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A DISTRIBUTIONAL VERSION OF THE FERENC LUKÁCS THEOREM

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Abstract. The theorem of F. Lukács determines the generalized jumps of a periodic, integrable function in terms of a logarithmic average of the partial sums of its conjugate Fourier series. Recently, F. Móricz gave a version of Lukács result for the Abel-Poisson means of the conjugate Fourier series, under an extended notion of jump. In this article we give a generalization that applies to periodic distributions under a much extended notion of jump, namely, that of distributional point values of Łojasiewicz. Our generalization is obtained by obtaining results on the local boundary behaviour of an analytic function with distributional boundary values near a point where the boundary generalized function has a jump.

1. Introduction

Let f be a function of period 2π, with Fourier series

$$f(x) \sim \frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \ . \tag{1.1}$$

Let

$$\sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx) \ , \tag{1.2}$$

be the conjugate series and let $\widetilde{s}_n(f,x)$ be the nth partial sum of (1.2). Then the Ferenc Lukács theorem [13], [21, Thm. 8.13] states that if f is integrable and if there exists a number $d = d_x(f)$ such that

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h |f(x+t) - f(x-t) - d| \, dt = 0 \, , \tag{1.3}$$

then

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$$\lim_{h\to 0^+} \frac{\widetilde{s}_n(f,x)}{\ln n} = -\frac{1}{\pi}d\,. \tag{1.4}$$

Recently, F. Móricz [14] proved a corresponding result for the Abel-Poisson means of the conjugate series, namely, if

$$\widetilde{f}(r,x) = \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx)\, r^k\ , \quad 0 \le r < 1\,, \tag{1.5}$$

the number, $d = d_x(f)$, defined by the limit

$$d = \lim_{h\to 0^+} \frac{1}{h}\int_0^h (f(x+t) - f(x-t))\ dt\,, \tag{1.6}$$

exists, and f is integrable, then

$$\lim_{r\to 1^-} \frac{\widetilde{f}(r,x)}{\ln(1-r)} = \frac{1}{\pi}d\,. \tag{1.7}$$

Our aim is to give a generalization of these results in two directions. First we consider the case when f is a periodic distribution, $f \in \mathcal{D}'(\mathbb{R})$. Second, we consider the case the case when d is the distributional point limit of the jump function $\psi_x(t) = f(x+t) - f(x-t)$ as $t \to 0^+$ in the sense of Łojasiewicz. The existence of the distributional jump d means that there exists $n \in \mathbb{N}$ and a primitive of order n of ψ_x, Ψ, with $\Psi^{(n)} = \psi_x$, such that Ψ is continuous near $t = 0$, and

$$\lim_{t\to 0} \frac{n!\Psi(t)}{t^n} = d\,. \tag{1.8}$$

The definition (1.6), although more general than (1.3), corresponds to the case $n = 1$ of the Łojasiewicz definition (1.8). It is interesting to observe that Fourier series having a *distributional* point value at a point have been characterized [3], while there is no corresponding result for ordinary functions.

The article is organized as follows. Section 2 gives some necessary background material from the distributional theory of asymptotic expansions [4, 7, 15, 18], and from the notion of distributional point values [12]. Section 3 considers an important question in the study of distributional point values: the definition of Łojasiewicz says that

$$f(x_0) = \gamma\,, \quad \text{distributionally,} \tag{1.9}$$

if and only if,

$$\lim_{\varepsilon\to 0} \langle f(x_0 + \varepsilon x), \phi(x)\rangle = \gamma \int_{-\infty}^{\infty} \phi(x)\ dx\,, \quad \forall\phi \in \mathcal{D}(\mathbb{R})\ . \tag{1.10}$$

It happens many times that the evaluation $\langle f(x_0 + \varepsilon x), \phi(x)\rangle$ is defined but $\phi \notin \mathcal{D}(\mathbb{R})$, for instance if f has compact support and ϕ is any smooth function. Then, one is interested in knowing if (1.10) still holds. Here we show that although (1.10) does not hold in general, there are extra conditions on f and ϕ under which it does. The results of Section 3 are used in Sections 4 and 5, respectively, to study the local boundary behavior of harmonic and analytic functions with distributional boundary values. The results of Sections 4 and 5, in turn, are then used in Section 6 to prove the announced generalization of the Lukács-Móricz theorem.

2. Preliminaries

In this section we explain the spaces of test functions and distributions needed in this paper. We also give a summary of the notion of Cesàro behavior of a distribution at infinity [4] and at a point [7, 12]. All of our functions and distributions are over one dimensional spaces.

The spaces of test functions $\mathcal{D}, \mathcal{E}$, and $\mathcal{S}$ and the corresponding spaces of distributions $\mathcal{D}'$, $\mathcal{E}'$, and $\mathcal{S}'$ are well-known [10, 11, 16]. In general [20] we call a topological vector space $\mathcal{A}$ a space of test functions if $\mathcal{D} \subset \mathcal{A} \subset \mathcal{E}$, the inclusions being strict, and if the derivative d/dx is a continuous operator of $\mathcal{A}$. Another useful space, particularly in the study of distributional asymptotic expansions [7, 15, 18], is $\mathcal{K}'$, dual of $\mathcal{K}$. A smooth function ϕ belongs to $\mathcal{K}$ if there is a constant γ such that $\phi^{(k)}(x) = O(|x|^{\gamma-k})$ as $|x| \to \infty$ for $k = 0, 1, 2, \ldots$, that is, if $\phi(x) = O(|x|^{\gamma})$ strongly. The space $\mathcal{K}$ is formed by the so-called GLS symbols [9]; the topology of $\mathcal{K}$ is given by the canonical seminorms. The space $\mathcal{K}'$ plays a fundamental role in the theory of summability of distributional evaluations [4]. The elements of $\mathcal{K}'$ are exactly the generalized functions that decay very rapidly at infinity in the distributional sense or, equivalently, in the Cesàro sense.

The Cesàro behavior of a distribution at infinity is studied by using the order symbols $O(x^{\alpha})$ and $o(x^{\alpha})$ in the Cesàro sense. If $f \in \mathcal{D}'(\mathbb{R})$ and $\alpha \in \mathbb{R} \setminus \{-1, -2, -3, \ldots\}$, we say that $f(x) = O(x^{\alpha})$ as $x \to \infty$ in the Cesàro sense and write

$$f(x) = O(x^{\alpha}) \text{ (C)}, \quad \text{as } x \to \infty \ , \tag{2.1}$$

if there exists $N \in \mathbb{N}$ such that every primitive F of order N of f, i.e., $F^{(N)} = f$, is an ordinary function for large arguments and satisfies the ordinary order relation

$$F(x) = p(x) + O(x^{\alpha+N}), \quad \text{as } x \to \infty \ , \tag{2.2}$$

for a suitable polynomial p of degree at most $N-1$. A similar definition applies to the little o symbol. The definitions when $x \to -\infty$ are clear. One can also consider the case when $\alpha = -1, -2, -3, \ldots$ [7, Def. 6.3.1].

The equivalent notations $f(x) = O(x^{-\infty})$ and $f(x) = o(x^{-\infty})$ mean that $f(x) = O(x^{-\beta})$ for each $\beta > 0$. It is shown in [4], [7, Thm. 6.7.1] that a distribution $f \in \mathcal{D}'$ is of rapid decay at $\pm\infty$ in the (C) sense,

$$f(x) = O(|x|^{-\infty}) \text{ (C) as } |x| \to \infty \,, \tag{2.3}$$

if and only if $f \in \mathcal{K}'$. Functions like $\sin x$, $J_0(x)$, or $x^2 e^{ix}$ belong to $\mathcal{K}'$ and thus are "distributionally small". The space $\mathcal{K}'$ is a distributional analogue of the space $\mathcal{S}$ of rapidly decreasing smooth functions [7, Section 2.9].

These ideas can be readily extended to the study of the local behavior of generalized functions [7, 18]. Actually, Łojasiewicz [12] defined the value of distribution $f \in \mathcal{D}'(\mathbb{R})$ at the point x_0 as the limit

$$f(x_0) = \lim_{\varepsilon \to 0} f(x_0 + \varepsilon x)\,, \tag{2.4}$$

if the limit exists in $\mathcal{D}'(\mathbb{R})$, that is, if

$$\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi(x) \rangle = f(x_0) \int_{-\infty}^{\infty} \phi(x)\, dx\,, \tag{2.5}$$

for each $\phi \in \mathcal{D}(\mathbb{R})$. It was shown by Łojasiewicz [12] that the existence of the distributional point value $\gamma = f\,(x_0)$ is equivalent to the existence of $n \in \mathbb{N}$, and a primitive of order n of f, that is $F^{(n)} = f$, which is continuous near $x = x_0$ and satisfies

$$\lim_{x \to x_0} \frac{n!F\,(x)}{(x - x_0)^n} = \gamma\,. \tag{2.6}$$

For example the generalized function $f(x) = \sin(1/x)$ is oscillatory near $x = 0$, however, it is easy to see that $f(0)$ exists and equals 0.

More generally, one could try to look for a representation of the form

$$f(x_0 + \varepsilon x) \sim \varepsilon^{\delta} g(x), \quad \text{as } \varepsilon \to 0\,, \tag{2.7}$$

in the space $\mathcal{D}'(\mathbb{R})$, where g is non-null. One can then show that g has to be homogeneous of order δ. When $f(x_0 + \varepsilon x) = o\left(\varepsilon^{\delta}\right)$, as $\varepsilon \to 0^+$, because of equivalencies similar to (2.6), we sometimes write $f(x_0 + x) = o\left(x^{\delta}\right)$ (C), as $x \to 0^+$ [7, Thm. 6.9.1].

If we consider the limit of $f\,(x_0 + \varepsilon x)$ in $\mathcal{D}'(\mathbb{R} \setminus \{0\})$, then we obtain the concept of the distributional limit of $f(x)$ at $x = x_0$. Thus $\lim_{x \to x_0} f(x) = L$ distributionally if

$$\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi(x) \rangle = L \int_{-\infty}^{\infty} \phi(x)\, dx, \quad \phi \in \mathcal{D}(\mathbb{R} \setminus \{0\})\,. \tag{2.8}$$

Notice that the distributional limit $\lim_{x \to x_0} f(x)$ can be defined for $f \in \mathcal{D}'(\mathbb{R} \setminus \{x_0\})$. If the point value $f(x_0)$ exists distributionally then the distributional limit $\lim_{x \to x_0} f(x)$ exists and equals $f(x_0)$. On the other hand, if $\lim_{x \to x_0} f(x) = L$ distributionally then there exist constants $a_0, \ldots, a_n$ such that $f(x) = f_0(x) + \sum_{j=0}^{n} a_j \delta^{(j)}(x - x_0)$, where the distributional point value $f_0(x_0)$ exists and equals L.

We may also consider lateral limits. We say that the distributional lateral value $f(x_0^+)$ exists if $f(x_0^+) = \lim_{\varepsilon\to 0^+} f(x_0+\varepsilon x)$ in $\mathcal{D}'(0,\infty)$, that is,

$$\lim_{\varepsilon\to 0^+} \langle f(x_0+\varepsilon x), \phi(x)\rangle = f(x_0^+)\int_0^\infty \phi(x)\,dx, \quad \phi\in\mathcal{D}(0,\infty)\,. \tag{2.9}$$

Similar definitions apply to $f(x_0^-)$. Notice that the distributional limit $\lim_{x\to x_0} f(x)$ exists if and only if the distributional lateral limits $f(x_0^-)$ and $f(x_0^+)$ exist and coincide.

3. About distributional point values

Let $f\in\mathcal{D}'(\mathbb{R})$. Suppose that f has the distributional point value γ, in the sense of Łojasiewicz [12], at $x=x_0$,

$$f(x_0) = \gamma, \quad \text{distributionally}\,, \tag{3.1}$$

namely,

$$\lim_{\varepsilon\to 0} \langle f(x_0+\varepsilon x), \phi(x)\rangle = \gamma\int_{-\infty}^{\infty} \phi(x)\;dx, \quad \forall\phi\in\mathcal{D}(\mathbb{R})\,. \tag{3.2}$$

Suppose now that f belongs to a smaller space of distributions, $f\in\mathcal{A}'(\mathbb{R})$, for instance, $\mathcal{A}'=\mathcal{E}'$ or $\mathcal{A}'=\mathcal{S}'$. Then, is it true that (3.2) remains valid if $\phi\in\mathcal{A}(\mathbb{R})$? The answer, in general, is that (3.2) will not hold if $\phi\notin\mathcal{D}'(\mathbb{R})$. As an example we may take

$$f(x) = \delta(x-1)\,, \quad \phi_0(x) = \frac{\sin x}{x}\,. \tag{3.3}$$

Then

$$f(0) = 0, \quad \text{distributionally}\,, \tag{3.4}$$

while

$$\langle f(\varepsilon x)\,;\phi_0(x)\rangle = \sin\varepsilon^{-1} \tag{3.5}$$

does not have a limit as $\varepsilon\to 0$, even though the integral $\int_{-\infty}^{\infty}\phi(x)\;dx$ exists. Actually, in this example, $\langle f(\varepsilon x), \phi(x)\rangle$ does not have a limit as $\varepsilon\to 0$ for most functions $\phi\in\mathcal{E}(\mathbb{R})$.

Thus, (3.2) does not hold for general ϕ, even if $\langle f(x_0+\varepsilon x), \phi(x)\rangle$ is defined for every $\varepsilon\neq 0$ and even if the right hand side of (3.2) exists. However, as we are going to show, (3.2) is valid in *some* cases. In particular, it holds if $f\in\mathcal{S}'(\mathbb{R})$ and $\phi\in\mathcal{S}(\mathbb{R})$.

Theorem 1. *Let $f\in\mathcal{D}'(\mathbb{R}), \phi\in\mathcal{E}(\mathbb{R})$. Suppose,*

$$f(x_0) = \gamma, \quad \textit{distributionally}, \tag{3.6}$$

$$f(x) = O\left(|x|^\beta\right)\ (\mathrm{C}), \quad \textit{as}\quad |x|\to\infty\,, \tag{3.7}$$

$$\phi(x) = O(|x|^\alpha)\,, \quad \textit{strongly, as}\quad |x|\to\infty\,. \tag{3.8}$$

If $\alpha < -1$ *and* $\alpha + \beta < -1$, *then*

$$\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi(x) \rangle = \gamma \int_{-\infty}^{\infty} \phi(x)\, dx\,. \tag{3.9}$$

Proof. We may suppose $x_0 = 0$, without any loss of generality. We may also suppose $\beta \neq -1, -2, -3 \ldots$ since after slightly increasing β the hypothesis $\alpha + \beta < -1$ is still satisfied. If (3.6) holds then [12], there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there exists a primitive of order n of f, $F_n^{(n)} = f$, such that F_n is continuous in a neighborhood of $x = 0$, and

$$\lim_{x \to 0} \frac{n! F_n(x)}{x^n} = \gamma\,. \tag{3.10}$$

On the other hand [7, Def. 6.3.1], (3.7) entails the existence of $n_1 \in \mathbb{N}$ such that if $n \geq n_1$ there are polynomials $p(x) = p_n(x)$, $q(x) = q_n(x)$ of degree $n - 1$ at most such that $F_n(x)$ is continuous for all x and

$$F_n(x) = p_n(x) + O\left(x^{\beta + n}\right), \quad x \to +\infty\,, \tag{3.11}$$

$$F_n(x) = q_n(x) + O\left(|x|^{\beta + n}\right), \quad x \to -\infty\,. \tag{3.12}$$

In general p_n and q_n do not vanish nor are they equal. However, a direct computation shows that (3.9) holds if ϕ satisfies (3.8) and $f(x) = \delta^{(j)}(x - a)$ for $a \neq 0$ and $j = 0, 1, 2, \ldots$. Therefore, by adding a suitable distribution whose support is a finite set that does not contain the origin, we may suppose that $p_n(x) = 0$. We could suppose also that $n \geq \max\{n_0, n_1\}$.

Observe now that (3.9) holds if *supp* ϕ is compact, because of (3.6). Hence our result would follow if we prove it under the additional assumption that *supp* $\phi \subseteq [1, \infty)$, since the case *supp* $\phi \subseteq (-\infty, -1]$ follows by symmetry, and each $\phi \in \mathcal{E}$ that satisfies (3.8) can be written as $\phi = \phi_1 + \phi_2 + \phi_3$, where *supp* $\phi_1 \subseteq (-\infty, -1]$, *supp* $\phi_2 \subseteq [1, \infty)$, and *supp* ϕ_3 is compact.

If *supp* $\phi \subseteq [1, \infty)$ and $p_n = 0$, then

$$\begin{aligned} \langle f(\varepsilon x), \phi(x) \rangle &= (-1)^n \left\langle G_n(\varepsilon x), x^n \phi^{(n)}(x) \right\rangle, \\ &= (-1)^n \int_{-\infty}^{\infty} G_n(\varepsilon x)\, x^n \phi^{(n)}(x)\, dx\,, \end{aligned} \tag{3.13}$$

where $G_n(x) = x^{-n} F_n(x)$ is continuous in $\mathbb{R}$, satisfies $G_n(0) = \gamma / n!$ and

$$|G_n(x)| \leq \begin{cases} M_1\,, & |x| \leq 1\,, \\ M_1 |x|^\beta\,, & |x| \geq 1\,, \end{cases} \tag{3.14}$$

for some constant M_1. There exists another constant M_2 such that

$$\left|\phi^{(n)}(x)\right| \leq M_2 |x|^{\alpha - n}\,, \quad \forall x \in \mathbb{R}\,. \tag{3.15}$$

Thus, if $|\varepsilon| \leq 1$,

$$\left|G_n(\varepsilon x)\, x^n \phi^{(n)}(x)\right| \leq M\, |x|^\rho\, H(x-1)\ , \tag{3.16}$$

where $\rho = \max\{\alpha, \alpha+\beta\}\,,\ M = M_1 M_2$, and H is the Heaviside function. But $\rho < -1$, and so $x^\rho H(x-1)$ belongs to $L^1(\mathbb{R})$ and thus we can use the Lebesgue dominated convergence theorem to obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\langle f(\varepsilon x), \phi(x)\rangle &= \lim_{\varepsilon\to 0}(-1)^n \int_{-\infty}^{\infty} G_n(\varepsilon x)\, x^n \phi^{(n)}(x)\ dx \\ &= \frac{(-1)^n \gamma}{n!}\int_{-\infty}^{\infty} x^n \phi^{(n)}(x)\ dx \\ &= \gamma \int_{-\infty}^{\infty}\phi(x)\ dx\,,\end{aligned}$$

as required. □

In particular, (3.9) holds if f is a tempered distribution and ϕ is a rapidly decreasing smooth function.

Corollary 1. *Let $f \in \mathcal{S}'(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$. If $f(x_0) = \gamma$ distributionally then*

$$\lim_{\varepsilon\to 0}\langle f(x_0+\varepsilon x), \phi(x)\rangle = \gamma\int_{-\infty}^{\infty}\phi(x)\ dx\,.$$

For our purposes, we need a generalization of Theorem 1 .

Theorem 2. *Let $f \in \mathcal{D}'(\mathbb{R})\,,\ \phi \in \mathcal{E}(\mathbb{R})\,.$ Suppose*

$$f(x_0+\varepsilon x) = o(\varepsilon^\kappa)\,,\quad \varepsilon \to 0^+\,, \tag{3.17}$$

in the space $\mathcal{D}'(\mathbb{R})\,,$ while

$$f(x) = O\left(|x|^\beta\right)\ (\mathrm{C}),\quad |x| \to \infty\,, \tag{3.18}$$

$$\phi(x) = O(|x|^\alpha)\ ,\quad \textit{strongly, as}\quad |x| \to \infty\,. \tag{3.19}$$

If $\alpha+\kappa < -1$ and $\alpha+\beta < -1$, then

$$\langle f(x_0+\varepsilon x), \phi(x)\rangle = o(\varepsilon^\kappa)\,,\quad \varepsilon \to 0^+. \tag{3.20}$$

Proof. We could give a proof similar to that of Theorem 1, but we prefer to derive Theorem 2 from Theorem 1. Indeed, if (3.17) is satisfied then the distribution

$$g_0(x) = \frac{f(x)}{|x-x_0|^\kappa}\,, \tag{3.21}$$

that belongs to $\mathcal{D}'(\mathbb{R}\setminus\{x_0\})\,,$ admits at least an extension $g \in \mathcal{D}'(\mathbb{R})$ that satisfies $g(x_0) = 0$ (if (3.17) is not necessarily satisfied, then g_0 would have many extensions to $\mathcal{D}'(\mathbb{R})$, but perhaps none that have a distributional value at $x = x_0$). The result then follows by writing $\phi = \phi_1 + \phi_2$, $supp\, \phi_1$ compact, $x_0 \notin supp\, \phi_2$:

$$\begin{aligned}\langle f(x_0 + \varepsilon x), \phi(x)\rangle &= \langle f(x_0 + \varepsilon x), \phi_1(x)\rangle \\ &\qquad + \varepsilon^{\kappa} \langle g(x_0 + \varepsilon x), |x - x_0|^{\kappa} \phi_2(x)\rangle \\ &= o(\varepsilon^{\kappa}) + \varepsilon^{\kappa} o(1) = o(\varepsilon^{\kappa}) ,\end{aligned}$$

the first bound because of the estimate (3.17), and the second because of Theorem 1. □

4. Boundary values of harmonic functions

Let $U(x, y)$ be harmonic in the upper half-plane, $y > 0$. Suppose U has distributional boundary values,

$$u(x) = \lim_{y \to 0} U(x, y) , \tag{4.1}$$

in $\mathcal{D}'(\mathbb{R})$. We want to study the relationship between the distributional behavior of $u(x)$ at $x = x_0$ and the limits of $U(x, y)$ as $(x, y) \to (x_0, 0)$; for ease of writing we shall take $x_0 = 0$.

Suppose first that $u(x) = 0$ in a neighborhood $(-\varsigma, \varsigma)$ of $x_0 = 0$. Then by applying the reflection principle [17, Section 4.5], [1, Section 3.4], U admits a harmonic extension to a region that contains $(-\varsigma, \varsigma)$. Therefore, U is real analytic at $(0, 0)$. It follows that if $(\widetilde{x}, \widetilde{y})$ is any fixed point with $\widetilde{y} > 0$, then $U(\varepsilon\widetilde{x}, \varepsilon\widetilde{y}) = O(\varepsilon)$, as $\varepsilon \to 0^+$. *The behavior of* $U(\varepsilon\widetilde{x}, \varepsilon\widetilde{y})$ *up to orders of magnitude smaller than* $O(\varepsilon)$ *depends only on the local behavior of* $u(x)$ *near* $x = 0$.

Therefore, we may suppose that u has compact support. Then we have the Poisson formula,

$$U(x, y) = \frac{1}{\pi} \left\langle u(t), \frac{y}{(t - x)^2 + y^2} \right\rangle . \tag{4.2}$$

So, if $\widetilde{y} > 0$,

$$U(\varepsilon\widetilde{x}, \varepsilon\widetilde{y}) = \frac{1}{\pi} \left\langle u(\varepsilon t), \frac{\widetilde{y}}{(t - \widetilde{x})^2 + \widetilde{y}^2} \right\rangle , \tag{4.3}$$

thus if $u(\varepsilon t) = o(\varepsilon^{\kappa})$, as $\varepsilon \to 0^+$, in the space $\mathcal{D}'(\mathbb{R})$, and $\kappa < 1$, then the Theorem 2 immediately yields $U(\varepsilon\widetilde{x}, \varepsilon\widetilde{y}) = o(\varepsilon^{\kappa})$, as $\varepsilon \to 0^+$. The estimate is uniform in $(\widetilde{x}, \widetilde{y}) \in K$, where K is any compact subset of the (open) upper half-plane. Clearly the estimates do not hold if $\kappa = 1$. Summarizing.

Theorem 3. *Let* $U(x, y)$ *be harmonic in the upper half-plane* $y > 0$, *with distributional boundary values* $u(x) = U(x, 0^+)$. *If*

$$u(\varepsilon x) = o(\varepsilon^{\kappa}) , \quad \varepsilon \to 0^+ , \tag{4.4}$$

in the space $\mathcal{D}'(\mathbb{R})$ *and*

$$\kappa < 1\,, \tag{4.5}$$

then

$$U(x,y) = o\left(\left(x^2+y^2\right)^{\kappa/2}\right), \quad as\ \ (x,y) \to (0,0)\ , \tag{4.6}$$

uniformly in any angular domain $y \geq m\,|x|$, *for* $m > 0$.

Using the Theorem 3, one may easily obtain the behavior of $U(x,y)$ as $(x,y) \to (0,0)$ if the behavior of $u(x)$ as $x \to 0$, *distributionally* (i.e., in the space $\mathcal{D}'(\mathbb{R})$), is known. In particular, if the distributional point value

$$u(x_0) = \gamma\,, \tag{4.7}$$

exists in the sense of Łojasiewicz, then

$$\lim_{(x,y)\to(x_0,0)} U(x,y) = \gamma\,. \tag{4.8}$$

This result holds even if u is a hyperfunction [19]. If, on the other hand, u has distributional limits from both sides of u, and no delta functions at x_0 present, in the sense that $\forall\phi \in \mathcal{D}(\mathbb{R})$

$$\lim_{\varepsilon\to 0^+} \langle u(x_0+\varepsilon x)\,, \phi(x)\rangle = \gamma_- \int_{-\infty}^{0} \phi(x)\ dx + \gamma_+ \int_0^\infty \phi(x)\ dx\,, \tag{4.9}$$

then

$$\lim_{\varepsilon\to 0} U(x_0+\varepsilon\widetilde{x}, \varepsilon\widetilde{y}) = \frac{\vartheta}{\pi}\gamma_- + \left(1-\frac{\vartheta}{\pi}\right)\gamma_+\,, \tag{4.10}$$

where $\vartheta = \arg(\widetilde{x}+i\widetilde{y})$, $0 < \vartheta < \pi$. Similarly, if

$$u(x_0+\varepsilon x) \sim \varepsilon^\kappa\left(\gamma_- x_-^\kappa + \gamma_+ x_+^\kappa\right), \quad \varepsilon \to 0^+, \tag{4.11}$$

that is, if $\forall\phi \in \mathcal{D}(\mathbb{R})$,

$$\lim_{\varepsilon\to 0^+} \frac{\langle u(x_0+\varepsilon x)\,, \phi(x)\rangle}{\varepsilon^\kappa} = \gamma_- \int_{-\infty}^{0} |x|^\kappa\,\phi(x)\ dx + \gamma_+ \int_0^\infty |x|^\kappa\,\phi(x)\ dx\,, \tag{4.12}$$

then, with $\widetilde{r}^2 = \widetilde{x}^2 + \widetilde{y}^2$,

$$\lim_{\varepsilon\to 0} \frac{U(x_0+\varepsilon\widetilde{x}, \varepsilon\widetilde{y})}{\varepsilon^\kappa} = \frac{\widetilde{r}^\kappa}{\sin\kappa\pi}\left(\gamma_+ \sin\kappa(\pi-\vartheta) + \gamma_- \sin\kappa\vartheta\right)\,, \tag{4.13}$$

if $\kappa < 1, \kappa \neq 0, -1, -2, -3, \ldots$. When $\kappa = -k = -1, -2, -3, \ldots$, we use the dispersion relations [7, (2.61)-(2.62)]

$$\frac{1}{(x \pm i0)^k} = \frac{1}{x^k} \mp \frac{i\pi\,(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}(x)\ , \tag{4.14}$$

where $\frac{1}{x^k}$ is the canonical regularization [7, pg. 69] to obtain that if

$$u\left(x_0+\varepsilon x\right)\sim\varepsilon^{-k}\left(\rho\,\delta^{(k-1)}\left(x\right)+\sigma\,x^{-k}\right),\ \ \varepsilon\rightarrow 0^{+},\tag{4.15}$$

that is, if $\forall\phi\in\mathcal{D}\left(\mathbb{R}\right)$,

$$\lim_{\varepsilon\rightarrow 0}\varepsilon^{k}\left\langle u\left(x_0+\varepsilon x\right),\phi\left(x\right)\right\rangle=\rho\phi^{(k-1)}\left(0\right)+\sigma\left\langle x^{-k},\phi\left(x\right)\right\rangle,\tag{4.16}$$

then

$$\lim_{\varepsilon\rightarrow 0}\varepsilon^{k}U\left(x_0+\varepsilon\widetilde{x}+\varepsilon\widetilde{y}\right)=\frac{\left(-1\right)^{k-1}\left(k-1\right)!\rho}{\pi}\Im m\left(\frac{1}{\widetilde{x}+i\widetilde{y}}\right)+\sigma\Re e\left(\frac{1}{\widetilde{x}+i\widetilde{y}}\right).\tag{4.17}$$

It should be remarked that Theorem 3, as well as formulas (4.8), (4.10), (4.13), and (4.17) have corresponding valid results for harmonic functions having distributional boundary values on a smooth contour or on a smooth arc. Indeed, we just have to use conformal mapping.

5. Boundary values of analytic functions

Let $f\in\mathcal{D}'\left(\mathbb{R}\right)$ with *supp* $f\subseteq[0,\infty)$. Suppose f is the jump of the sectionally analytic function $F\left(z\right),\ z\in\mathbb{C}\setminus\mathbb{R}$, that admits an analytic continuation to $z\in\mathbb{C}\setminus[0,\infty)$, across the real axis,

$$f\left(x\right)=F\left(x+i0\right)-F\left(x-i0\right),$$

where the two limits

$$F\left(x\pm i0\right)=\lim_{\varepsilon\rightarrow 0^{+}}F\left(x\pm i\varepsilon\right),\tag{5.1}$$

exist in $\mathcal{D}'\left(\mathbb{R}\right)$.

Then, if

$$f\left(\varepsilon x\right)=o\left(\varepsilon^{\kappa}\right),\ \ \varepsilon\rightarrow 0^{+},\tag{5.2}$$

in the space $\mathcal{D}'\left(\mathbb{R}\right)$ and $\kappa<0$, we can conclude that

$$F\left(z\right)=o\left(\left|z\right|^{\kappa}\right),\ \ z\rightarrow 0,\ \ z\in\mathbb{C}\setminus[0,\infty).\tag{5.3}$$

Actually (5.3) is uniform in any sector $\varsigma<\arg z<2\pi-\varsigma$, for any ς with $0<\varsigma<2\pi$. To prove (5.3) we decompose f as $f=f_1+f_2$, where f_1 has compact support and where $0\notin$ *supp* f_2. Then $F\left(z\right)=F_1\left(z\right)+F_2\left(z\right)$, where

$$F_1\left(z\right)=\frac{1}{2\pi i}\left\langle f_1\left(x\right),\frac{1}{x-z}\right\rangle,\tag{5.4}$$

while $F_2=F-F_1$. Since 0 does not belong to the support of f_2, the jump of F_2, it follows [1, Thm. 3.14], [2, Section 5.8], [6], that F_2 admits an analytic continuation to a neighborhood of $z=0$. Therefore, $F_2\left(z\right)=O\left(1\right),\ z\rightarrow 0$.

The behavior of $F_1(z)$ as $z \to 0$ in $\mathbb{C} \setminus [0, \infty)$ follows directly from Theorem 2. Indeed, let $\xi \in \mathbb{C}$, $|\xi| = 1, \xi \neq 1$, then if $z = \varepsilon\xi$, we obtain

$$F_1(\varepsilon\xi) = \frac{1}{2\pi i}\left\langle f_1(x), \frac{1}{x - \varepsilon\xi}\right\rangle = \frac{1}{2\pi i}\left\langle f(\varepsilon x), \frac{1}{x - \xi}\right\rangle. \tag{5.5}$$

In this case $f(x) = o(\varepsilon^{\kappa})$, as $\varepsilon \to 0^+$ in the space $\mathcal{D}'(\mathbb{R})$, $f(x) = O\left(x^{\beta}\right)$ (C), $x \to \infty$ for any $\beta \in \mathbb{R}$, while $\phi(x) = o(x^{\alpha})$ strongly as $x \to \infty$ for any $\alpha > -1$. If we take β negative and α close to -1 we obtain $\alpha + \kappa < -1$, $\alpha + \beta < -1$. Therefore, $F_1(\varepsilon\xi) = o(\varepsilon^{\kappa})$, $\varepsilon \to 0^+$, and (5.3) follows. That (5.3) is uniform in ξ for $\varsigma < \arg\xi < 2\pi - \varsigma$, whenever $0 < \varsigma < 2\pi$ is clear.

Suppose now that $\kappa = 0$, that is,

$$f(\varepsilon x) = o(1), \quad \varepsilon \to 0, \text{ in the space } \mathcal{D}'(\mathbb{R}). \tag{5.6}$$

Then the Theorem 2 cannot be applied, but we may proceed by observing that distributional limits can be differentiated, thus,

$$\varepsilon f'(\varepsilon x) = o(1), \quad \varepsilon \to 0, \quad \text{distributionally,}$$

or

$$f'(\varepsilon x) = o\left(\varepsilon^{-1}\right), \quad \varepsilon \to 0, \quad \text{distributionally.} \tag{5.7}$$

But if the jump of F is f then the jump of F' is f'. Thus, from (5.3) with $\kappa = -1$ we obtain

$$F'(z) = o\left(|z|^{-1}\right), \quad z \to 0, \quad z \in \mathbb{C} \setminus [0, \infty). \tag{5.8}$$

Integrating this order relation we obtain

$$F(z) = o\left(\ln|z|^{-1}\right), \quad z \to 0, \quad z \in \mathbb{C} \setminus [0, \infty), \tag{5.9}$$

uniformly in $\varsigma < \arg z < 2\pi - \varsigma$ whenever $0 < \varsigma < 2\pi$.

Therefore, we obtain the ensuing result.

Theorem 4. *Let $f \in \mathcal{D}'(\mathbb{R})$ be the jump of the sectionally analytic function $F(z)$, $z \in \mathbb{C} \setminus [0, \infty)$. Suppose*

$$f(x) = \sum_{j=1}^{n} \gamma_j x^{\kappa_j} + \sum_{k=1}^{m} (\rho_k \delta^{(k-1)}(x) + \sigma_k x^{-k}) + \gamma_0 + o(1) \text{ (C)}, \tag{5.10}$$

as $x \to 0^+$, *where* $\kappa_1 < \kappa_2 < \cdots < \kappa_n < 0$, $\kappa_j \neq -1, -2, -3, \ldots$. *Then*

$$F(z) = \sum_{j=1}^{n} \frac{\gamma_j}{1 - e^{2\pi i \kappa_j}} z^{\kappa_j} + \sum_{k=1}^{m} \frac{i\rho_k \ln z + (-1)^k (k-1)! \sigma_k}{2\pi z^k} + \frac{\gamma_0 \ln z^{-1}}{2\pi i} + o\left(\ln |z|^{-1}\right), \tag{5.11}$$

as $z \to 0$ *in* $\mathbb{C} \setminus [0, \infty)$, *uniformly in* $\varsigma < \arg z < 2\pi - \varsigma$ *whenever* $0 < \varsigma < 2\pi$.

When $\kappa > 0$, the leading asymptotic behavior of $F(z)$ in (5.3) will depend globally on f, not just on its local behavior near $x = 0$.

Observe also that the corresponding estimates near an endpoint of any smooth arc C hold. Indeed, using a conformal map, we obtain that if $F(z)$ is analytic in $\mathbb{C} \setminus C$ and has the distributional jump $f(\xi)$ along C, that satisfies $f(\xi) = o(|\xi - a|^\kappa)$ (C), as $\xi \to a$, where a is an endpoint of C, then if $\kappa < 0$,

$$F(z) = o(|z - a|^\kappa), \quad z \to a, \quad z \in \mathbb{C} \setminus C, \tag{5.12}$$

while if $\kappa = 0$,

$$F(z) = o\left(\ln |z - a|^{-1}\right), \quad z \to a \quad z \in \mathbb{C} \setminus C, \tag{5.13}$$

the order relations being uniform for the angular non-tangential approach to C. Results similar to the Theorem 4 also follow.

We also have the following useful result. We use the notation $\operatorname{sgn} x$ for the signum function, $x/|x|$ for $x \neq 0$.

Theorem 5. *Let F be analytic in the upper half-plane, with distributional boundary values $f(x) = F(x + i0)$. Suppose f has a distributional symmetric jump d at $x = 0$, in the sense that*

$$f(\varepsilon x) - f(-\varepsilon x) = d \operatorname{sgn} x + o(1), \quad \varepsilon \to 0^+, \tag{5.14}$$

in the space $\mathcal{D}'(\mathbb{R})$. Then,

$$F(z) = \frac{d}{\pi i} \ln z^{-1} + o(|\ln z|), \quad \textit{as} \quad z \to 0, \tag{5.15}$$

in the upper-half plane, uniformly in $\varsigma < \arg z < \pi - \varsigma$ *whenever* $0 < \varsigma < 2\pi$.

Proof. Let $\omega = z^2$ and $G(\omega) = F(z)$. Then G is analytic in $\mathbb{C} \setminus [0, \infty)$, the distributional boundary limits $G(x \pm i0)$ exist, and coincide in the negative semi-axis $x < 0$. Thus $g(x) = G(x + i0) - G(x - i0)$ vanishes for $x < 0$ and equals $f(t) - f(-t)$, $x = t^2$, for $x > 0$, and from (5.14), has the distributional lateral limit d as $x \to 0^+$. Therefore, using (5.10),

$$\begin{aligned} F(z) &= G(\omega) \\ &= \frac{d}{2\pi i} \ln \omega^{-1} + o(|\ln \omega|) \\ &= \frac{d}{\pi i} \ln z^{-1} + o(|\ln z|) \,, \end{aligned}$$

as $\omega \to 0$ in $\mathbb{C} \setminus [0, \infty)$, or, what is the same, as $z \to 0$ in $\Im m\, z > 0$. □

It is important to point out what Theorem 5 does not say. Let F be a sectionally analytic function, defined in $\mathbb{C} \setminus \mathbb{R}$. Suppose the distributional boundary limits $F(x \pm i0)$ exist, and let $f(x) = F(x + i0) - F(x - i0)$ be the corresponding jump. Then the existence of the distributional symmetric jump of $f(x)$ as $x \to 0$ does not imply that F satisfies an asymptotic approximation of the type (5.15). Indeed, just take F analytic in the upper half-plane and zero in the lower half-plane!

When (5.14) holds as a lateral limit, that is, in the space $\mathcal{D}'(\mathbb{R} \setminus \{0\})$ then there exists a polynomial P, whose constant term we may take to be 0, such that

$$F(z) = P\left(\frac{1}{z}\right) + \frac{d}{\pi i} \ln z^{-1} + o(|\ln z|)\,, \quad \text{as } z \to 0\,. \tag{5.16}$$

Using a suitable conformal mapping one can prove the equivalent of (5.16) for other geometries. In particular, if $F(z)$ is analytic in $|z| < 1$, has distributional boundary values $f(\theta) = \lim_{r \to 1^-} F\left(re^{i\theta}\right)$, and f has a distributional symmetric lateral limit, $d = \lim_{\theta \to 0} \left(f(\theta) - f(-\theta)\right)$, then

$$F(z) = P\left(\frac{1}{z-1}\right) + \frac{d \,\ln (z-1)^{-1}}{\pi i} + o(|\ln (z-1)|)\,, \quad z \to 1\,, \tag{5.17}$$

where P is a polynomial. In case the odd function $f(\theta) - f(-\theta)$ does not contain any delta function at $\theta = 0$, we can conclude that P is just a constant, that can be taken to be 0.

The Theorem 5 has an interesting generalization. If F is analytic in a sector $\theta_1 < \arg z < \theta_2$, has distributional boundary values at $\arg z = \theta_1$ and $\arg z = \theta_2$, $f_1(r) = f\left(re^{i\theta_1}\right)$ and $f_2(r) = f\left(re^{i\theta_2}\right)$, respectively, and if $f_1(r) - f_2(r)$ has a distributional limit d as $r \to 0^+$ and no delta functions at the corner, then

$$F(z) = \frac{d \,\ln z^{-1}}{(\theta_1 - \theta_0)\, i} + o(|\ln z|)\,, \quad \text{as} \quad z \to 0\,, \tag{5.18}$$

in the sector. Theorem 4 corresponds to the case the angle aperture is $\theta_1 - \theta_0 = 2\pi$, while (5.15) covers the case $\theta_1 - \theta_0 = \pi$. The proof is basically the same as that of Theorem 5.

We emphasize that we do not assume the existence of the distributional limits of $f_1(r)$ and $f_2(r)$ as $r \to 0^+$, but the existence of the distributional limit of $f_1(r) - f_2(r)$. Actually, more is true, since if the distributional limits $f_1(0^+)$ and $f_2(0^+)$ exist then they must coincide, and thus the jump has to vanish!. Indeed, by using conformal mapping, it is enough to see it in the case of the unit disc, and (5.17) shows that unless P is a constant and $d = 0$ the lateral limits do not exist. Therefore, (5.17) gives another proof of the following beautiful result [5, Thm. 3.1] .

Theorem 6. *Let f be analytic inside a region bounded by a smooth contour C. Suppose F has distributional boundary values, $f(\xi)$, as $z \to \xi \in C$. Let $\xi_0 \in C$ and suppose the distributional lateral limits $f\left(\xi_0^-\right)$ and $f\left(\xi_0^+\right)$ both exist. Then $f\left(\xi_0^-\right) = f\left(\xi_0^+\right)$, the distributional point value $f(\xi_0)$ exists, and equals this common value.*

6. The Ferenc Lukács theorem

Let $f \in \mathcal{D}'(\mathbb{R})$ be periodic, of period 2π, with Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) \ . \tag{6.1}$$

Let

$$\psi_x(\theta) = \frac{1}{2}\left(f(x+\theta) - f(x-\theta)\right) \ . \tag{6.2}$$

Observe that as function of θ, $\psi_x(\theta)$ is odd and has the Fourier sine series representation

$$\psi_x(\theta) = -\sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx) \sin k\theta \,. \tag{6.3}$$

Suppose now that the distributional lateral limit $\psi_x(0^+)$ exists, so that $\psi_x(0^-) = -\psi_x(0^+)$ also exists, and that $\psi_x(\theta)$ contains no delta functions at $x = 0$. This means that

$$\psi_x(\varepsilon\theta) = d \operatorname{sgn}\theta + o(1)\,, \quad \varepsilon \to 0^+, \quad \text{in the space } \mathcal{D}'(\mathbb{R}) \ , \tag{6.4}$$

where $d = \psi_x(0^+)$ is half the value of the jump of ψ_x at $\theta = 0$. Equivalently, there exists $n \in \mathbb{N}$ and a primitive of order n of ψ_x, Ψ_x, with $\Psi_x^{(n)}(\theta) = \psi_x(\theta)$, that is continuous near $\theta = 0$, and satisfies

$$\lim_{\theta \to 0} \frac{n!\Psi_x(\theta)}{\theta^n \operatorname{sgn}\theta} = d \,. \tag{6.5}$$

Observe that the existence of the distributional lateral limits of $\psi_x(\theta)$ at $\theta = 0$ does not imply the existence of the distributional lateral limits of $f(t)$ at $t = x$. Similarly, the fact that $\psi_x(\theta)$ contains no delta functions at $\theta = 0$

does not imply that the same holds for $f(t)$ at $t = x$. The function $\psi_x(\theta)$ captures the *symmetric* jump structure of $f(t)$ around $t = x$.

Let us now consider the conjugate Abel-Poisson means of f and ψ_x :

$$\widetilde{f}(r,t) = \sum_{k=1}^{\infty} (a_k \sin kt - b_k \cos kt)\, r^k\,, \tag{6.6}$$

$$\widetilde{\psi_x}(r,\theta) = \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx) \cos k\theta \; r^k\,, \tag{6.7}$$

for $0 \le r < 1$. Then for $t = x$ and $\theta = 0$ we obtain

$$\widetilde{f}(r,x) = \widetilde{\psi_x}(r,0)\ . \tag{6.8}$$

Therefore, the behavior of the conjugate Abel-Poisson means $\widetilde{f}(r,x)$ as $r \to 1^-$ depends only on the *symmetric* behavior of $f(t)$ about $t = x$.

Next, we define

$$G(\zeta) = \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx)\, \zeta^k, \quad |\zeta| < 1\,. \tag{6.9}$$

The analytic function G has the distributional boundary value $g(\theta) = \lim_{r\to 1} g\left(re^{i\theta}\right)$ given by

$$\begin{aligned} g(\theta) &= \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx)\, e^{ik\theta} \\ &= \widetilde{\psi_x}(\theta) - i\psi_x(\theta)\ , \end{aligned} \tag{6.10}$$

where $\widetilde{\psi_x}(\theta) = \widetilde{\psi_x}(1,\theta)$. The distribution g has the distributional symmetric jump function

$$\{g(\theta) - g(-\theta)\} = -2i\psi_x(\theta)\ ,$$

that has the lateral limit $-2id$ as $\theta \to 0^+$, and that contains no delta functions at $\theta = 0$. Then formula (5.17) yields

$$\begin{aligned} G(\zeta) &= \frac{-2id}{2\pi i} \ln(\zeta - 1)^{-1} + o(|\ln(\zeta - 1)|) \\ &= -\frac{d}{\pi} \ln(\zeta - 1)^{-1} + o(|\ln(\zeta - 1)|)\,, \quad \zeta \to 1\,. \end{aligned} \tag{6.11}$$

But if $\zeta = r \in \mathbb{R}$ then

$$G(r) = \widetilde{\psi_x}(r,0)\ , \tag{6.12}$$

and thus

$$\lim_{r\to 1^-} \frac{\widetilde{\psi}_x(r,0)}{\ln(1-r)} = \frac{d}{\pi}\,, \tag{6.13}$$

which, because of (6.8), is the Móricz-Lukács formula. We thus have proved the following result.

Theorem 7. *Let $f \in \mathcal{D}'(\mathbb{R})$ be periodic of period 2π. Let $x \in \mathbb{R}$, such that f has a distributional symmetric jump of magnitude d at x, without any delta functions present, that is,*

$$f(x+\varepsilon\theta) - f(x-\varepsilon\theta) = d\operatorname{sgn}\theta + o(1)\,,\quad \varepsilon \to 0^+, \tag{6.14}$$

in $\mathcal{D}'(\mathbb{R})$. If $\widetilde{f}(r,x)$ is the Abel-Poisson mean of the conjugate Fourier series of f then

$$\lim_{r\to 1^-} \frac{\widetilde{f}(r,x)}{\ln(1-r)} = \frac{d}{\pi}\,. \tag{6.15}$$

More generally, we could just assume that the odd generalized function $\psi_x(\theta)$ has a distributional limit d as $\theta \to 0^+$, so that $\psi_x(\theta)$ could have a sum of (odd) derivatives of the Dirac delta function concentrated at $\theta = 0$. Then (5.17) yields

$$G(\zeta) = P\left(\frac{1}{\zeta-1}\right) - \frac{d}{\pi}\ln(\zeta-1)^{-1} + o(|\ln(\zeta-1)|)\,,\quad \zeta \to 1\,. \tag{6.16}$$

Therefore, if we use the notion of the finite part of a limit [7, Section 2.4; exercise 2.3.4], the limit of the expression obtained by removing the singular part, we obtain the next, more complete generalization of the Móricz-Lukács formula.

Theorem 8. *Let $f \in \mathcal{D}'(\mathbb{R})$ be periodic of period 2π. Let $x \in \mathbb{R}$, such that $f(x+\theta) - f(x-\theta)$ has a distributional limit d as $\theta \to 0^+$, that is,*

$$f(x+\varepsilon\theta) - f(x-\varepsilon\theta) = d\operatorname{sgn}\theta + o(1)\,,\quad \varepsilon \to 0^+, \tag{6.17}$$

in $\mathcal{D}'(\mathbb{R}\setminus\{0\})$. If $\widetilde{f}(r,x)$ is the Abel-Poisson mean of the conjugate Fourier series of f and F.p. *stands for the finite part, then*

$$\text{F.p.}\lim_{r\to 1^-} \frac{\widetilde{f}(r,x)}{\ln(1-r)} = \frac{d}{\pi}\,. \tag{6.18}$$

It is to be observed that Fejér gave the first result in the subject of determining the jumps of a function in terms of the partial sums of its Fourier series [8], and that later Zygmund [21, 9.11, Chap. III, pg. 108] gave a corresponding theorem for the Abel-Poisson means of the Fourier series. It is a simple matter to derive a generalization of the Fejér-Zygmund result as a corollary of our analysis.

Theorem 9. *Let $f \in \mathcal{D}'(\mathbb{R})$ be periodic of period 2π. Let $x \in \mathbb{R}$, such that $f(x+\theta) - f(x-\theta)$ has a distributional limit d as $\theta \to 0^+$, that is,*

$$f(x+\varepsilon\theta) - f(x-\varepsilon\theta) = d \operatorname{sgn}\theta + o(1), \quad \varepsilon \to 0^+, \tag{6.19}$$

in $\mathcal{D}'(\mathbb{R}\setminus\{0\})$. If $f_{,x}(r,x)$ is the Abel-Poisson mean of the Fourier series of the derivative $f_{,x} = f'(x)$ then

$$\text{F.p.} \lim_{r\to 1^-} (1-r) f_{,x}(r,x) = \frac{d}{\pi}. \tag{6.20}$$

Proof. Indeed, an easy computation shows that $f_{,x}(r,x) = -rG'(r)$, and thus (6.20) will follow immediately if we can show that (6.16) can be differentiated with respect to z. Actually, by subtracting an appropriate term we may suppose that $P = 0$ and that $d = 0$, so that it is enough to show that if

$$g(\varepsilon\theta) - g(-\varepsilon\theta) = o(1), \quad \text{as } \varepsilon \to 0^+, \tag{6.21}$$

in $\mathcal{D}'(\mathbb{R})$ then

$$G'(\zeta) = o\left(\frac{1}{\zeta - 1}\right), \quad \text{as } \zeta \to 1. \tag{6.22}$$

But distributional limits can be differentiated, so taking the second derivative of (6.21) we obtain

$$g''(\varepsilon\theta) - g''(-\varepsilon\theta) = o\left(\frac{1}{\varepsilon^2}\right), \quad \text{as } \varepsilon \to 0^+, \tag{6.23}$$

in $\mathcal{D}'(\mathbb{R})$ and therefore our analysis at the beginning of Section 5 yields

$$G''(\zeta) = o\left(\frac{1}{(\zeta-1)^2}\right), \quad \text{as } \zeta \to 1, \tag{6.24}$$

since g'' is the distributional boundary value of G''. Hence (6.22) follows by integration. ☐

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