounded on Q_n^+ for all $n \in \mathbb{N}$.*

From Lemma 10.7 it follows that $PX(D, -1, T)$ is subspace of $E(D)$. Denote by $L(E^+)$ the space of real-valued functions α on E^+ satisfying the following condition: $\alpha\big|_{E_n^+}$ is bounded for any $n \in \mathbb{N}$.

Let us construct a mapping Δ between the spaces $E(D)$ and $L(E^+)$. Let $\varphi \in E(D)$. For each $i \in \mathbb{N}$ we define the function $\alpha_i : E_i^+ \to \mathbb{R}$ by induction as follows: $\alpha_1 \equiv \varphi\big|_{E_1^+}$, and if the values $\alpha_1, \dots, \alpha_n$ have already been defined, then we set

$$\alpha_{n+1} = (\varphi - \sum_{i=1}^{n} \varphi_{\alpha_i})\big|_{Q_{n+1}^+}(w), \quad w \in Q_{n+1}^+. \tag{10.9}$$

Here φ_{α_i} are pseudocharacters introduced by the formula

$$\varphi_{\alpha_i} = \sum_{w \in E_i^+} \alpha(w)\widehat{\delta}_w. \tag{10.10}$$

Now we define the function $\alpha = \Delta(\varphi)$ via its restriction to E_i^+ by setting $\alpha\big|_{E_i^+} = \alpha_i$.

Theorem 10.8. *Δ is an isomorphism between the linear spaces $E(D)$ and $L(E^+)$.*

Denote by $L(E^+, \Theta)$ a subspace of $L(E^+)$, consisting of functions $\alpha \in L(E^+)$ such that the quantities

$$\left| \int_{E^+} \alpha d\Theta_{u,v} \right| \quad u, v \in D$$

are uniformly bounded.

Theorem 10.9. 1) *The mapping Δ establishes an isomorphism between linear spaces $PX(D, -1, T)$ and $L(E^+, \Theta^T)$.*

2) *Each element φ from the space $PX(D,-1,T)$ is uniquely representable in the form*

$$\varphi = \sum_{w \in E^+} \alpha(w)\delta_w, \quad \textit{where} \quad \alpha \in L(E^+, \Theta^T)$$

or in the form

$$\varphi = \sum_{w \in E^+} \beta(w)\widehat{\delta}_w, \quad \textit{where} \quad \beta(w) = \frac{\alpha(w)}{\delta_w(w)}.$$

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