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GENERAL SUMMABILITY FACTOR THEOREMS AND APPLICATIONS

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ABSTRACT. We obtain sufficient and (different) necessary conditions for the series $\sum a_n$, which is absolutely summable of order k by a triangular matrix method A, to be such that $\sum a_n\lambda_n$ is absolutely summable of order k by a triangular matrix B. As corollaries we obtain a number of inclusion theorems.

In a recent paper the authors [3] obtained sufficient conditions for a series $\sum a_n$ which is absolutely summable of order k by a weighted mean method to be such that $\sum a_n\lambda_n$ is absolutely summable of order k by a triangular matrix method. In this paper we establish a more general summability factor theorem involving two lower triangular matrices. Using these results we obtain a number of corollaries.

Let T be a lower triangular matrix, $\{s_n\}$ a sequence. Then

$$T_n := \sum_{\nu=0}^{n} t_{n\nu}s_\nu.$$

A series $\sum a_n$ is said to be summable $|T|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1}|T_n - T_{n-1}|^k < \infty. \tag{1}$$

We may associate with T two lower triangular matrices $\overline{T}$ and $\hat{T}$ as follows:

$$\bar{t}_{n\nu} := \sum_{r=\nu}^{n} t_{nr}, \qquad n, \nu = 0, 1, 2, \ldots,$$

and

$$\hat{t}_{n\nu} := \bar{t}_{n\nu} - \bar{t}_{n-1,\nu}, \qquad n = 1, 2, 3, \ldots.$$

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With $s_n := \sum_{i=0}^{n} \lambda_i a_i$.

$$y_n := \sum_{i=0}^{n} t_{ni} s_i = \sum_{i=0}^{n} t_{ni} \sum_{\nu=0}^{i} \lambda_\nu a_\nu$$
$$= \sum_{\nu=0}^{n} \lambda_\nu a_\nu \sum_{i=\nu}^{n} t_{ni} = \sum_{\nu=0}^{n} \bar{t}_{n\nu} \lambda_\nu a_\nu$$

and

$$Y_n := y_n - y_{n-1} = \sum_{\nu=0}^{n} (\bar{t}_{n\nu} - \bar{t}_{n-1,\nu}) \lambda_\nu a_\nu = \sum_{\nu=0}^{n} \hat{t}_{n\nu} \lambda_\nu a_\nu. \tag{2}$$

We shall call T a triangle if T is lower triangular and $t_{nn} \neq 0$ for each n. The notation $\Delta_\nu \hat{a}_{n\nu}$ means $\hat{a}_{n\nu} - \hat{a}_{n,\nu+1}$.

Theorem 1 of this paper represents the first time that two arbitrary triangles have been used in a summability factor theorem for absolute summability of order $k > 1$. By restricting A and B to be specific matrices we obtain summability factor theorems for specific classes of matrices, such as weighted means and the Cèsaro matrix of order 1. By setting each $\lambda_n = 1$ we obtain a number of inclusion theorems.

The notation $\lambda \in (|A|_k, |B|_k)$ will be used to represent the statement that, if $\sum a_n$ is summable $|A|_k$, then $\sum a_n \lambda_n$ is summable $|B|_k$.

Theorem 1. *Let $\{\lambda_n\}$ be a sequence of constants, A and B triangles satisfying*

(i) $\dfrac{|b_{nn}|}{|a_{nn}|} = O\Big(\dfrac{1}{|\lambda_n|}\Big)$,

(ii) $|a_{nn} - a_{n+1,n}| = O(|a_{nn} a_{n+1,n+1}|)$,

(iii) $\sum_{\nu=0}^{n-1} |\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)| = O(|b_{nn}\lambda_n|)$,

(iv) $\sum_{n=\nu+1}^{\infty} (n|b_{nn}\lambda_n|)^{k-1} |\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)| = O(\nu^{k-1}|b_{\nu\nu}\lambda_\nu|^k)$,

(v) $\sum_{\nu=0}^{n-1} |b_{\nu\nu}||\hat{b}_{n,\nu+1}\lambda_{\nu+1}| = O(|b_{nn}\lambda_{n+1}|)$,

(vi) $\sum_{n=\nu+1}^{\infty} (n|b_{nn}\lambda_{n+1}|)^{k-1} |\hat{b}_{n,\nu+1}| = O((\nu|b_{\nu\nu}\lambda_{\nu+1}|)^{k-1})$,

(vii) $\sum_{\nu=1}^{\infty} \nu^{k-1} |\lambda_{\nu+1} X_\nu|^k = O(1)$,

(viii) $\sum_{n=1}^{\infty} n^{k-1} \Big| \sum_{\nu=2}^{n} \hat{b}_{n\nu} \lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \Big|^k = O(1)$

where X_ν, X_i and $\hat{a}'_{\nu i}$ are defined latter, in formulas (4) *and* (5).

Then $\lambda \in (|A|_k, |B|_k)$.

Proof. If y_n denotes the nth term of the B-transform of a sequence $\{s_n\}$, then

$$y_n = \sum_{i=0}^{n} b_{ni} s_i = \sum_{i=0}^{n} b_{ni} \sum_{\nu=0}^{i} \lambda_\nu a_\nu$$

$$= \sum_{\nu=0}^{n} \lambda_\nu a_\nu \sum_{i=\nu}^{n} b_{ni} = \sum_{\nu=0}^{n} \bar{b}_{n\nu} \lambda_\nu a_\nu.$$

$$y_{n-1} = \sum_{\nu=0}^{n-1} \bar{b}_{n-1,\nu} \lambda_\nu a_\nu.$$

$$Y_n := y_n - y_{n-1} = \sum_{\nu=0}^{n} \hat{b}_{n\nu} \lambda_\nu a_\nu, \tag{3}$$

where $s_n = \sum_{i=0}^{n} \lambda_i a_i$.

Let x_n denote the n-th term of the A-transform of a series $\sum a_n$. Then

$$X_n := x_n - x_{n-1} = \sum_{\nu=0}^{n} \hat{a}_{n\nu} a_\nu. \tag{4}$$

Since $\hat{A}$ is a triangle, it has a unique two-sided inverse, which we shall denote by $\hat{A}'$. Thus we may solve (4) for a_n to obtain

$$a_n = \sum_{\nu=0}^{n} \hat{a}'_{n\nu} X_\nu. \tag{5}$$

Substituting (5) into (3) yields

$$\begin{aligned} Y_n &= \sum_{\nu=0}^{n} \hat{b}_{n\nu} \lambda_\nu a_\nu = \sum_{\nu=0}^{n} \hat{b}_{n\nu} \lambda_\nu \left(\sum_{i=0}^{\nu} \hat{a}'_{\nu i} X_i \right) \\ &= \sum_{\nu=0}^{n} \hat{b}_{n\nu} \lambda_\nu \left(\sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i + \hat{a}'_{\nu,\nu-1} X_{\nu-1} + \hat{a}'_{\nu\nu} X_\nu \right) \\ &= \sum_{\nu=0}^{n} \hat{b}_{n\nu} \lambda_\nu \hat{a}'_{\nu\nu} X_\nu + \sum_{\nu=1}^{n} \hat{b}_{n\nu} \lambda_\nu \hat{a}'_{\nu,\nu-1} X_{\nu-1} \end{aligned}$$

$$
\begin{aligned}
&\qquad + \sum_{\nu=2}^{n} \hat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \\
&= \hat{b}_{nn}\lambda_n \hat{a}'_{nn} X_n + \sum_{\nu=0}^{n-1} \hat{b}_{n\nu}\lambda_\nu \hat{a}'_{\nu\nu} X_\nu + \sum_{\nu=0}^{n-1} \hat{b}_{n,\nu+1}\lambda_{\nu+1}\hat{a}'_{\nu+1,\nu} X_\nu \\
&\qquad + \sum_{\nu=2}^{n} \hat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \\
&= \frac{b_{nn}}{a_{nn}}\lambda_n X_n + \sum_{\nu=0}^{n-1} \big(\hat{b}_{n\nu}\lambda_\nu a'_{\nu\nu} + \hat{b}_{n,\nu+1}\lambda_{\nu+1}\hat{a}'_{\nu+1,\nu}\big) X_\nu \\
&\qquad + \sum_{\nu=2}^{n} \hat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \\
&= \frac{b_{nn}}{a_{nn}}\lambda_n X_n + \sum_{\nu=0}^{n-1} \big(\hat{b}_{n\nu}\lambda_\nu a'_{\nu\nu} + \hat{b}_{n,\nu+1}\lambda_{\nu+1} a'_{\nu\nu} - \hat{b}_{n,\nu+1}\lambda_{\nu+1} a'_{\nu\nu} \\
&\quad + \hat{b}_{n,\nu+1}\lambda_{\nu+1}\hat{a}'_{\nu+1,\nu}\big) X_\nu + \sum_{\nu=2}^{n} \hat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}_{\nu i} X_i \\
&= \frac{b_{nn}}{a_{nn}}\lambda_n X_n + \sum_{\nu=0}^{n-1} \frac{\Delta_\nu\big(\hat{b}_{n\nu}\lambda_\nu\big)}{a_{\nu\nu}} X_\nu \\
&\quad + \sum_{\nu=0}^{n-1} \hat{b}_{n,\nu+1}\lambda_{\nu+1}\big(a'_{\nu\nu} + \hat{a}'_{\nu+1,\nu}\big) X_\nu + \sum_{\nu=2}^{n} \hat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i. \qquad (6)
\end{aligned}
$$

Using the fact that

$$
a'_{\nu\nu} + \hat{a}'_{\nu+1,\nu} = \frac{1}{a_{\nu\nu}}\Big(\frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu+1,\nu+1}}\Big), \qquad (7)
$$

and substituting (7) into (6), we have

$$
\begin{aligned}
Y_n &= \frac{b_{nn}}{a_{nn}}\lambda_n X_n + \sum_{\nu=0}^{n-1} \frac{\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}} X_\nu + \sum_{\nu=0}^{n-1} \hat{b}_{n,\nu+1}\lambda_{\nu+1}\Big(\frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}}\Big) X_\nu \\
&\quad + \sum_{\nu=2}^{n} \hat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \\
&= T_{n1} + T_{n2} + T_{n3} + T_{n4}.
\end{aligned}
$$

By Minkowski's inequality it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1}|T_{ni}|^k < \infty, \qquad i = 1, 2, 3, 4.$$

Using (i)

$$\begin{aligned}\sum_{n=1}^{\infty} n^{k-1}|T_{n1}|^k &= \sum_{n=1}^{\infty} n^{k-1}\Big|\frac{b_{nn}}{a_{nn}}\lambda_n X_n\Big|^k \\ &= O(1)\sum_{n=1}^{\infty} n^{k-1}|X_n|^k = O(1),\end{aligned}$$

since $\sum a_n$ is summable $|A|_k$.

Using (i), (iii), (iv) and Hölder's inequality,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{k-1}|T_{n2}|^k &= \sum_{n=1}^{\infty} n^{k-1}\Big|\sum_{\nu=0}^{n-1}\frac{\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}}X_\nu\Big|^k \\ &\le \sum_{n=1}^{\infty} n^{k-1}\Big\{\sum_{\nu=0}^{n-1}(|a_{\nu\nu}|^{-1}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)||X_\nu|\Big\}^k \\ &= O(1)\sum_{n=1}^{\infty} n^{k-1}\Big[\sum_{\nu=0}^{n-1}|b_{\nu\nu}\lambda_\nu|^{-1}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)||X_\nu|\Big]^k \\ &= O(1)\sum_{n=1}^{\infty} n^{k-1}\Big(\sum_{\nu=0}^{n-1}|b_{\nu\nu}\lambda_\nu|^{-k}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)||X_\nu|^k\Big)\times \\ &\qquad \times\Big(\sum_{\nu=0}^{n-1}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)|\Big)^{k-1} \\ &= O(1)\sum_{n=1}^{\infty}(n|b_{nn}\lambda_n|)^{k-1}\sum_{\nu=0}^{n-1}|b_{\nu\nu}\lambda_\nu|^{-k}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)||X_\nu|^k \\ &= O(1)\sum_{\nu=1}^{\infty}|b_{\nu\nu}\lambda_\nu|^{-k}|X_\nu|^k\sum_{n=\nu+1}^{\infty}(n|b_{nn}\lambda_n|)^{k-1}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)| \\ &= O(1)\sum_{\nu=1}^{\infty}|b_{\nu\nu}\lambda_\nu|^{-k}|X_\nu|^k\nu^{k-1}|b_{\nu\nu}\lambda_\nu|^k \\ &= O(1)\sum_{\nu=1}^{\infty}\nu^{k-1}|X_\nu|^k = O(1).\end{aligned}$$

Using (ii), (v), (vi), (vii) and Hölder's inequality,

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{k-1}|T_{n3}|^k &= \sum_{n=1}^{\infty} n^{k-1}\Big|\sum_{\nu=0}^{n-1}\hat{b}_{n,\nu+1}\lambda_{\nu+1}\Big(\frac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}}\Big)X_\nu\Big|^k\\
&\le \sum_{n=1}^{\infty} n^{k-1}\Big(\sum_{\nu=0}^{n-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}|\Big|\frac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}}\Big||X_\nu|\Big)^k\\
&= O(1)\sum_{n=1}^{\infty} n^{k-1}\Big(\sum_{\nu=0}^{n-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}||X_\nu|\Big)^k\\
&= O(1)\sum_{n=1}^{\infty} n^{k-1}\Big(\sum_{\nu=0}^{n-1}\Big(\frac{|b_{\nu\nu}|}{|b_{\nu\nu}|}\Big)|\hat{b}_{n,\nu+1}\lambda_{\nu+1}||X_\nu|\Big)^k\\
&= O(1)\sum_{n=1}^{\infty} n^{k-1}\sum_{\nu=0}^{n-1}|b_{\nu\nu}|^{1-k}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}||X_\nu|^k\times\\
&\qquad\times\Big(\sum_{\nu=0}^{n-1}|b_{\nu\nu}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}|\Big)^{k-1}\\
&= O(1)\sum_{n=1}^{\infty}(n|b_{nn}\lambda_{n+1}|)^{k-1}\sum_{\nu=0}^{n-1}|b_{\nu\nu}|^{1-k}|\hat{b}_{n,\nu+1}||X_\nu\lambda_{\nu+1}|^k\\
&= O(1)\sum_{\nu=0}^{\infty}|b_{\nu\nu}|^{1-k}||\lambda_{\nu+1}||X_\nu|^k\sum_{n=\nu+1}^{\infty}(n|b_{nn}\lambda_{n+1}|)^{k-1}|\hat{b}_{n,\nu+1}|\\
&= O(1)\sum_{\nu=0}^{\infty}|b_{\nu\nu}|^{1-k}||\lambda_{\nu+1}||X_\nu|^k\nu^{k-1}|b_{\nu\nu}\lambda_{\nu+1}|^{k-1}\\
&= O(1)\sum_{\nu=0}^{\infty}\nu^{k-1}|\lambda_{\nu+1}X_\nu|^k = O(1).
\end{aligned}$$

From (viii),

$$\sum_{n=1}^{\infty} n^{k-1}|T_{n4}|^k = \sum_{n=1}^{\infty} n^{k-1}\Big|\sum_{\nu=2}^{n}\hat{b}_{n\nu}\lambda_\nu\sum_{i=0}^{\nu-2}\hat{a}'_{\nu i}X_i\Big|^k = O(1).$$

□

A weighted mean matrix is a lower triangular matrix with entries p_k/P_n, $0 \le k \le n$, where $P_n := \sum_{k=0}^{n} p_k$.

Corollary 1. *Let λ_n be a sequence of constants, $\{p_n\}$ a sequence of positive constants, B a triangle satisfying*

(i) $P_n|b_{nn}| = O(p_n/|\lambda_n|)$,

(ii) $\sum_{\nu=0}^{n-1} |\Delta_\nu(\lambda_\nu \hat{b}_{n\nu})| = O(|b_{nn}\lambda_n|)$,

(iii) $\sum_{n=\nu+1}^{\infty} (n|b_{nn}\lambda_n|)^{k-1}|\Delta_\nu(\lambda_\nu \hat{b}_{n\nu})| = O(\nu^{k-1}|\lambda_\nu b_{\nu\nu}|^k)$,

(iv) $\sum_{\nu=0}^{n-1} |b_{\nu\nu}\hat{b}_{n,\nu+1}\lambda_{\nu+1}| = O(|b_{nn}\lambda_{n+1}|)$,

(v) $\sum_{n=\nu+1}^{\infty} (n|b_{nn}\lambda_{n+1}|)^{k-1}|\hat{b}_{n,\nu+1}| = O((\nu|b_{\nu\nu}\lambda_{\nu+1}|)^{k-1})$.

(vi) $\sum_{\nu=1}^{\infty} \nu^{k-1}|\lambda_{\nu+1}X_\nu|^k = O(1)$.

Then $\lambda \in (|\overline{N}, p_n|_k, |B|_k)$.

Proof. Conditions (i), (iii) - (vii) of Theorem 1 reduce to conditions (i) - (vi), respectively of Corollary 1.

With $A = (\overline{N}, p_n)$,

$$a_{nn} - a_{n+1,n} = \frac{p_n}{P_n} - \frac{p_n}{P_{n+1}} = \frac{p_n p_{n+1}}{P_n P_{n+1}} = a_{nn}a_{n+1,n+1},$$

and condition (ii) of Theorem 1 is automatically satisfied.

A matrix A is said to be factorable if $a_{nk} = b_n c_k$ for each n and k.

Since A is a weighted mean matrix, $\hat{A}$ is a factorable triangle and, as has been shown in [4], its inverse is bidiagonal. Therefore condition (viii) of Theorem 1 is trivially satisfied. □

Corollary 2. *Let* λ_n *be a sequence of constants,* $\{p_n\}$ *a sequence of positive constants,* A *a triangle satisfying*

(i) $p_n/(P_n|a_{nn}|) = O(1/|\lambda_n|)$,

(ii) $|a_{nn} - a_{n+1,n}| = O(|a_{nn}a_{n+1,n+1}|)$,

(iii) $\sum_{\nu=0}^{n-1} |\Delta_\nu(\lambda_\nu P_{\nu-1})| = O(P_{n-1}|\lambda_n|)$,

(iv) $|\Delta_\nu(P_{\nu-1}\lambda_\nu)| \sum_{n=\nu+1}^{\infty} \Big(\frac{np_n|\lambda_n|}{P_n}\Big)^{k-1} \frac{p_n}{P_n P_{n-1}} = O\Big(\nu^{k-1}\Big(\frac{p_\nu|\lambda_\nu|}{P_\nu}\Big)^k\Big)$,

(v) $\sum_{\nu=0}^{n-1} p_\nu |\lambda_{\nu+1}| = O(P_{n-1}\lambda_{n+1})$,

(vi) $\sum_{n=\nu+1}^{\infty} n^{k-1}\Big(\frac{np_n\lambda_{n+1}}{P_n}\Big)^{k-1}\frac{p_n}{P_nP_{n-1}} = O\bigg(\frac{(\nu p_\nu|\lambda_{\nu+1}|)^{k-1}}{P_\nu^k}\bigg)$,

(vii) $\sum_{\nu=1}^{\infty} \nu^{k-1}|\lambda_{\nu+1}X_\nu|^k = O(1)$,

(viii) $\sum_{n=\nu+1}^{\infty} n^{k-1}\Big(\frac{p_n}{P_nP_{n-1}}\Big)^k \Big|\sum_{\nu=2}^{n}\lambda_\nu P_{\nu-1}\sum_{i=0}^{\nu-2}\hat{a}'_{\nu i}X_i\Big|^k = O(1)$.

Then $\lambda \in (|A|_k, |\overline{N}, p_n|_k)$.

Proof. With $B = (\overline{N}, p_n)$, conditions (i) - (viii) of Theorem 1 reduce to conditions (i) - (viii), respectively of Corollary 2, since

$$\hat{b}_{n\nu} = \frac{p_nP_{\nu-1}}{P_nP_{n-1}}.$$

□

Corollary 3. *Let* $q_n = 1$ *for each* $n, \{p_n\}$ *a positive sequence satisfying conditions* (iii)-(vi) *of Corollary* 2,

(i) $\frac{np_n|\lambda_n|}{P_n} = O(1)$,

(ii) $\sum_{\nu=1}^{\infty} \nu^{k-1}|\lambda_\nu)X_\nu|^k| = O(1)$.

Then $\lambda \in (|C, 1|_k, |\overline{N}, p_n|_k)$.

Proof. With $A = (C, 1)$, condition (i) of Corollary 2 becomes condition (i) of Corollary 3.

Note that

$$a_{nn} - a_{n+1,n} = \frac{p_n}{P_n} - \frac{p_n}{P_{n+1}} = \frac{p_np_{n+1}}{P_nP_{n+1}} = a_{nn}a_{n+1,n+1},$$

and condition (ii) of Corollary 2 is automatically satisfied.

Since the inverse of $(C, 1)$ is bidiagonal, condition (viii) of Corollary 2 is automatically satisfied. □

Corollary 4. *Let* $\{p_n\}$ *be a positive sequence,* $q_n = 1$ *for each* n, *satisfying*

(i) $\dfrac{P_n|\lambda_n|}{np_n} = O(1)$,

(ii) $\displaystyle\sum_{\nu=0}^{n-1} |\Delta_\nu(\nu\lambda_\nu)| = O(n|\lambda_n|)$,

(iii) $\displaystyle|\Delta_\nu(\nu\lambda_\nu)| \sum_{n=\nu+1}^{\infty} \frac{|\lambda_n|^{k-1}}{n(n+1)} = O\Big(\frac{|\lambda_\nu|^k}{\nu}\Big)$,

(iv) $\displaystyle\sum_{\nu=0}^{n-1} |\lambda_{\nu+1}| = O(n|\lambda_{n+1}|)$,

(v) $\displaystyle\sum_{n=\nu+1}^{\infty} \frac{|\lambda_{n+1}|^k}{n(n+1)^k} = O\Big(\Big(\frac{|\lambda_{\nu+1}|}{\nu}\Big)^{k-1}\Big)$,

(vi) $\displaystyle\sum_{\nu=1}^{\infty} \nu^{k-1}|\lambda_{\nu+1}X_\nu|^k = O(1)$.

Then $\lambda \in |\overline{N}, p_n|_k, |C,1|_k)$.

With $B = (C,1)$, the conditions of Corollary 1 reduce to those of Corollary 4.

We now turn our attention to obtaining necessary conditions.

Theorem 2. *Let A and B be two lower triangular matrices with A satisfying*

$$\sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu \hat{a}_{n\nu}|^k = O(|a_{\nu\nu}|^k). \tag{8}$$

Then necessary conditions for $\lambda \in (A|_k, |B|_k)$ are

(i) $|b_{\nu\nu}\lambda_\nu| = O(|a_{\nu\nu}|)$,

(ii) $\displaystyle\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu \hat{b}_{n\nu}\lambda_\nu|^k\Big)^{1/k} = O(|a_{\nu\nu}|\nu^{1-1/k}\Big)$,

(iii) $\displaystyle\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k = O\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{a}_{n,\nu+1}|^k\Big)$.

Proof. For $k \geq 1$ define

$$A^* = \Big\{ \{a_i\} : \sum a_i \text{ is summable } |A|_k \Big\},$$
$$B^* = \Big\{ \{b_i\} : \sum b_i\lambda_i \text{ is summable } |B|_k \Big\}.$$

With Y_n and X_n as defined by (3) and (4), the spaces A^* and B^* are BK-spaces, with norms given by

$$\|a\|_1 = \Big\{|X_0|^k + \sum_{n=1}^{\infty} n^{k-1}|X_n|^k\Big\}^{1/k} \tag{9}$$

and

$$\|a\|_2 = \Big\{|Y_0|^k + \sum_{n=1}^{\infty} n^{k-1}|Y_n|^k\Big\}^{1/k}, \tag{10}$$

respectively.

From the hypothesis of the theorem, $\|a\|_1 < \infty$ implies that $\|a\|_2 < \infty$. The inclusion map $i : A^* \to B^*$ defined by $i(x) = x$ is continuous, since A^* and B^* are BK-spaces. Applying the closed graph theorem, there exists a constant $K > 0$ such that

$$\|a\|_2 \leq K\|a\|_1. \tag{11}$$

Let e_n denote the n-th coordinate vector. From (3) and (4), with $\{a_n\}$ defined by $a_n = e_n - e_{n+1}, n = \nu$, $a_n = 0$ otherwise, we have

$$X_n = \begin{cases} 0, & n < \nu, \\ \hat{a}_{n\nu}, & n = \nu, \\ \Delta_\nu \hat{a}_{n\nu}, & n > \nu, \end{cases}$$

and

$$Y_n = \begin{cases} 0, & n < \nu, \\ \hat{b}_{n\nu}, & n = \nu, \\ \Delta_\nu(\hat{b}_{n\nu}\lambda_\nu), & n > \nu. \end{cases}$$

From (9) and (10),

$$\|a\|_1 = \Big\{\nu^{k-1}|a_{\nu\nu}|^k + \sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu \hat{a}_{n\nu}|^k\Big\}^{1/k},$$

and

$$\|a\|_2 = \Big\{\nu^{k-1}|b_{\nu\nu}|^k + \sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu \hat{b}_{n\nu}|^k\Big\}^{1/k},$$

recalling that $\hat{b}_{\nu\nu} = \bar{b}_{\nu\nu} = b_{\nu\nu}$.

From (11), using (8), we obtain

$$\nu^{k-1}|b_{\nu\nu}\lambda_\nu|^k + \sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu|^k$$

$$\leq K^k\Big(\nu^{k-1}|a_{\nu\nu}|^k + \sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu \hat{a}_{n\nu}|^k\Big)$$
$$\leq K^k\Big(\nu^{k-1}|a_{\nu\nu}|^k + O(1)|a_{\nu\nu}|^k\Big)$$
$$= O\big(|a_{\nu\nu}|^k(\nu^{k-1}+1)\big)$$
$$= O(\nu^{k-1}|a_{\nu\nu}|^k).$$

The above inequality will be true if and only if each term on the left hand side is $O(\nu^{k-1}|a_{\nu\nu}|^k)$. Using the first term,

$$\nu^{k-1}|b_{\nu\nu}\lambda_\nu|^k = O(\nu^{k-1}|a_{\nu\nu}|^k),$$

which implies that $|b_{\nu\nu}\lambda_\nu| = O(|a_{\nu\nu}|)$, and (i) is necessary.

Using the second term we obtain

$$\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)|^k\Big)^{1/k} = O(\nu^{1-1/k}|a_{\nu\nu}|),$$

which is condition (ii).

If we now define $a_n = e_{n+1}$ for $n = \nu$, $a_n = 0$ otherwise, then, from (3) and (4) we obtain

$$X_n = \begin{cases} 0, & n \leq \nu, \\ \hat{a}_{n,\nu+1}, & n > \nu, \end{cases}$$

and

$$Y_n = \begin{cases} 0, & n \leq \nu, \\ \hat{b}_{n,\nu+1}\lambda_{\nu+1}, & n > \nu. \end{cases}$$

The corresponding norms are

$$\|a\|_1 = \Big\{\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{a}_{n,\nu+1}|^k\Big\}^{1/k}$$

and

$$\|a\|_2 = \Big\{\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k\Big\}^{1/k}.$$

Applying (11),

$$\Big\{\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k\Big\}^{1/k} \leq K\Big\{\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{a}_{n,\nu+1}|^k\Big\}^{1/k},$$

which implies condition (iii). □

Corollary 5. *Let B be a lower triangular matrix, $\{p_n\}$ a sequence satisfying*

$$\sum_{n=\nu+1}^{\infty} n^{k-1}\Big(\frac{p_n}{P_nP_{n-1}}\Big)^k = O\Big(\frac{1}{P_\nu^k}\Big). \tag{12}$$

Then necessary conditions for $\lambda \in (|\overline{N},p_n|_k, |B|_k)$ are

(i) $|b_{\nu\nu}\lambda_\nu| = O\Big(\frac{p_\nu}{P_\nu}\Big)$,

(ii) $\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)|^k\Big)^{1/k} = O\Big(\nu^{1-1/k}\frac{p_\nu}{P_\nu}\Big)$,

(iii) $\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k = O(1)$.

Proof. With $A = (\overline{N},p_n)$, equation (8) becomes (12), and conditions (i) - (iii) of Theorem 2 become conditions (i) - (iii) of Corollary 10, respectively. □

Corollary 6. *Let $1 \le k < \infty, \{p_n\}$ a positive sequence. Then $\lambda \in (|\overline{N},p_n|, |B|_k)$ if and only if*

(i) $|b_{\nu\nu}\lambda_\nu|\frac{P_\nu}{p_\nu} = O(\nu^{1/k-1})$,

(ii) $\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)|^k\Big)^{1/k} = O\Big(\frac{p_\nu}{P_\nu}\Big)$,

(iii) $\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k\Big)^{1/k} = O(1)$.

Every summability factor theorem becomes an inclusion theorem by setting each $\lambda_n = 1$.

Corollary 7. *Let A and B be triangles satisfying*

(i) $\frac{|a_{nn}|}{|b_{nn}|} = O(1)$,

(ii) $\Big|\frac{b_{n+1,n} - b_{nn}}{b_{nn}b_{n+1,n+1}}\Big| = O(1)$,

(iii) $\sum_{\nu=0}^{n-1} |\Delta_\nu \hat{a}_{n\nu}| = O(|a_{nn}|)$,

(iv) $\sum_{n=\nu+1}^{\infty} (n|a_{nn}|)^{k-1}|\Delta_\nu \hat{a}_{n\nu}| = O(\nu^{k-1}|a_{\nu\nu}|^k)$,

(v) $\sum_{\nu=0}^{n-1} |a_{\nu\nu}||\hat{a}_{n,\nu+1}| = O(|a_{nn}|)$,

(vi) $\sum_{n=\nu+1}^{\infty} (n|a_{nn}|)^{k-1}|\hat{a}_{n,\nu+1}| = O((\nu|a_{\nu\nu}|)^{k-1})$,

(vii) $\sum_{n=1}^{\infty} n^{k-1}\Big|\sum_{\nu=2}^{n} \hat{a}_{n,\nu} \sum_{r=1}^{r-2} b'_{\nu r} X_r|^k = O(1)$.

Then $\sum a_n$ *summable* $|B|_k$ *implies that it is summable* $|A|_k, k \geq 1$.

Corollary 7 is Theorem 1 of [3].

Corollary 8. *Let* $\{p_n\}$ *be a positive sequence,* T *a nonnegative triangle satisfying*

(i) $t_{ni} \geq t_{n+1,i}, \qquad n \geq i,\ i = 0, 1, \ldots,$

(ii) $P_n t_{nn} = O(p_n)$,

(iii) $\bar{t}_{n0} = \bar{t}_{n-1,0},\ n = 1, 2, \ldots,$

(iv) $\sum_{\nu=1}^{n-1} t_{\nu\nu}|\hat{t}_{n,\nu}| = O(t_{nn})$,

(v) $\sum_{n=\nu+1}^{\infty} (nt_{nn})^{k-1}|\Delta_\nu \hat{t}_{n\nu}| = O(\nu^{k-1}t_{\nu\nu}^k)$,

(vi) $\sum_{n=\nu+1}^{\infty} (nt_{nn})^{k-1}|\hat{t}_{n,\nu}| = O((\nu t_{\nu\nu})^{k-1}$.

Then $\sum a_n$ *summable* $|\overline{N}, p_n|_k$ *implies* $\sum a_n$ *is summable* $|T|_k, k \geq 1$.

Proof. Since each $\lambda_n = 1$, condition (vi) of Corollary 1 simply states that $\sum a_n$ is summable $|\overline{N}, p_n|_k$.

Condition (i) of Corollary 1 reduces to condition (ii) of Corollary 6.

Note that

$$\begin{aligned}
\Delta_\nu \hat{t}_{n\nu} &= \hat{t}_{n\nu} - \hat{t}_{n,\nu+1} = \bar{t}_{n\nu} - \bar{t}_{n-1,\nu} - \bar{t}_{n,\nu+1} + \bar{t}_{n-1,\nu+1} \\
&= \sum_{i=\nu}^{n} t_{ni} - \sum_{i=\nu}^{n-1} t_{n-1,i} - \sum_{i=\nu+1}^{n} t_{ni} + \sum_{i=\nu+1}^{n-1} t_{n-1,i} \\
&= t_{n\nu} - t_{n-1,\nu} \geq 0.
\end{aligned}$$

Therefore, from (i) and (iii) of Corollary 6,

$$\sum_{\nu=0}^{n-1} |\Delta_\nu \hat{t}_{n\nu}| = \sum_{\nu=0}^{n-1} |t_{n\nu} - t_{n-1,\nu}| = \sum_{\nu=0}^{n-1} t_{n-1,\nu} - \sum_{\nu=0}^{n-1} t_{n\nu}$$
$$= \bar{t}_{n-1,0} - \bar{t}_{n0} + t_{nn} = t_{nn},$$

and condition (ii) of Corollary 1 is satisfied.

Condition (iii) of Corollary 1 reduces to condition (v) of Corollary 6.

Using condition (ii) of Corollary 1, condition (iv) of Corollary 6, and the fact that condition (iii) of Corollary 6 implies that $\hat{t}_{n0} = 0$,

$$\sum_{\nu=0}^{n-1} t_{\nu\nu}\hat{t}_{n,\nu+1} = \sum_{\nu=0}^{n-1} t_{\nu\nu}(\hat{t}_{n,\nu+1} - \hat{t}_{n\nu}) + \sum_{\nu=0}^{n-1} t_{\nu\nu}\hat{t}_{n\nu}$$
$$= \sum_{\nu=0}^{n-1} t_{\nu\nu}|\Delta_\nu \hat{t}_{n\nu}| + \sum_{\nu=0}^{n-1} t_{n\nu}\hat{t}_{n\nu} = O(t_{nn}),$$

and condition (iv) of Corollary 1 is satisfied.

Using condition (iv) of Corollary 1 and condition (v) of Corollary 6,

$$\sum_{n=\nu+1}^{\infty} (nt_{nn})^{k-1}\hat{t}_{n,\nu+1} = \sum_{n=\nu+1}^{\infty} (nt_{nn})^{k-1}|\Delta_\nu \hat{t}_{n\nu}| + \sum_{n=\nu+1}^{\infty} (nt_{nn})^{k-1}\hat{t}_{n\nu}$$
$$= O((\nu t_{\nu\nu})^{k-1}),$$

and condition (v) of Corollary 1 is satisfied. □

Remark 1. Corollary 6 is equivalent to the corrected version of the Theorem in [1], which appears in [2].

Corollary 9. *Let A and B be two lower triangular matrices, A satisfing (8). Necessary conditions for $\sum a_n$ summable $|A|_k$ to imply that $\sum a_n$ is summable $|B|_k$ are*

(i) $|b_{\nu\nu}| = O(|a_{\nu\nu}|)$,

(ii) $\displaystyle\sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu \hat{b}_{n\nu}|^k = O(|a_{\nu\nu}|^k \nu^{k-1})$,

(iii) $\displaystyle\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{b}_{n,\nu+1}|^k = O\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{a}_{n,\nu+1}|^k\Big)$.

To prove the corollary simply put $\lambda_n = 1$ in Theorem 2.

Corollary 10. *Let B be a lower triangular matrix, A a weighted mean matrix with $\{p_n\}$ a sequence satisfying (8). Then necessary conditions for $\sum a_n$ summable $|\overline{N}, p_n|_k$ to imply that $\sum a_n$ is summable $|B|_k$ are*

(i) $\dfrac{P_\nu |b_{\nu\nu}|}{p_\nu} = O(1)$,

(i) $\displaystyle\sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu \hat{b}_{n\nu}|^k = O\Big(\nu^{k-1}\Big(\frac{p_\nu}{P_\nu}\Big)^k\Big)$,

(iii) $\displaystyle\sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu \hat{b}_{n,\nu+1}|^k = O(1)$.

To prove the corollary set $\lambda_n = 1$ in Corollary 5.

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