

CHARACTERIZATIONS OF MEASURABILITY-PRESERVING ERGODIC TRANSFORMATIONS

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ABSTRACT. Let (S, \mathfrak{A}, μ) be a finite measure space and let $\phi: S \rightarrow S$ be a transformation which preserves the measure μ . The purpose of this paper is to give some (measure theoretical) necessary and sufficient conditions for the transformation ϕ to be measurability-preserving ergodic with respect to μ . The obtained results extend well-known results for invertible ergodic transformations and complement the previous work of R.E. Rice on measurability-preserving strong-mixing transformations.

1. INTRODUCTION

The aim of this paper is to present characterizations of measurability-preserving ergodic transformations of a finite measure space which are based on the ergodic concepts in measure set-theoretic form and the well-known results for invertible and ergodic measure-preserving transformations (see, e.g., [1], pp. 14–21 and [11], Ch. 1).

Suppose (S, \mathfrak{A}, μ) is a finite measure space. A transformation $\phi: S \rightarrow S$ is called: (i) *measurable* (μ -*measurable*) if, for any A in \mathfrak{A} , the inverse image $\phi^{-1}(A)$ is in \mathfrak{A} ; (ii) *measure-preserving* if ϕ is measurable and $\mu(\phi^{-1}(A)) = \mu(A)$ for any A in \mathfrak{A} ; (iii) *ergodic* if the only members A of \mathfrak{A} with $\phi^{-1}(A) = A$ satisfy $\mu(A) = 0$ or $\mu(S \setminus A) = 0$; (iv) (*strong-*) *mixing* (with respect to μ) if ϕ is μ -measurable and

$$\lim_{n \rightarrow \infty} \mu(\phi^{-n}(A) \cap B) = \frac{\mu(A)\mu(B)}{\mu(S)} \quad (1)$$

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for any two μ - measurable subsets A, B of S . We say that the transformation $\phi : S \rightarrow S$ is *invertible* if ϕ is one-to-one and such that $\phi(A)$ is μ - measurable whenever A is μ - measurable subset of S .

If ϕ is a strong-mixing transformation of a finite measure space (S, \mathfrak{A}, μ) , then, as is well-known, ϕ is both measure-preserving and ergodic. Furthermore, if $\phi : S \rightarrow S$, in addition (to being strong-mixing on S with respect to μ), is invertible, then (1) is equivalent to (the well-known result):

$$\lim_{n \rightarrow \infty} \mu(\phi^n(A) \cap B) = \frac{\mu(A)\mu(B)}{\mu(S)} \quad (2)$$

for any μ - measurable subsets A, B of S .

Investigations have shown, however, that many important consequences of (2) persist in the absence of invertibility (see [8] and [9], § 11.2 - 11.4) and/or the property of strong - mixing (see [10], [5] and [6]). The following result (the most useful result of these investigations for the goals of this paper) is due to R.E. Rice ([8], Theorem 1):

Theorem A. *Let ϕ be a strong-mixing transformation on the normalised measure space (probability space) (S, \mathfrak{A}, μ) . If ϕ is forward measurable, i.e., if $\phi(A)$ is μ - measurable whenever A is μ - measurable subset of S , then for any μ - measurable subsets A, B of S ,*

$$\lim_{n \rightarrow \infty} \mu(\phi^n(A) \cap B) = \mu(B) \lim_{n \rightarrow \infty} \mu(\phi^n(A)). \quad (3)$$

Theorem A has many consequences which are of interest because of the extreme simplicity of both their mathematical and physical realizations. These consequences have great relevance in the discussion of the recurrence paradox of Statistical Mechanics (see [3], [4], [7], [9] and [10]). It is therefore interesting to investigate how the conclusions of Theorem A must be modified when the forward measurable transformation ϕ (i.e., the transformation ϕ which preserves μ - measurability) is assumed to have properties weaker than strong-mixing. In this direction we consider a case when the forward measurable transformation ϕ is assumed to have measure-preserving and ergodic properties. Such transformations we will call *measurability - preserving ergodic transformations*. Note that these transformations are generalizations of the invertible ergodic transformations (they are not necessarily one - to - one).

An example of a measurability - preserving ergodic transformations which is not invertible is given by $\phi(x) = \{2x\}$ (the fractional part of $2x$) on the half - open unit interval $S: = [0, 1)$, where \mathfrak{A} consist of the Borel subsets of S , with Lebesgue measure for μ . In this case we have $\phi([0, \frac{1}{2})) = S$ and therefore it is not generally true for noninvertible ergodic transformations that $\mu(\phi(A)) = \mu(A)$, even when $\phi(A)$ is measurable.

In the sequel we give a full treatment of the class of all measurability-preserving ergodic transformations of a finite measure space, giving their characterizations which also extend well-known results for invertible ergodic transformations and represent the corresponding analogues of the above result (3) of Rice for measurability-preserving strong-mixing transformations.

2. MAIN RESULTS

We begin with the consideration of a very general and subtle (measure set-theoretic) characterization of measurability-preserving ergodic transformations, which represents the corresponding analogue of well-known results for ergodic transformations (not necessarily invertible and not necessarily having the property of being measurability-preserving) (see [1], pp. 14, 19, 30–32, 36–38; [2], pp. 11–16 and [11], pp. 19–40).

Definition 1. Suppose that (S, \mathfrak{A}, μ) is a finite measure space.

(a) A transformation $\phi: S \rightarrow S$ is *measurability-preserving* (preserves μ -measurability) if, for any A in \mathfrak{A} , the image $\phi(A)$ is in \mathfrak{A} .

(b) We say that $\phi: S \rightarrow S$ is a *measurability-preserving ergodic transformation* if ϕ is measurability-preserving and ergodic measure-preserving.

Theorem 1. *Let (S, \mathfrak{A}, μ) be a finite measure space and let $\phi: S \rightarrow S$ be a transformation which preserves the measure μ and μ -measurability. Then the following statements are equivalent:*

- (i) ϕ is ergodic.
- (ii) The only members B of \mathfrak{A} with $\mu(B \Delta \phi(B)) = 0$ satisfy $\mu(B) = 0$ or $\mu(S \setminus B) = 0$ (where Δ is the symbol for the symmetric difference of sets).
- (iii) For all $A \in \mathfrak{A}$ with $\mu(A) > 0$ we have $\mu(\bigcup_{n=1}^{\infty} \phi^n(A)) = \mu(S)$.
- (iv) For all $A, B \in \mathfrak{A}$ with $\mu(A) > 0$ and $\mu(B) > 0$ there is a positive integer n such that

$$\mu(A \cap \phi^n(B)) > 0.$$

Proof. (i) \Rightarrow (ii). Let B in \mathfrak{A} and $\mu(B \Delta \phi(B)) = 0$. For each n in the set of natural numbers $\mathbb{N}(= \{1, 2, 3, \dots\})$ we have

$$\begin{aligned} \phi(B) \Delta \phi^{-n}(\phi(B)) &\subseteq \bigcup_{i=0}^{n-1} [\phi^{-i}(\phi(B)) \Delta \phi^{-(i+1)}(\phi(B))] \\ &= \bigcup_{i=0}^{n-1} [\phi^{-i}(\phi(B)) \Delta \phi^{-1}(\phi(B))] \end{aligned}$$

and hence $\mu(\phi(B) \Delta \phi^{-n}(\phi(B))) \leq n\mu(\phi(B) \Delta \phi^{-1}(\phi(B)))$.

However, since $\mu(B \Delta \phi(B)) = 0$, it follows that $\mu(\phi(B) \setminus B) = 0$ and hence $\mu(\phi(B) \setminus \phi^{-1}(\phi(B))) = 0$ (because $\phi^{-1}(\phi(B)) \supseteq B$ implies $\phi(B) \setminus \phi^{-1}(\phi(B)) \subseteq \phi(B) \setminus B$). Since

$$\begin{aligned}\phi^{-1}(\phi(B)) &= [\phi^{-1}(\phi(B)) \setminus \phi(B)] \cup [\phi(B) \cap \phi^{-1}(\phi(B))], \\ \phi(B) &= [\phi(B) \setminus \phi^{-1}(\phi(B))] \cup [\phi(B) \cap \phi^{-1}(\phi(B))]\end{aligned}$$

and $\mu(\phi^{-1}(\phi(B))) = \mu(\phi(B))$, we have $\mu(\phi^{-1}(\phi(B)) \setminus \phi(B)) = 0$, whence

$$\mu(\phi(B) \Delta \phi^{-1}(\phi(B))) = 0.$$

Therefore

$$\mu(\phi(B) \Delta \phi^{-n}(\phi(B))) = 0 \tag{4}$$

for each $n \in \mathbb{N}_0 (= \{0, 1, 2, 3, \dots\})$.

Let $E := \limsup_{n \rightarrow \infty} \phi^{-n}(\phi(B))$. By (4) we have

$$\begin{aligned}\mu(\phi(B) \Delta \bigcup_{i=n}^{\infty} \phi^{-i}(\phi(B))) &\leq \mu\left(\bigcup_{i=n}^{\infty} (\phi(B) \Delta \phi^{-i}(\phi(B)))\right) \\ &\leq \sum_{i=n}^{\infty} \mu(\phi(B) \Delta \phi^{-i}(\phi(B))) = 0\end{aligned} \tag{5}$$

for each $n \in \mathbb{N}_0$. Since the sequence $(\bigcup_{i=n}^{\infty} \phi^{-i}(\phi(B)) : n \in \mathbb{N}_0)$ is decreasing and $\mu(\bigcup_{i=n}^{\infty} \phi^{-i}(\phi(B))) = \mu(\phi(B))$ for each $n \in \mathbb{N}_0$, we have $\mu(E) = \mu(\phi(B))$. Also we know that

$$\begin{aligned}\phi^{-1}(E) &= \bigcap_{n=0}^{\infty} \left[\bigcup_{i=n}^{\infty} \phi^{-1}(\phi^{-i}(\phi(B))) \right] = \bigcap_{n=0}^{\infty} \left[\bigcup_{i=n}^{\infty} \phi^{-(i+1)}(\phi(B)) \right] \\ &= \bigcap_{n=0}^{\infty} \left[\bigcup_{j=n+1}^{\infty} \phi^{-j}(\phi(B)) \right] = E.\end{aligned}$$

Thus we have obtained a set E with $\phi^{-1}(E) = E$ and $\mu(\phi(B) \Delta E) = 0$. By ergodicity of ϕ we must have $\mu(E) = 0$ or $\mu(S \setminus E) = 0$ and hence $\mu(\phi(B)) = 0$ or $\mu(S \setminus \phi(B)) = 0$, whence $\mu(B) = 0$ or $\mu(S \setminus B) = 0$ since $\mu(\phi(B) \Delta B) = 0$.

(ii) \Rightarrow (iii). Let A in \mathfrak{A} with $\mu(A) > 0$ and let $B = \bigcup_{n=1}^{\infty} \phi^n(A)$. Then we have $\phi(B) \subseteq B \subseteq \phi^{-1}(\phi(B))$, whence by $\mu(\phi^{-1}(\phi(B))) = \mu(\phi(B))$, $\mu(\phi(B)) = \mu(B)$. Therefore $\mu(B \Delta \phi(B)) = 0$, and hence by (ii), $\mu(B) = 0$ or $\mu(B) = \mu(S)$. However, we cannot have $\mu(B) = 0$ because $\phi(A) \subseteq B$ and $\mu(\phi(A)) = \mu(\phi^{-1}(\phi(A))) \geq \mu(A) > 0$. Therefore $\mu(B) = \mu(S)$.

(iii) \Rightarrow (iv). Let $A, B \in \mathfrak{A}$ with $\mu(A) > 0$, $\mu(B) > 0$. By (iii) we have $\mu(\bigcup_{n=1}^{\infty} \phi^n(B)) = \mu(S)$ so that $0 < \mu(A) = \mu(A \cap \bigcup_{n=1}^{\infty} \phi^n(B)) =$

$\mu(\bigcup_{n=1}^{\infty} A \cap \phi^n(B))$. This implies that $\mu(A \cap \phi^n(B)) > 0$ for some positive integer n .

(iv) \Rightarrow (i). Suppose that A in \mathfrak{A} and $\phi^{-1}(A) = A$. If $0 < \mu(A) < \mu(S)$, then $\mu(S \setminus A) > 0$. Next, from $\phi^{-1}(A) = A$ it follows that $\phi(A) = \phi(\phi^{-1}(A)) \subseteq A$, whence $\phi^n(A) \subseteq A$ for all $n \in \mathbb{N}$. Hence $0 = \mu(A \cap (S \setminus A)) = \mu(\phi^n(A) \cap (S \setminus A))$ for all $n \in \mathbb{N}$, which contradicts (iv). This completes the proof of Theorem 1. \square

Remark 1. Since $\bigcup_{n=N}^{\infty} \phi^n(A) = \phi^N(\bigcup_{m=0}^{\infty} \phi^m(A))$ for every natural number N and, by (ii) \Rightarrow (iii) in the proof of Theorem 1, $\mu(\phi(\bigcup_{n=1}^{\infty} \phi^n(A))) = \mu(\bigcup_{n=1}^{\infty} \phi^n(A))$, it follows that we could replace (iii) in Theorem 1 by the statement “For every $A \in \mathfrak{A}$ with $\mu(A) > 0$ and every natural number N we have $\mu(\bigcup_{n=N}^{\infty} \phi^n(A)) = \mu(S)$ ”. Consequently we could replace (iv) in Theorem 1 by “For every $A, B \in \mathfrak{A}$ with $\mu(A) > 0$, $\mu(B) > 0$ and every natural number N there exists $n > N$ with $\mu(A \cap \phi^n(B)) > 0$ ”. The statement (iii) in Theorem 1 may be restated as follows: *In the case of a measurability-preserving ergodic transformation $\phi: S \rightarrow S$ (i.e., in the case of a measurability-preserving ergodic dynamical system with discrete time $(S, \mathfrak{A}, \mu, \phi)$), for any measurable set $A (A \subseteq S)$ of positive measure, the orbit $\{\phi^n(A)\}_{n=0}^{\infty}$ is the entire phase space S with the exception of a set of zero measure.*

Following the previous characterization, another useful characterization of the measurability-preserving ergodic transformations is given by the next theorem, which extends the above result of Rice (i.e., Theorem A) from strong-mixing transformations on a normalised measure space to ergodic measure-preserving transformations of a finite measure space and also generalizes well-known results for invertible (ergodic or mixing) transformations.

Theorem 2. *Let (S, \mathfrak{A}, μ) be a finite measure space and let $\phi: S \rightarrow S$ be a transformation which preserves the measure μ and μ -measurability. Then ϕ is ergodic if and only if*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(A \cap \phi^i(B)) = \frac{\mu(A)}{\mu(S)} \cdot \lim_{n \rightarrow \infty} \mu(\phi^n(B)) \quad (6)$$

holds for all $A, B \in \mathfrak{A}$.

Proof. (i) Suppose ϕ is ergodic. Then from the Birkhoff ergodic theorem (cf. [11], Theorem 1.14, p. 34) one can deduce

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(\phi^i(x)) = \frac{\mu(A)}{\mu(S)} \quad \text{a.e.,}$$

where χ_A denote characteristic function of $A \in \mathfrak{A}$, whence, integrating over $B \in \mathfrak{A}$ and applying the dominated convergence theorem yields

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(\phi^{-i}(A) \cap B) = \frac{\mu(A)\mu(B)}{\mu(S)} \quad (7)$$

for all $A, B \in \mathfrak{A}$ because $\chi_A(\phi^i(x)) = \chi_{\phi^{-i}(A)}(x)$. Since

$$\phi^{-n}(A \cap \phi^n(B)) = \phi^{-n}(A) \cap \phi^{-n}(\phi^n(B)) \supseteq \phi^{-n}(A) \cap B \quad (8)$$

for all subsets A, B of S and every natural number n , it follows that (because ϕ preserves measure μ), for all $A, B \in \mathfrak{A}$,

$$\mu(A \cap \phi^n(B)) = \mu(\phi^{-n}(A \cap \phi^n(B))) \geq \mu(\phi^{-n}(A) \cap B). \quad (9)$$

Next, by (7) and (9), for every non-negative integer k and every $A, B \in \mathfrak{A}$ we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(A \cap \phi^i(B)) &= \liminf_{m \rightarrow \infty} \frac{1}{m+k} \sum_{i=0}^{m+k-1} \mu(A \cap \phi^i(B)) \\ &= \liminf_{m \rightarrow \infty} \frac{1}{m+k} \sum_{j=0}^{m-1} \mu(A \cap \phi^{k+j}(B)) = \liminf_{m \rightarrow \infty} \frac{1}{m} \sum_{j=0}^{m-1} \mu(A \cap \phi^j(\phi^k(B))) \\ &\geq \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{j=0}^{m-1} \mu(\phi^{-j}(A) \cap \phi^k(B)) = \frac{\mu(A)}{\mu(S)} \mu(\phi^k(B)). \end{aligned} \quad (10)$$

From

$$\phi^{-1}(\phi^n(B)) = \phi^{-1}[\phi(\phi^{n-1}(B))] \supseteq \phi^{n-1}(B)$$

we have, for all $n \in \mathbb{N}$,

$$\mu(S) \geq \mu(\phi^n(B)) = \mu(\phi^{-1}(\phi^n(B))) \geq \mu(\phi^{n-1}(B)), \quad (11)$$

whence we obtain that the sequence $(\mu(\phi^n(B)))_{n=1}^{\infty}$ is bounded and non-decreasing. Thus the $\lim_{n \rightarrow \infty} \mu(\phi^n(B))$ exists. Now, using (10), for all μ -measurable subsets $A, B \subseteq S$, we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(A \cap \phi^i(B)) \geq \frac{\mu(A)}{\mu(S)} \lim_{k \rightarrow \infty} \mu(\phi^k(B)). \quad (12)$$

Since A was an arbitrary μ -measurable subset of S , (12) must hold with A replaced by its complement, $S \setminus A$. Hence

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu((S \setminus A) \cap \phi^i(B)) \geq \frac{\mu(S \setminus A)}{\mu(S)} \liminf_{k \rightarrow \infty} \mu(\phi^k(B)).$$

Consequently, since

$$\mu(A \cap \phi^i(B)) = \mu(\phi^i(B)) - \mu((S \setminus A) \cap \phi^i(B)),$$

we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(A \cap \phi^i(B)) &= \lim_{n \rightarrow \infty} \mu(\phi^n(B)) \\ &\quad - \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu((S \setminus A) \cap \phi^i(B)) \\ &\leq \lim_{n \rightarrow \infty} \mu(\phi^n(B)) - \frac{\mu(S \setminus A)}{\mu(S)} \lim_{n \rightarrow \infty} \mu(\phi^n(B)) \\ &= \left(1 - \frac{\mu(S \setminus A)}{\mu(S)}\right) \lim_{n \rightarrow \infty} \mu(\phi^n(B)) = \frac{\mu(A)}{\mu(S)} \lim_{n \rightarrow \infty} \mu(\phi^n(B)). \end{aligned}$$

This taken together with (12) gives (6).

(ii) Conversely, suppose the Cesàro convergence property (6) holds. Let $\phi^{-1}(B) = B$, $B \in \mathfrak{A}$. Put $A = S \setminus B$ in (6). Since $\phi(B) = \phi(\phi^{-1}(B)) \subseteq B$, we have, for all $j \in \mathbb{N}$,

$$\phi^j(B) \subseteq \phi^{j-1}(B) \subseteq B \quad (13)$$

and hence $\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(A \cap \phi^i(B)) = 0$, whence (6) yields

$$\text{either } \mu(S \setminus B) = 0 \quad \text{or} \quad \lim_{n \rightarrow \infty} \mu(\phi^n(B)) = 0.$$

If $\mu(S \setminus B) \neq 0$, then it follows from (11)* and (13) that, for all $n \in \mathbb{N}$,

$$0 \leq \mu(\phi^n(B)) \leq \mu(B) \leq \mu(\phi^n(B))$$

holds and hence $\lim_{n \rightarrow \infty} \mu(\phi^n(B)) = 0$ implies $\mu(B) = 0$ and Theorem 2 is proved. \square

Corollary 1. *Let (S, \mathfrak{A}, μ) be a probability space and let $\phi: S \rightarrow S$ be an invertible measure-preserving transformation with respect to μ . Then ϕ is ergodic if and only if*

$$\forall (A, B \in \mathfrak{A}) \quad \frac{1}{n} \sum_{i=0}^{n-1} \mu(A \cap \phi^i(B)) \rightarrow \mu(A)\mu(B), \quad \text{as } (n \rightarrow \infty). \quad (14)$$

Proof. If ϕ is invertible, then $\mu(\phi^n(B)) = \mu(B)$ for any $B \in \mathfrak{A}$ and any $n \in \mathbb{N}$. Now since $\mu(S) = 1$, the convergence property (14) follows from (6) and the corollary is proved. \square

*The property (11) holds, not only for ergodic transformations, but also for all transformations ϕ which preserve the measure μ and μ -measurability.

Remark 2. It follows at once from the proof of Theorem 2 that, for every ergodic transformation ϕ of the probability space (S, \mathfrak{A}, μ) which preserves μ - measurability, the following inequality holds for all $A, B \in \mathfrak{A}$:

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(A \cap \phi^i(B)) \geq \mu(A)\mu(B). \quad (15)$$

Namely, since now $\mu(S) = 1$, putting $k = 0$ in (10) gives (12).

Corollary 2. *Let (S, \mathfrak{A}, μ) be a probability space and ϕ a transformation on S that is ergodic with respect to μ . Then:*

(i) (Sempi [10]). *For all A, B in \mathfrak{A} , one has*

$$\limsup_{n \rightarrow \infty} \mu(\phi^{-n}(A) \cap B) \geq \mu(A)\mu(B). \quad (16)$$

(ii) *If, in addition, ϕ preserves μ - measurability, then*

$$\limsup_{n \rightarrow \infty} \mu(A \cap \phi^n(B)) \geq \limsup_{n \rightarrow \infty} \mu(\phi^{-n}(A) \cap B) \geq \mu(A)\mu(B) \quad (17)$$

for all $A, B \in \mathfrak{A}$.

Proof. The property (16) follows from (7), and property (17) follows from (7) and (9), since $\mu(S) = 1$ for the probability space (S, \mathfrak{A}, μ) and since

$$\liminf_{n \rightarrow \infty} a_n \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} a_i \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} a_i \leq \limsup_{n \rightarrow \infty} a_n$$

for every sequence (a_n) of real numbers. □

3. FINAL COMMENTS

There is considerable evidence (see the proof of Theorem 2) to support a conjecture that our results for measurability - preserving ergodic transformations (i.e., for measurability - preserving ergodic dynamical systems with discrete time) can be extended to measurability - preserving ergodic dynamical systems with continuous time (see [2], pp. 6–26).

Next, notice that some of our results (in measure set-theoretical form) can be expressed in functional form (giving a characterization of ergodicity in terms of a unitary operator on the Hilbert space $L^2(S, \mathfrak{A}, \mu)$ or in terms of the induced operator on the Banach space $L^p(S, \mathfrak{A}, \mu)$, ($p \geq 1$)). This is useful when checking whether or not examples have the measurability-preserving ergodic properties (see [1], pp. 14, 19, 30–38 and [11], pp. 19–40).

Theorem 2 also motivates the following conjectures (open problems).

Problem 1. Let (S, \mathfrak{A}, μ) be a normalised measure space and let $\phi: S \rightarrow S$ be a transformation which preserves the measure μ and μ - measurability. Prove or disprove: *If $\lim_{n \rightarrow \infty} \mu(\phi^n(A) \cap B) = \mu(B) \lim_{n \rightarrow \infty} \mu(\phi^n(A))$ holds for all $A, B \in \mathfrak{A}$, then ϕ is strong-mixing (i.e., prove or disprove that the converse statement of Theorem A / Theorem 1 in [8] / holds).*

Problem 2. Let (S, \mathfrak{A}, μ) be a finite measure space and let $\phi: S \rightarrow S$ be a transformation which preserves the measure μ . As usual, if, in addition,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \left| \mu(\phi^{-i}(A) \cap B) - \frac{\mu(A)\mu(B)}{\mu(S)} \right| = 0$$

holds for all $A, B \in \mathfrak{A}$, then ϕ is called *weak-mixing with respect to μ* (see [11], pp. 39–52; and also [5] and [10]). *Does Theorem 2 remain valid when “ergodic” is replaced by “weak-mixing” and “the Cesàro convergence property (6)” is replaced by “the strong Cesàro convergence property*

$$\lim_{J(A,B) \not\equiv n \rightarrow \infty} \mu(A \cap \phi^n(B)) = \frac{\mu(A)}{\mu(S)} \cdot \lim_{n \rightarrow \infty} \mu(\phi^n(B)),$$

where $J(A, B)$ is the subset of \mathbb{N}_0 of density zero”?

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REFERENCES

- [1] J. R. Brown, *Ergodic Theory and Topological Dynamics*, Academic Press, New York–San Francisco–London, 1976.
- [2] I. P. Cornfeld, S.V. Fomin and Ya.G. Sinai, *Ergodic Theory*, Springer Verlag, New York–Heidelberg–Berlin, 1982.
- [3] T. Erber, B. Schweizer and A. Sklar, *Mixing transformations on metric spaces*, Comm. Math. Phys., 29 (1973), 311–317.
- [4] T. Erber and A. Sklar, *Macroscopic irreversibility as a manifestation of micro-instabilities*, in Modern Developments in Thermodynamics (B. Gal-Or, ed.), pp. 281–301, Israel Universities Press and J. Wiley and Sons, Jerusalem–New York, 1974.
- [5] H. Fatkić, *Note on weakly mixing transformations*, Aequationes Math., 43 (1992), 38–44.
- [6] H. Fatkić, *On probabilistic metric spaces and ergodic transformations*, Doctoral dissertation, Univ. Sarajevo, 2000, 170 pp.; Rad. Mat., 10 (2) (2001), 261–263.
- [7] P. Johnson and A. Sklar, *Recurrence and dispersion under iteration of Čebyšev polynomials*, J. Math. Anal. Appl., 54 (1976), 752–771.
- [8] R. E. Rice, *On mixing transformations*, Aequationes Math., 17 (1978), 104–108.
- [9] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, Elsevier North–Holland, New York, 1983.
- [10] C. Sempì, *On weakly mixing transformations on metric spaces*, Rad. Mat., 1 (1985), 3–7.

- [11] P. Walters, *An Introduction to Ergodic Theory*, (Graduate texts in mathematics, Vol. 79), Springer–Verlag, New York–Heidelberg–Berlin, 1982.

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