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TWO EXPONENTIAL FORMULAS FOR $\alpha-$TIMES INTEGRATED SEMIGROUPS ($\alpha \in \mathbb{R}^+$)

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Abstract. In this paper X is a Banach space, $(S(t))_{t\geq 0}$ is non-degenerate $\alpha-$times integrated, exponentially bounded semigroup on X ($\alpha \in \mathbb{R}^+$), $M \geq 0$ and $\omega_0 \in \mathbb{R}$ are constants such that $\|S(t)\| \leq Me^{\omega_0 t}$ for all $t \geq 0$, γ is any positive constant greater than ω_0, Γ is the Gamma-function, $(C,\beta) - \lim$ is the Cesàro-β limit. Here we prove that

$$\lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left(\frac{n+1}{s}\right)^{n+1} R^{n+1}\left(\frac{n+1}{s}, A\right) x\, ds = S(T)x,$$

for every $x \in X$, and the limit is uniform in $T > 0$ on any bounded interval. Also we prove that

$$S(t)x = \frac{1}{2\pi i}(C,\beta) - \lim_{\omega\to\infty} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} \frac{R(\lambda, A)x}{\lambda^\alpha}\, d\lambda,$$

for every $x \in X$, $\beta > 0$ and $t \geq 0$.

1. Introduction

Once integrated exponentially bounded semigroups of operators on a Banach space were introduced and investigated in [1], [2], [3], [7] and studied by Arendt, Kellermann, Hieber, Thieme and many others. The $n-$times integrated exponentially bounded semigroups of operators, $n \in \mathbb{N}$, on a Banach space were introduced and investigated in [4] by Neubrander. The $\alpha-$times integrated exponentially bounded semigroups of operators, $\alpha \in \mathbb{R}^+$, on a Banach space were investigated in [9] , by Mijatović, Pilipović and Vajzović. Some exponential formulas for C_0-semigroups of operators on a Banach

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space X are given and proved in [6]. These formulas are the motivation for our further analysis.

2. Preliminaries from the theory of $\alpha-$times integrated semigroup $(\alpha \in \mathbb{R}^+)$

We refer to [9] for the notion of $\alpha-$ times integrated semigroups $(\alpha \in \mathbb{R}^+)$. We denote by X a Banach space with the norm $\|\cdot\|$; $L(X) = L(X, X)$ is the space of bounded linear operators from X into X.

Definition 2.1. Let $(S(t))_{t\geq 0}$ be a strongly continuous family of operators in $L(X)$ and $\alpha \in \mathbb{R}^+$. Then, $(S(t))_{t\geq 0}$ is called an $\alpha-$times integrated semigroup if $S(0) = 0$ and the following holds

$$S(t)S(s) = \frac{1}{\Gamma(\alpha)}\left[\int_t^{t+s} (t+s-r)^{\alpha-1} S(r)dr - \int_0^s (t+s-r)^{\alpha-1} S(r)dr\right],$$

for every $t,\ s \geq 0$. $(S(t))_{t\geq 0}$ is called non-degenerate if $S(t)x = 0$ for all $t \geq 0$ implies $x = 0$. If there exist constants $M \geq 0$ and $\omega \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, then $(S(t))_{t\geq 0}$ is called an $\alpha-$ times integrated, exponentially bounded semigroup.

Theorem 2.1. *Let $\alpha \in \mathbb{R}^+$, $S : [0,\infty) \to L(X)$ be a strongly continuous family, exponentially bounded at infinity (i.e. it satisfies $\|S(t)\| \leq Me^{\omega t}$ for $t \geq 0$ and some constants $M \geq 0$ and $\omega \in \mathbb{R}$), and $R(\lambda) = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)dt$, $Re\,\lambda > \omega$. Then, $R(\lambda)$, $Re\,\lambda > \omega$, is a pseudoresolvent (i.e. it satisfies the resolvent equation $R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu)$) if and only if*

$$S(t)S(s) = \frac{1}{\Gamma(\alpha)}\left[\int_t^{t+s} (t+s-r)^{\alpha-1} S(r)dr - \int_0^s (t+s-r)^{\alpha-1} S(r)dr\right],$$

for every $t,\ s \geq 0$.

Let $(S(t))_{t\geq 0}$ be an $\alpha-$ times integrated semigroup, $\alpha \in \mathbb{R}^+$. Let $R(\lambda) = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)dt$, $\mathrm{Re}\,\lambda > \omega$. Here we take the branch of the function λ^α for which $1^\alpha := 1$. Then, by the resolvent equation, $\mathrm{Ker}\, R(\lambda)$ is independent of $\mathrm{Re}\,\lambda > \omega$. Hence, by the uniqueness theorem, $R(\lambda)$ is injective if and only if $(S(t))_{t\geq 0}$ is non-degenerate. In this case there exists a unique operator A satisfying $(\omega, \infty) \subset \rho(A)(\rho(A)$ is the resolvent set of $A)$ such that $R(\lambda) = (\lambda I - A)^{-1}$ for all λ with $\mathrm{Re}\,\lambda > \omega$. This operator is called the generator of $(S(t))_{t\geq 0}$.

Definition 2.2. Let $\alpha \in \mathbb{R}^+$. An operator A is the generator of an $\alpha-$times integrated, exponentially bounded semigroup $(S(t))_{t\geq 0}$ if and only if $(a, \infty) \subset \rho(A)$ for some $a \in \mathbb{R}$ and $R(\lambda, A)x = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)xdt$, $x \in X, \mathrm{Re}\,\lambda > a$.

The following theorems (exponential formulas) hold for C_0- semigroups (see [6]).

Theorem 2.2. *Let $T(t), t \geq 0$, be a C_0- semigroup on X. If A is the infinitesimal generator of $T(t), t \geq 0$, then*

$$T(t)x = \lim_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n} x = \lim_{n\to\infty} \left[\frac{n}{t} R\left(\frac{n}{t}, A\right)\right]^n x,$$

for every $x \in X$, $t \geq 0$, and the limit is uniform on any bounded interval $[a, b] \subset [0, \infty)$.

Theorem 2.3. *Let $T(t), t \geq 0$, be a C_0- semigroup on X such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ (for suitable constants $M \geq 1$ and $\omega \geq 0$). If A is the infinitesimal generator of $T(t), t \geq 0$, then*

$$T(t)x = (C,1) - \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} R(\lambda, A)\, x \, d\lambda,$$

for every $x \in X,\ t \geq 0,\ \gamma > \omega$. Here $(C,1)-\lim$ means the Cesàro - 1 limit.

We generalize these theorems for $\alpha-$times integrated semigroups ($\alpha \in \mathbb{R}^+$).

3. Exponential formulas for $\alpha-$times integrated semigroups ($\alpha \in \mathbb{R}^+$)

First of all, we need two lemmas.

Lemma 3.1. *Let $\alpha \in \mathbb{R}$. Then,*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! \binom{\alpha}{n-k} \sum_{i=0}^{k} \binom{k}{i} i! \binom{\alpha+i-1}{i} a^{k-i} = (-1)^n a^n$$

for all $n \in \mathbb{N}$ and for all $a \in \mathbb{R}$.

Proof. Let $n \in \mathbb{N}$ be fixed and $a \in \mathbb{R}$. The expression on the left side of the equation designate with $A(n)$. Obviously, $A(n)$ is a polynomial of degree n in the variable a, i.e., $A(n) = \sum_{l=0}^{n} A_l a^l$. Using the substitution $k - i = l$, we obtain for every $l \in \{0, 1, 2, \ldots, n\}$:

$$\begin{aligned} A_l &= \sum_{k=l}^{n} (-1)^k \binom{n}{k} (n-k)! \binom{\alpha}{n-k} \binom{k}{k-l} (k-l)! \binom{\alpha+k-l-1}{k-l} \\ &= \frac{n!}{l!} \sum_{k=l}^{n} (-1)^k \binom{\alpha}{n-k} \binom{\alpha+k-l-1}{k-l}. \end{aligned}$$

The substitution $k-l=s$, leads us to the next equality

$$A_l=(-1)^l\frac{n!}{l!}\sum_{s=0}^{n-l}(-1)^s\binom{\alpha}{n-l-s}\binom{\alpha+s-1}{s},\ l\in\{0,1,2,\ldots,n\}.$$

For $l=n$ we have that $A_n=(-1)^n$. We want to prove that $A_l=0$ for $l=0,1,2,\ldots,n-1$. If we take $n-l=m$, then we need to prove that

$$\sum_{s=0}^{m}(-1)^s\binom{\alpha}{m-s}\binom{\alpha+s-1}{s}=0,\ \text{ for } m=1,2,\ldots,n.$$

Consider now the Taylor's series of the functions x^α and $x^{-\alpha}$ in a neighborhood of $x=1$. We have

$$x^\alpha=\sum_{k=0}^{\infty}\binom{\alpha}{k}(x-1)^k$$

and

$$x^{-\alpha}=\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+k-1}{k}(x-1)^k,\quad x\in(0,2).$$

These series converge absolutely on the interval $(0,2)$. Therefore, for all $x\in(0,2)$ we have

$$\begin{aligned}1\equiv x^\alpha\cdot x^{-\alpha} &= \sum_{k=0}^{\infty}\binom{\alpha}{k}(x-1)^k\cdot\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+k-1}{k}(x-1)^k\\ &= 1+\sum_{m=1}^{\infty}a_m(x-1)^m,\end{aligned}$$

where $a_m=\sum_{s=0}^{m}(-1)^s\binom{\alpha}{m-s}\binom{\alpha+s-1}{s}$. Hence, $a_m=0$ for all $m=1,2,\ldots$. □

Lemma 3.2. *If* Γ *is the Gamma - function, then*

$$\lim_{n\to\infty}\frac{(n+1)^\alpha}{n!}\Gamma(n+1-\alpha)=1.$$

Proof. Let $n_0>\alpha,\ n_0\in\mathbb{N}$ and $n>n_0$. Then

$$\begin{aligned}&\frac{(n+1)^\alpha\,\Gamma(n+1-\alpha)}{n!}=\\ &=\frac{(n+1)^\alpha}{n!}(n-\alpha)(n-\alpha-1)\ldots(n_0-\alpha)\,\Gamma(n_0-\alpha)\\ &=\frac{(n+1)^\alpha}{n!}\frac{(n-\alpha)(n-\alpha-1)\ldots(n_0-\alpha)\,\Gamma(n_0-\alpha)}{(n-n_0)^{n_0-\alpha}(n-n_0)!}(n-n_0)^{n_0-\alpha}(n-n_0)!\end{aligned}$$

$$= \left(\frac{n+1}{n-n_0}\right)^{\alpha} \frac{(n-\alpha)(n-\alpha-1)\dots(n_0-\alpha)\,\Gamma(n_0-\alpha)}{(n-n_0)^{n_0-\alpha}(n-n_0)!} \frac{(n-n_0)^{n_0}}{n(n-1)\dots(n-n_0+1)}.$$

All of the factors on the right side converges to 1 as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \frac{(n+1)^{\alpha}}{n!} \Gamma(n+1-\alpha) = 1.$$

□

Theorem 3.1. *Let $(S(t))_{t\geq 0}$ be non-degenerate $\alpha-$ times integrated, exponentially bounded semigroup on a Banach space X ($\alpha \in \mathbb{R}^+$), and let A be its generator. Then*

$$\lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left(\frac{n+1}{s}\right)^{n+1} R^{n+1}\left(\frac{n+1}{s}, A\right) x\, ds = S(T)\,x,$$

for every $x \in X$, and the limit is uniform in $T > 0$ on any bounded interval $[a, b] \subset [0, \infty)$.

Remark 3.1. In particular, for $\alpha = 1$, the assertion of this theorem was recently proved in [8].

Proof. It is well known that

$$R(\lambda, A) = (\lambda I - A)^{-1} = \lambda^{\alpha} \int_0^{\infty} e^{-\lambda t} S(t)\, dt. \tag{1}$$

Since

$$\frac{d^n}{d\lambda^n}\left[\lambda^{\alpha} \int_0^{\infty} e^{-\lambda t} S(t)\, dt\right] =$$

$$= \sum_{k=0}^{n} \binom{n}{k} (n-k)! \binom{\alpha}{n-k} (-1)^k \lambda^{\alpha-n+k} \int_0^{\infty} t^k e^{-\lambda t} S(t)\, dt,$$

by putting $\lambda = \frac{n+1}{s}$, it follows from (1) that

$$\frac{d^n}{d\lambda^n}\left[R(\lambda, A)\right]_{\lambda=\frac{n+1}{s}} = \sum_{k=0}^{n} \binom{n}{k} (n-k)! \binom{\alpha}{n-k} \cdot \cdot (-1)^k \left(\frac{n+1}{s}\right)^{\alpha-n+k} \int_0^{\infty} t^k e^{-\frac{n+1}{s}t} S(t)\, dt. \tag{2}$$

But,

$$\frac{d^n}{d\lambda^n}R(\lambda, A) = (-1)^n n! R^{n+1}(\lambda, A),\;\; n \in \mathbb{N},\;\; \lambda \in \rho(A), \tag{3}$$

and therefore from (2), and (3) it follows that

$$\begin{aligned} R^{n+1}\left(\frac{n+1}{s}, A\right) &= \frac{(-1)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}(n-k)!\binom{\alpha}{n-k}\cdot \\ &\quad \cdot (-1)^k\left(\frac{n+1}{s}\right)^{\alpha-n+k}\int_0^{\infty} t^k e^{-\frac{n+1}{s}t}S(t)\,dt. \end{aligned} \tag{4}$$

Consider now the integral

$$\begin{aligned} I &= \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left(\frac{n+1}{s}\right)^{n+1} R^{n+1}\left(\frac{n+1}{s}, A\right) ds \\ &= \frac{(-1)^n}{n!\Gamma(\alpha)}\sum_{k=0}^{n}\binom{n}{k}(n-k)!\binom{\alpha}{n-k}(-1)^k\int_0^{\infty} t^k S(t)\cdot \\ &\quad \cdot\int_0^T (T-s)^{\alpha-1}\left(\frac{n+1}{s}\right)^{\alpha+k+1} e^{-\frac{n+1}{s}t} ds\,dt. \end{aligned} \tag{5}$$

First of all, consider the inside integral I_{int}. By substituting $(n+1)\frac{t}{s} = u$, we have

$$\begin{aligned} I_{\text{int}} &= \int_0^T (T-s)^{\alpha-1}\left(\frac{n+1}{s}\right)^{\alpha+k+1} e^{-\frac{n+1}{s}t} ds \\ &= \frac{n+1}{t^{\alpha+k}}\int_{(n+1)t/T}^{\infty}(Tu-(n+1)t)^{\alpha-1}u^k e^{-u}du. \end{aligned}$$

The substitution $u - \frac{(n+1)t}{T} = z$ gives

$$\begin{aligned} I_{\text{int}} &= \frac{n+1}{t^{\alpha+k}}\int_0^{\infty} z^{\alpha-1}T^{\alpha-1}\left(z+\frac{(n+1)t}{T}\right)^k e^{-z-\frac{(n+1)t}{T}}dz \\ &= \frac{(n+1)T^{\alpha-k-1}}{t^{\alpha+k}e^{(n+1)t/T}}\int_0^{\infty} z^{\alpha-1}e^{-z}\left[Tz+(n+1)t\right]^k dz. \end{aligned}$$

Using the binomial formula and the next property of the Gamma - function:

$$\Gamma(\alpha+i) = i!\binom{\alpha+i-1}{i}\Gamma(\alpha),$$

we obtain

$$I_{\text{int}} = \frac{(n+1)T^{\alpha-k-1}}{t^{\alpha+k}e^{(n+1)t/T}} \sum_{i=0}^{k} \binom{k}{i} [(n+1)t]^{k-i} T^i \int_0^\infty z^{\alpha+i-1} e^{-z} dz$$

$$= \frac{(n+1)T^{\alpha-1}}{t^{\alpha+k}e^{(n+1)t/T}} \sum_{i=0}^{k} \binom{k}{i} \left[\frac{(n+1)t}{T}\right]^{k-i} \Gamma(\alpha+i) \tag{6}$$

$$= \frac{(n+1)T^{\alpha-1}\Gamma(\alpha)}{t^{\alpha+k}e^{(n+1)t/T}} \sum_{i=0}^{k} \binom{k}{i} i! \binom{\alpha+i-1}{i} \left[\frac{(n+1)t}{T}\right]^{k-i}.$$

Now (5), and (6) imply

$$I = \frac{(-1)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} (n-k)! \binom{\alpha}{n-k} (-1)^k (n+1)T^{\alpha-1}$$

$$\int_0^\infty t^{-\alpha} e^{-(n+1)t/T} S(t) \sum_{i=0}^{k} \binom{k}{i} i! \binom{\alpha+i-1}{i} \left(\frac{(n+1)t}{T}\right)^{k-i} dt$$

$$= \frac{(-1)^n(n+1)T^{\alpha-1}}{n!} \int_0^\infty t^{-\alpha} e^{-(n+1)t/T} S(t) \sum_{k=0}^{n} \binom{n}{k} (n-k)! \cdot$$

$$\cdot \binom{\alpha}{n-k} (-1)^k \sum_{i=0}^{k} \binom{k}{i} i! \binom{\alpha+i-1}{i} \left(\frac{(n+1)t}{T}\right)^{k-i} dt. \tag{7}$$

By Lemma 3.1 and (7), we obtain for $a = \frac{(n+1)t}{T}$:

$$I = \frac{(-1)^n(n+1)T^{\alpha-1}}{n!} \int_0^\infty t^{-\alpha} e^{-(n+1)t/T} S(t) (-1)^n \left(\frac{(n+1)t}{T}\right)^n dt$$

$$= \frac{(n+1)^{n+1}T^{\alpha-n-1}}{n!} \int_0^\infty t^{n-\alpha} e^{-(n+1)t/T} S(t) dt. \tag{8}$$

Using the substitution $\frac{(n+1)t}{T} = z$, we have further

$$I = \frac{(n+1)^{n+1}T^{\alpha-n-1}}{n!} \int_0^\infty \left(\frac{Tz}{n+1}\right)^{n-\alpha} e^{-z} S\left(\frac{zT}{n+1}\right) \frac{T}{n+1} dz. \tag{9}$$

Hence, we have that

$$I = \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left(\frac{n+1}{s}\right)^{n+1} R^{n+1}\left(\frac{n+1}{s}, A\right) ds =$$
$$= \frac{(n+1)^\alpha}{n!} \int_0^\infty S\left(\frac{zT}{n+1}\right) z^{n-\alpha} e^{-z} dz. \tag{10}$$

Fix $\varepsilon > 0$ and choose $\delta \in (0,\ \mathrm{T})$ such that for

$$(n+1)\left(1-\frac{\delta}{T}\right) < z < (n+1)\left(1+\frac{\delta}{T}\right), \quad T > 0, \quad n \in \mathbb{N},$$

we have

$$\left\| S\left(\frac{zT}{n+1}\right) x - S(T)x \right\| < \varepsilon, \quad x \in X.$$

Put for $x \in X$, $T > 0$, $n \in \mathbb{N}$:

$$J = \frac{(n+1)^\alpha}{n!} \int_0^\infty \left[S\left(\frac{zT}{n+1}\right) x - S(T)x \right] e^{-z} z^{n-\alpha} dz = J_1 + J_2 + J_3, \tag{11}$$

where

$$J_1 = \frac{(n+1)^\alpha}{n!} \int_0^{(n+1)\left(1-\frac{\delta}{T}\right)} \left[S\left(\frac{zT}{n+1}\right) x - S(T)x \right] e^{-z} z^{n-\alpha} dz,$$

$$J_2 = \frac{(n+1)^\alpha}{n!} \int_{(n+1)\left(1-\frac{\delta}{T}\right)}^{(n+1)\left(1+\frac{\delta}{T}\right)} \left[S\left(\frac{zT}{n+1}\right) x - S(T)x \right] e^{-z} z^{n-\alpha} dz,$$

$$J_3 = \frac{(n+1)^\alpha}{n!} \int_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} \left[S\left(\frac{zT}{n+1}\right) x - S(T)x \right] e^{-z} z^{n-\alpha} dz.$$

We will estimate each of these integrals. We have

$$\|J_1\| \leq \frac{(n+1)^\alpha}{n!} \int_0^{(n+1)\left(1-\frac{\delta}{T}\right)} \left\| S\left(\frac{zT}{n+1}\right) x - S(T)x \right\| e^{-z} z^{n-\alpha} dz.$$

We know that $(S(t))_{t \geq 0}$ is an exponentially bounded family of operators, i.e. there exist constants $M \geq 0$ and $\omega_0 \in \mathbb{R}$ such that $\|S(t)\| \leq M e^{\omega_0 t}$, for all $t \geq 0$. Therefore,

$$\|J_1\| \leq \frac{(n+1)^\alpha}{n!} M \|x\| \int_0^{(n+1)\left(1-\frac{\delta}{T}\right)} \left[e^{\frac{\omega_0 z T}{n+1}} + e^{\omega_0 T}\right] e^{-z} z^{n-\alpha} dz = S_1 + S_2,$$

where

$$S_1 = \frac{(n+1)^\alpha}{n!} M \|x\| \int_0^{(n+1)\left(1-\frac{\delta}{T}\right)} e^{-z\left(1-\frac{\omega_0 T}{n+1}\right)} z^{n-\alpha} dz$$

and

$$S_2 = \frac{(n+1)^\alpha}{n!} M \|x\| e^{\omega_0 T} \int_0^{(n+1)\left(1-\frac{\delta}{T}\right)} e^{-z} z^{n-\alpha} dz.$$

Let us estimate S_1. Take $z \frac{n+1-\omega_0 T}{n+1} = u$. Then the integral S_1 becomes

$$S_1 = \frac{(n+1)^\alpha}{n!} M \|x\| \int_0^{(n+1-\omega_0 T)\left(1-\frac{\delta}{T}\right)} e^{-u} \left(\frac{n+1}{n+1-\omega_0 T} u\right)^{n-\alpha} \frac{n+1}{n+1-\omega_0 T} du$$

$$= \frac{(n+1)^{n+1} M \|x\|}{n!(n+1-\omega_0 T)^{n-\alpha+1}} \int_0^{(n+1-\omega_0 T)\left(1-\frac{\delta}{T}\right)} e^{-u} u^{n-\alpha} du.$$

The function $f(u) = e^{-u} u^{n-\alpha}$ ($u \in \mathbb{R}$) takes its maximum value at the point $u = n - \alpha$. For sufficiently large n and fixed δ, $n - \alpha$ is greater than $(n+1-\omega_0 T)\left(1-\frac{\delta}{T}\right)$. Note, the function f is increasing in the interval $\left[0, (n+1-\omega_0 T)\left(1-\frac{\delta}{T}\right)\right]$. Using these facts, we obtain

$$S_1 \leq \frac{(n+1)^{n+1} M \|x\|}{n!(n+1-\omega_0 T)^{n-\alpha+1}} (n+1-\omega_0 T) \left(1 - \frac{\delta}{T}\right)$$

$$\cdot \frac{\left[(n+1-\omega_0 T)\left(1-\frac{\delta}{T}\right)\right]^{n-\alpha}}{e^{(n+1-\omega_0 T)\left(1-\frac{\delta}{T}\right)}} = \frac{(n+1)^{n+1} M \|x\| \left(1-\frac{\delta}{T}\right)^{n-\alpha+1}}{n! e^{(n+1-\omega_0 T)\left(1-\frac{\delta}{T}\right)}}.$$

For large n, Stirling's formula implies

$$S_1 \leq \frac{e^n (n+1)^{n+1} M \|x\| \left(1-\frac{\delta}{T}\right)^{n-\alpha+1}}{n^n \sqrt{2\pi n} \cdot e^{n\left(1-\frac{\delta}{T}\right)} e^{(1-\omega_0 T)\left(1-\frac{\delta}{T}\right)}}$$

$$= \frac{M \|x\|}{\sqrt{2\pi} \left(1-\frac{\delta}{T}\right)^{\alpha-1} e^{(1-\omega_0 T)\left(1-\frac{\delta}{T}\right)}} \left(1 + \frac{1}{n}\right)^n \frac{n+1}{\sqrt{n}} \left[\left(1 - \frac{\delta}{T}\right) e^{\frac{\delta}{T}}\right]^n.$$

The function $g(x) = (1-x)e^x$, $x \in \mathbb{R}$, attains the global maximum 1 at the point $x = 0$. Since $0 < \delta < T$, we have $\left(1-\frac{\delta}{T}\right) e^{\frac{\delta}{T}} < 1$ and $\left[\left(1-\frac{\delta}{T}\right) e^{\frac{\delta}{T}}\right]^n \to 0$

as $n \to \infty$. Also, $\frac{n+1}{\sqrt{n}}\left[\left(1-\frac{\delta}{T}\right)e^{\frac{\delta}{T}}\right]^n \to 0$ as $n \to \infty$. So we obtain that $S_1 \to 0$ as $n \to \infty$, and the limit is uniform in $T > 0$ on any bounded interval.

Let us estimate $S_2 = \frac{(n+1)^\alpha}{n!} M \|x\| e^{\omega_0 T} \int_0^{(n+1)\left(1-\frac{\delta}{T}\right)} e^{-z} z^{n-\alpha} dz$.

The function $f(z) = e^{-z} z^{n-\alpha}$ $(z \in \mathbb{R})$ takes its maximum at the point $z = n - \alpha$. For sufficiently large n and fixed δ, $n - \alpha$ belongs to the interval

$$\left[(n+1)\left(1-\frac{\delta}{T}\right),\ (n+1)\left(1+\frac{\delta}{T}\right)\right].$$

Hence, the function f is increasing in the interval $\left[0, (n+1)\left(1-\frac{\delta}{T}\right)\right]$. Thus,

$$\begin{aligned}
S_2 &\leq \frac{(n+1)^\alpha}{n!} M \|x\| e^{\omega_0 T} (n+1)\left(1-\frac{\delta}{T}\right) \frac{\left[(n+1)\left(1-\frac{\delta}{T}\right)\right]^{n-\alpha}}{e^{(n+1)\left(1-\frac{\delta}{T}\right)}} \\
&= \frac{M \|x\| e^{\omega_0 T}\left(1-\frac{\delta}{T}\right)^{1-\alpha}}{e^{1-\frac{\delta}{T}}} \frac{(n+1)^{n+1}}{n! e^n} \left[\left(1-\frac{\delta}{T}\right) e^{\frac{\delta}{T}}\right]^n .
\end{aligned}$$

Using Stirling's formula, for sufficiently large n, we obtain

$$\begin{aligned}
S_2 &\leq \frac{M \|x\| e^{\omega_0 T}\left(1-\frac{\delta}{T}\right)^{1-\alpha}}{e^{1-\frac{\delta}{T}}} \frac{(n+1)^{n+1}}{n^n \sqrt{2\pi n}} \left[\left(1-\frac{\delta}{T}\right) e^{\frac{\delta}{T}}\right]^n \\
&= \frac{M \|x\| e^{\omega_0 T}\left(1-\frac{\delta}{T}\right)^{1-\alpha}}{\sqrt{2\pi}\cdot e^{1-\frac{\delta}{T}}} \left(1+\frac{1}{n}\right)^n \frac{n+1}{\sqrt{n}} \left[\left(1-\frac{\delta}{T}\right) e^{\frac{\delta}{T}}\right]^n .
\end{aligned}$$

So we obtain that $S_2 \to 0$ as $n \to \infty$, and the limit is uniform in $T > 0$ on any bounded interval. Hence,

$$\|J_1\| \to 0 \quad \text{as} \quad n \to \infty. \tag{12}$$

Now, we will estimate the integral J_2.

$$\begin{aligned}
\|J_2\| &\leq \frac{(n+1)^\alpha}{n!} \int_{(n+1)\left(1-\frac{\delta}{T}\right)}^{(n+1)\left(1+\frac{\delta}{T}\right)} \left\| S\left(\frac{zT}{n+1}\right) x - S(T)x \right\| e^{-z} z^{n-\alpha} dz \\
&< \varepsilon \frac{(n+1)^\alpha}{n!} \int_{(n+1)\left(1-\frac{\delta}{T}\right)}^{(n+1)\left(1+\frac{\delta}{T}\right)} e^{-z} z^{n-\alpha} dz \\
&< \varepsilon \frac{(n+1)^\alpha}{n!} \int_0^\infty e^{-z} z^{n-\alpha} dz = \varepsilon \frac{(n+1)^\alpha}{n!} \Gamma(n+1-\alpha).
\end{aligned}$$

From Lemma 3.2 we see that $\lim\limits_{n\to\infty} \frac{(n+1)^\alpha}{n!}\Gamma(n+1-\alpha) = 1$. This implies $\|J_2\| \le \varepsilon$ for large n. Because ε is an arbitrary small number we conclude that

$$\|J_2\| \to 0 \quad \text{as} \quad n \to \infty. \tag{13}$$

Let us estimate the integral J_3.

$$\|J_3\| \le \frac{(n+1)^\alpha}{n!} \int\limits_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} \left\| S\left(\frac{zT}{n+1}\right) x - S(T)x \right\| e^{-z} z^{n-\alpha} dz$$

$$\le \frac{(n+1)^\alpha}{n!} M \|x\| \int\limits_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} \left(e^{\frac{\omega_0 zT}{n+1}} + e^{\omega_0 T}\right) e^{-z} z^{n-\alpha} dz = S_3 + S_4,$$

where

$$S_3 = \frac{(n+1)^\alpha}{n!} M \|x\| \int\limits_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-z\left(1-\frac{\omega_0 T}{n+1}\right)} z^{n-\alpha} dz \quad \text{and}$$

$$S_4 = \frac{(n+1)^\alpha}{n!} M \|x\| e^{\omega_0 T} \int\limits_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-z} z^{n-\alpha} dz.$$

Let us estimate S_3. Take $z\frac{n+1-\omega_0 T}{n+1} = u$. Then the integral S_3 becomes

$$S_3 = \frac{(n+1)^\alpha}{n!} M \|x\| \int\limits_{(n+1-\omega_0 T)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-u} \left(\frac{n+1}{n+1-\omega_0 T} u\right)^{n-\alpha} \frac{n+1}{n+1-\omega_0 T} du$$

$$= \frac{(n+1)^{n+1} M \|x\|}{n!(n+1-\omega_0 T)^{n-\alpha+1}} \int\limits_{(n+1-\omega_0 T)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-u} u^{n-\alpha} du.$$

Consider the integral

$$\int\limits_{(n+1-\omega_0 T)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-u} u^{n-\alpha} du.$$

We have

$$\int\limits_{(n+1-\omega_0 T)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-u} u^{n-\alpha} du = \int\limits_{(n+1-\omega_0 T)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-u(1-\eta)} e^{-u\eta} u^{n-\alpha} du, \ \text{ for } 0 < \eta < 1.$$

The function $h(u) = e^{-u\eta}u^{n-\alpha}$, $u \in \mathbb{R}$, has a maximum at the point $u = \frac{n-\alpha}{\eta}$. This maximum equals $h\left(\frac{n-\alpha}{\eta}\right) = \frac{e^{-(n-\alpha)}(n-\alpha)^{n-\alpha}}{\eta^{n-\alpha}}$. Thus, we obtain

$$\begin{aligned}\int\limits_{(n+1-\omega_0 T)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-u}u^{n-\alpha}du &= \int\limits_{(n+1-\omega_0 T)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-u(1-\eta)}e^{-u\eta}u^{n-\alpha}du \\ &< \frac{e^{-(n-\alpha)}(n-\alpha)^{n-\alpha}}{\eta^{n-\alpha}} \int\limits_{(n+1-\omega_0 T)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-u(1-\eta)}du \\ &= \frac{e^{-(n-\alpha)}(n-\alpha)^{n-\alpha}}{\eta^{n-\alpha}} \cdot \frac{e^{(\eta-1)(n+1-\omega_0 T)\left(1+\frac{\delta}{T}\right)}}{1-\eta}.\end{aligned}$$

Using Stirling's formula, for sufficiently large n, we obtain

$$\begin{aligned}S_3 &\leq \frac{(n+1)^{n+1}e^n M\|x\|}{n^n\sqrt{2\pi n}(n+1-\omega_0 T)^{n-\alpha+1}} \frac{e^{-(n-\alpha)}(n-\alpha)^{n-\alpha}}{\eta^{n-\alpha}} \frac{e^{(\eta-1)(n+1-\omega_0 T)\left(1+\frac{\delta}{T}\right)}}{1-\eta} \\ &= \frac{M\|x\| e^{\alpha}\eta^{\alpha}}{(1-\eta)\sqrt{2\pi n}\cdot e^{(1-\omega_0 T)\left(1+\frac{\delta}{T}\right)(1-\eta)}} \left(\frac{n+1}{n}\right)^{\alpha} \left(\frac{n+1}{n+1-\omega_0 T}\right)^{n-\alpha+1} \cdot \\ &\quad \cdot \left(\frac{n-\alpha}{n}\right)^{n-\alpha} \frac{1}{\eta^n e^{n\left(1+\frac{\delta}{T}\right)(1-\eta)}}.\end{aligned}$$

Notice that $\left(\frac{n+1}{n}\right)^{\alpha} \to 1$, $\left(\frac{n+1}{n+1-\omega_0 T}\right)^{n-\alpha+1} \to e^{\omega_0 T}$ and $\left(\frac{n-\alpha}{n}\right)^{n-\alpha} \to e^{-\alpha}$, as $n \to \infty$.

If we can prove that $\eta^n e^{n\left(1+\frac{\delta}{T}\right)(1-\eta)} \to \infty$ as $n \to \infty$, then $S_3 \to 0$ as $n \to \infty$. Since

$$\eta^n e^{n\left(1+\frac{\delta}{T}\right)(1-\eta)} = e^{n\left[\ln\eta + \left(1+\frac{\delta}{T}\right)(1-\eta)\right]},$$

it is enough to choose η such that

$$\ln\eta + \left(1+\frac{\delta}{T}\right)(1-\eta) > 0.$$

Since, $\ln\eta = \ln(1+(\eta-1))$ and $\frac{\eta-1}{\eta} < \ln(1+(\eta-1)) < \eta - 1$, we obtain

$$\ln\eta + \left(1+\frac{\delta}{T}\right)(1-\eta) > \frac{\eta-1}{\eta} + \left(1+\frac{\delta}{T}\right)(1-\eta) = (1-\eta)\left(1+\frac{\delta}{T}-\frac{1}{\eta}\right).$$

But, the last inequality holds for $\frac{1}{1+\frac{\delta}{T}} < \eta < 1$. Hence, by choosing $\eta \in \left(\frac{1}{1+\frac{\delta}{T}},\ 1\right)$, we can conclude that $S_3 \to 0$ as $n \to \infty$. Moreover, the limit is uniform in $T > 0$ on any bounded interval.

Let us estimate

$$S_4 = \frac{(n+1)^\alpha}{n!} M \|x\| e^{\omega_0 T} \int\limits_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-z} z^{n-\alpha} dz.$$

If $\psi \in \left(\frac{1}{1+\frac{\delta}{T}},\ 1\right)$, then $\psi^n e^{n\left(1+\frac{\delta}{T}\right)(1-\psi)} \to \infty$ as $n \to \infty$. Since

$$\begin{aligned} \int\limits_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-z} z^{n-\alpha} dz &= \int\limits_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-z(1-\psi)} e^{-z\psi} z^{n-\alpha} dz \\ &< \frac{e^{-(n-\alpha)}(n-\alpha)^{n-\alpha}}{\psi^{n-\alpha}} \int\limits_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-z(1-\psi)} dz \\ &= \frac{e^{-(n-\alpha)}(n-\alpha)^{n-\alpha}}{\psi^{n-\alpha}} \cdot \frac{e^{(\psi-1)(n+1)\left(1+\frac{\delta}{T}\right)}}{1-\psi}, \end{aligned}$$

we conclude that

$$S_4 < \frac{(n+1)^\alpha}{n!} M \|x\| e^{\omega_0 T} \frac{e^{-(n-\alpha)}(n-\alpha)^{n-\alpha}}{\psi^{n-\alpha}} \cdot \frac{e^{(\psi-1)(n+1)\left(1+\frac{\delta}{T}\right)}}{1-\psi}.$$

Using Stirling's formula, for sufficiently large n, we obtain

$$S_4 < \frac{M \|x\| e^{\omega_0 T} e^\alpha \psi^\alpha}{(1-\psi)\sqrt{2\pi n} \cdot e^{\left(1+\frac{\delta}{T}\right)(1-\psi)}} \left(\frac{n+1}{n}\right)^\alpha \left(\frac{n-\alpha}{n}\right)^{n-\alpha} \frac{1}{\psi^n e^{n\left(1+\frac{\delta}{T}\right)(1-\psi)}}.$$

We know that $\left(\frac{n+1}{n}\right)^\alpha \to 1$, $\left(\frac{n-\alpha}{n}\right)^{n-\alpha} \to e^{-\alpha}$ and $\psi^n e^{n\left(1+\frac{\delta}{T}\right)(1-\psi)} \to \infty$, as $n \to \infty$. Hence, $S_4 \to 0$ as $n \to \infty$, and, therefore

$$\|J_3\| \to 0 \quad \text{as} \quad n \to \infty. \tag{14}$$

This limit is uniform in $T > 0$ on any bounded interval. Finally, by (11), (12), (13), and (14) we conclude that

$$J = \frac{(n+1)^\alpha}{n!} \int\limits_0^\infty \left[S\left(\frac{zT}{n+1}\right) x - S(T)x\right] e^{-z} z^{n-\alpha} dz \to 0, \quad \text{as} \quad n \to \infty. \tag{15}$$

Since, by Lemma 3.2, $\lim\limits_{n\to\infty} \frac{(n+1)^\alpha}{n!}\Gamma(n+1-\alpha) = 1$, using (10), and (15) we obtain

$$\lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int\limits_0^T (T-s)^{\alpha-1} \left(\frac{n+1}{s}\right)^{n+1} \left[R\left(\frac{n+1}{s}, A\right)\right]^{n+1} x\, ds$$

$$= \lim_{n\to\infty} \frac{(n+1)^\alpha}{n!} \int_0^\infty S\left(\frac{zT}{n+1}\right) x \cdot e^{-z} z^{n-\alpha} dz = S(T)\, x,$$

for every $x \in X$, and this limit is uniform in $T > 0$. □

Definition 3.1. Let $f(\omega)$ be a function on $[0,\infty)$ with values in a complex Banach space X, such that for every $\lambda > 0$, $e^{-\lambda\omega} f(\omega) \in L\left([0,\infty), X\right)$ ($L\left([0,\infty), X\right)$ is the space of linear bounded functions from $[0,\infty)$ into X). Then, for $\beta > 0$, the Cesàro-β limit of the function $f(\omega)$ as $\omega \to \infty$ is defined as follows

$$(C,\beta) - \lim_{\omega\to\infty} f(\omega) := \lim_{T\to\infty} \frac{\beta}{T^\beta} \int_0^T (T-\omega)^{\beta-1} f(\omega)\, d\omega.$$

The next result is well-known (for example, see [6]).

Theorem 3.2. *If for some $\alpha \geq 0$: $(C,\alpha) - \lim_{\omega\to\infty} f(\omega) = a$, then for every $\beta > \alpha$ $(C,\beta) - \lim_{\omega\to\infty} f(\omega) = a$.*

Lemma 3.3. *Let $0 < \beta < 1$ and $s \geq \pi$. Then*

$$\int_0^1 (1-u)^{\beta-1} \sin(su) du \leq \frac{M_1}{s^\beta} \quad (M_1 - \textit{some constant}).$$

Proof. Obviously,

$$\int_0^1 (1-u)^{\beta-1} \sin(su)\, du = \int_0^1 \frac{\sin(1-v)s}{v^{1-\beta}}\, dv$$
$$= \sin s \int_0^1 \frac{\cos(vs)}{v^{1-\beta}}\, dv - \cos s \int_0^1 \frac{\sin(vs)}{v^{1-\beta}}\, dv.$$

Therefore, it is sufficient to prove that

$$\left| \int_0^1 \frac{\cos(vs)}{v^{1-\beta}}\, dv \right| \leq \frac{K_1}{s^\beta} \quad \text{and} \quad \left| \int_0^1 \frac{\sin(vs)}{v^{1-\beta}}\, dv \right| \leq \frac{K_2}{s^\beta},$$

where K_1 and K_2 are some constants.

Both of these integrals can be estimated in a similar manner. Therefore, we estimate only $\int_0^1 \frac{\sin(vs)}{v^{1-\beta}} dv$. We have

$$\int_0^1 \frac{\sin(vs)}{v^{1-\beta}}\, dv = \int_0^{\pi/s} \frac{\sin(vs)}{v^{1-\beta}}\, dv + \sum_{k=1}^{k_0-1} \int_{k\pi/s}^{(k+1)\pi/s} \frac{\sin(vs)}{v^{1-\beta}}\, dv + \int_{k_0\pi/s}^{1} \frac{\sin(vs)}{v^{1-\beta}}\, dv, \tag{16}$$

where k_0 is a natural number such that $\frac{k_0\pi}{s} \le 1 < \frac{(k_0+1)\pi}{s}$. Since

$$\sup_{s\in(0,\,\pi]} \left| s^\beta \int_0^1 (1-u)^{\beta-1} \sin(su)\, du \right| < \infty,$$

it is enough to assume that $s \ge \pi$ and that k_0 is an odd natural number.

Obviously,

$$\left| \int_{k_0\pi/s}^{1} \frac{\sin(vs)}{v^{1-\beta}}\, dv \right| \le \int_{k_0\pi/s}^{1} \frac{dv}{v^{1-\beta}} \le \int_{k_0\pi/s}^{(k_0+1)\pi/s} \frac{dv}{v^{1-\beta}} \le \frac{1}{\left(\frac{k_0\pi}{s}\right)^{1-\beta}} \cdot \frac{\pi}{s}.$$

Hence, it follows that

$$\left| \int_{k_0\pi/s}^{1} \frac{\sin(vs)}{v^{1-\beta}}\, dv \right| \le \left(\frac{\pi}{s}\right)^\beta. \tag{17}$$

Similarly,

$$\left| \int_0^{\pi/s} \frac{\sin(vs)}{v^{1-\beta}}\, dv \right| \le \int_0^{\pi/s} \frac{dv}{v^{1-\beta}} = \frac{1}{\beta}\left(\frac{\pi}{s}\right)^\beta. \tag{18}$$

Further, we have

$$\begin{aligned}\sum_{k=1}^{k_0-1} \int_{k\pi/s}^{(k+1)\pi/s} \frac{\sin(vs)}{v^{1-\beta}}\, dv &= \sum_{k=1}^{k_0-1} \int_0^{\pi/s} \frac{\sin s\left(v+\frac{k\pi}{s}\right)}{\left(v+\frac{k\pi}{s}\right)^{1-\beta}}\, dv \\ &= \sum_{k=1}^{k_0-1} (-1)^k \int_0^{\pi/s} \frac{\sin(vs)}{\left(v+\frac{k\pi}{s}\right)^{1-\beta}}\, dv \\ &= \int_0^{\pi/s} \sin(vs) \sum_{k=1}^{k_0-1} \frac{(-1)^k}{\left(v+\frac{k\pi}{s}\right)^{1-\beta}}\, dv.\end{aligned}$$

Therefore we have

$$\sum_{k=1}^{k_0-1} \int\limits_{k\pi/s}^{(k+1)\pi/s} \frac{\sin(vs)}{v^{1-\beta}}\, dv = \int\limits_{0}^{\pi/s} \sin(vs) \sum_{k=1}^{k_0-1} \frac{(-1)^k}{\left(v+\frac{k\pi}{s}\right)^{1-\beta}}\, dv. \tag{19}$$

Now we will estimate the sum $\sum_{k=1}^{k_0-1} \frac{(-1)^k}{\left(v+\frac{k\pi}{s}\right)^{1-\beta}}$. Clearly

$$\left|\sum_{k=1}^{k_0-1} \frac{(-1)^k}{\left(v+\frac{k\pi}{s}\right)^{1-\beta}}\right| = \sum_{i=0}^{i_0} \left[\frac{1}{\left(v+(2i+1)\frac{\pi}{s}\right)^{1-\beta}} - \frac{1}{\left(v+(2i+2)\frac{\pi}{s}\right)^{1-\beta}}\right],$$

where $i_0 = \frac{k_0-3}{2}$.

Using Lagrange's mean value formula we obtain (for some $\theta \in (0,1)$) :

$$\begin{aligned}
\left|\sum_{k=1}^{k_0-1} \frac{(-1)^k}{\left(v+\frac{k\pi}{s}\right)^{1-\beta}}\right| &= (1-\beta)\frac{\pi}{s} \sum_{i=0}^{i_0} \frac{1}{\left(v+(2i+1)\frac{\pi}{s}+\theta\frac{\pi}{s}\right)^{2-\beta}} \\
&\le (1-\beta)\frac{\pi}{s} \sum_{i=0}^{i_0} \frac{1}{\left((2i+1)\frac{\pi}{s}\right)^{2-\beta}} \\
&= (1-\beta)\left(\frac{\pi}{s}\right)^{\beta-1} \sum_{i=0}^{i_0} \frac{1}{(2i+1)^{2-\beta}} \\
&\le (1-\beta)\left(\frac{\pi}{s}\right)^{\beta-1} \sum_{i=0}^{\infty} \frac{1}{(2i+1)^{2-\beta}}.
\end{aligned}$$

This inequality combined with (19) gives

$$\left|\sum_{k=1}^{k_0-1} \int\limits_{k\pi/s}^{(k+1)\pi/s} \frac{\sin(vs)}{v^{1-\beta}} dv\right| \le (1-\beta)\left(\frac{\pi}{s}\right)^{\beta} \sum_{i=0}^{\infty} \frac{1}{(2i+1)^{2-\beta}}.$$

The assertion of our lemma now follows from (17) and (18). □

Theorem 3.3. *Let $(S(t))_{t\ge 0}$ be an $\alpha-$ times integrated, exponentially bounded semigroup defined on a Banach space X $(\alpha \in \mathbb{R}^+)$. Let $M \ge 0$ and $\omega_0 \in \mathbb{R}$ satisfy $\|S(t)\| \le Me^{\omega_0 t}$, for all $t \ge 0$. Let $0<\beta<1$. If $\gamma > \max(\omega_0, 0)$, $x \in X$ and $t \ge 0$, then we have*

$$S(t)x = \frac{1}{2\pi i}(C,\beta) - \lim_{\omega\to\infty} \int\limits_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} \frac{R(\lambda, A)x}{\lambda^\alpha}\, d\lambda,$$

and the limit is uniform in t on any bounded interval $[a,b] \subset [0,\infty)$.

Proof. Let $\gamma > \max(\omega_0,\ 0)$. By Definition 3.1, for any fixed $x \in X,\ t \geq 0$ we have

$$\frac{1}{2\pi i}(C,\beta) - \lim_{\omega\to\infty} \int\limits_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} \frac{R(\lambda, A)x}{\lambda^\alpha}\, d\lambda$$

$$= \lim_{T\to\infty} \frac{\beta}{T^\beta} \int\limits_0^T (T-\omega)^{\beta-1}\, d\omega \frac{1}{2\pi i} \int\limits_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} \frac{R(\lambda, A)x}{\lambda^\alpha}\, d\lambda \tag{20}$$

$$= \lim_{T\to\infty} \frac{\beta}{T^\beta} \int\limits_0^T (T-\omega)^{\beta-1}\, d\omega \frac{1}{2\pi} \int\limits_{-\omega}^{\omega} e^{(\gamma+i\tau)t} \frac{R(\gamma+i\tau, A)x}{(\gamma+i\tau)^\alpha}\, d\tau.$$

We interchange the order of integration and obtain the expression :

$$\lim_{T\to\infty} \frac{\beta}{2\pi T^\beta} \Bigg[\int\limits_{-T}^{0} e^{(\gamma+i\tau)t} \frac{R(\gamma+i\tau, A)x}{(\gamma+i\tau)^\alpha}\, d\tau \int\limits_{-\tau}^{T} (T-\omega)^{\beta-1}\, d\omega$$

$$+ \int\limits_0^T e^{(\gamma+i\tau)t} \frac{R(\gamma+i\tau, A)x}{(\gamma+i\tau)^\alpha}\, d\tau \int\limits_{\tau}^{T} (T-\omega)^{\beta-1}\, d\omega \Bigg]$$

$$= \lim_{T\to\infty} \frac{1}{2\pi} \Bigg[\int\limits_{-T}^{0} \left(1+\frac{\tau}{T}\right)^\beta e^{(\gamma+i\tau)t} \frac{R(\gamma+i\tau, A)x}{(\gamma+i\tau)^\alpha}\, d\tau$$

$$+ \int\limits_0^T \left(1-\frac{\tau}{T}\right)^\beta e^{(\gamma+i\tau)t} \frac{R(\gamma+i\tau, A)x}{(\gamma+i\tau)^\alpha}\, d\tau \Bigg].$$

Because $\frac{R(\gamma+i\tau,A)x}{(\gamma+i\tau)^\alpha} = \int_0^\infty e^{-(\gamma+i\tau)s} S(s)x ds$, we obtain

$$\lim_{T\to\infty} \frac{1}{2\pi} \int\limits_{-T}^{T} \left(1-\frac{|\tau|}{T}\right)^\beta e^{(\gamma+i\tau)t} d\tau \int\limits_0^\infty e^{-(\gamma+i\tau)s} S(s)x\, ds =$$

$$= \lim_{T\to\infty} \frac{1}{2\pi} \Bigg[\int\limits_{-T}^{T} \left(1-\frac{|\tau|}{T}\right)^\beta e^{(\gamma+i\tau)t} d\tau \int\limits_0^\infty e^{-(\gamma+i\tau)s} \left(S(s)x - S(t)x\right) ds$$

$$+ S(t)x \int\limits_{-T}^{T} \left(1-\frac{|\tau|}{T}\right)^\beta e^{(\gamma+i\tau)t} d\tau \int\limits_0^\infty e^{-(\gamma+i\tau)s} ds \Bigg]. \tag{21}$$

We will prove that the limit given in (21) equals $S(t)x$. If we put

$$I_1 = \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right)^{\beta} e^{(\gamma+i\tau)t} d\tau \int_0^{\infty} e^{-(\gamma+i\tau)s} ds = \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right)^{\beta} \frac{e^{(\gamma+i\tau)t}}{\gamma + i\tau}\, d\tau,$$

and

$$I_2 = \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right)^{\beta} e^{(\gamma+i\tau)t}\, d\tau \int_0^{\infty} e^{-(\gamma+i\tau)s}\, (S(s)x - S(t)x)\, ds,$$

then, it suffices to prove that $I_1 \to 2\pi$ and $I_2 \to 0$, as $T \to \infty$. We have

$$I_1 = \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right)^{\beta} \frac{e^{(\gamma+i\tau)t}}{\gamma + i\tau}\, d\tau = \int_0^{T} \left(1 - \frac{\tau}{T}\right)^{\beta} \left[\frac{e^{(\gamma+i\tau)t}}{\gamma + i\tau} + \frac{e^{(\gamma-i\tau)t}}{\gamma - i\tau}\right] d\tau$$

$$= e^{\gamma t} \int_0^{T} \left(1 - \frac{\tau}{T}\right)^{\beta} \frac{2\gamma \cos(\tau t) + 2\tau \sin(\tau t)}{\gamma^2 + \tau^2}\, d\tau.$$

Now we will show that

$$\int_0^{T} \left(1 - \frac{\tau}{T}\right)^{\beta} \frac{\tau \sin(\tau t)}{\gamma^2 + \tau^2}\, d\tau \to \int_0^{\infty} \frac{\tau \sin(\tau t)}{\gamma^2 + \tau^2}\, d\tau$$

and

$$\int_0^{T} \left(1 - \frac{\tau}{T}\right)^{\beta} \frac{\cos(\tau t)}{\gamma^2 + \tau^2}\, d\tau \to \int_0^{\infty} \frac{\cos(\tau t)}{\gamma^2 + \tau^2}\, d\tau, \tag{22}$$

as $T \to \infty$. Let $J(T) = \int_0^T \left(1 - \frac{\tau}{T}\right)^{\beta} \frac{\tau \sin(\tau t)}{\gamma^2+\tau^2} d\tau$ and $J = \int_0^{\infty} \frac{\tau \sin(\tau t)}{\gamma^2+\tau^2} d\tau$. Fix $\eta > 0$ and after that select an natural number N_0 such that for all $N, N' \geq N_0$ the following relation holds : $\left|\int_N^{N'} \frac{\tau \sin(\tau t)}{\gamma^2+\tau^2} d\tau\right| < \frac{\eta}{3}$. Then we obtain $\left|\int_N^{\infty} \frac{\tau \sin(\tau t)}{\gamma^2+\tau^2} d\tau\right| \leq \frac{\eta}{3}$ for every $N \geq N_0$.

If $T > N_0$, then we have

$$J(T) - J = \int_0^{N_0} \left(1 - \frac{\tau}{T}\right)^{\beta} \frac{\tau \sin(\tau t)}{\gamma^2 + \tau^2}\, d\tau + \int_{N_0}^{T} \left(1 - \frac{\tau}{T}\right)^{\beta} \frac{\tau \sin(\tau t)}{\gamma^2 + \tau^2}\, d\tau$$

$$- \int_0^{N_0} \frac{\tau \sin(\tau t)}{\gamma^2 + \tau^2}\, d\tau - \int_{N_0}^{\infty} \frac{\tau \sin(\tau t)}{\gamma^2 + \tau^2}\, d\tau. \tag{23}$$

Further, we have

$$|J(T)-J| \leq \left| \int_0^{N_0} \left(1-\frac{\tau}{T}\right)^{\beta} \frac{\tau \sin(\tau t)}{\gamma^2+\tau^2}\, d\tau - \int_0^{N_0} \frac{\tau \sin(\tau t)}{\gamma^2+\tau^2}\, d\tau \right| + \left| \int_{N_0}^{T} \left(1-\frac{\tau}{T}\right)^{\beta} \frac{\tau \sin(\tau t)}{\gamma^2+\tau^2}\, d\tau \right| + \left| \int_{N_0}^{\infty} \frac{\tau \sin(\tau t)}{\gamma^2+\tau^2}\, d\tau \right|. \quad (24)$$

The function $f(\tau) = \left(1-\frac{\tau}{T}\right)^{\beta}$ is decreasing on the interval $[N_0,\ T]$. Therefore, by the second mean value theorem of integral calculus, we obtain

$$\int_{N_0}^{T} \left(1-\frac{\tau}{T}\right)^{\beta} \frac{\tau \sin(\tau t)}{\gamma^2+\tau^2}\, d\tau = \left(1-\frac{N_0}{T}\right)^{\beta} \int_{N_0}^{\xi} \frac{\tau \sin(\tau t)}{\gamma^2+\tau^2}\, d\tau,$$

where $\xi \in [N_0,\ T]$. Then we have

$$\left| \int_{N_0}^{T} \left(1-\frac{\tau}{T}\right)^{\beta} \frac{\tau \sin(\tau t)}{\gamma^2+\tau^2}\, d\tau \right| = \left(1-\frac{N_0}{T}\right)^{\beta} \left| \int_{N_0}^{\xi} \frac{\tau \sin(\tau t)}{\gamma^2+\tau^2}\, d\tau \right| < \frac{\eta}{3}\left(1-\frac{N_0}{T}\right)^{\beta} < \frac{\eta}{3}.$$

This, together with (24) shows that

$$|J(T)-J| \leq \left| \int_0^{N_0} \left[\left(1-\frac{\tau}{T}\right)^{\beta} - 1 \right] \frac{\tau \sin(\tau t)}{\gamma^2+\tau^2}\, d\tau \right| + \frac{2\eta}{3},$$

for $T > N_0$. Further, it follows that

$$\begin{aligned} |J(T)-J| &\leq \int_0^{N_0} \left| \left(1-\frac{\tau}{T}\right)^{\beta} - 1 \right| \cdot \left| \frac{\tau \sin(\tau t)}{\gamma^2+\tau^2} \right| d\tau + \frac{2\eta}{3} \\ &= \int_0^{N_0} \left[1 - \left(1-\frac{\tau}{T}\right)^{\beta} \right] \cdot \left| \frac{\tau \sin(\tau t)}{\gamma^2+\tau^2} \right| d\tau + \frac{2\eta}{3} \\ &\leq \left[1 - \left(1-\frac{N_0}{T}\right)^{\beta} \right] \int_0^{N_0} \left| \frac{\tau \sin(\tau t)}{\gamma^2+\tau^2} \right| d\tau + \frac{2\eta}{3}. \end{aligned}$$

It is clear that $1-\left(1-\frac{N_0}{T}\right)^\beta \to 0$ as $T\to\infty$. Therefore, one can find $T_0 \geq N_0$ such that

$$\left[1-\left(1-\frac{N_0}{T}\right)^\beta\right]\int\limits_0^{N_0}\left|\frac{\tau\sin(\tau t)}{\gamma^2+\tau^2}\right|d\tau < \frac{\eta}{3}$$

for every $T > T_0$. Hence, for every $T > T_0$ we have $|J(T)-J| < \eta$. Because $\eta > 0$ is an arbitrary real number, we conclude that $J(T)\to J$ as $T\to\infty$. By the same method, it can be proved that

$$\int\limits_0^T\left(1-\frac{\tau}{T}\right)^\beta\frac{\cos(\tau t)}{\gamma^2+\tau^2}\,d\tau \to \int\limits_0^\infty\frac{\cos(\tau t)}{\gamma^2+\tau^2}\,d\tau \quad\text{as}\quad T\to\infty.$$

It is well known that

$$\int\limits_0^\infty\frac{\gamma\cos(\tau t)}{\gamma^2+\tau^2}\,d\tau = \frac{\pi}{2}e^{-\gamma t}\quad\text{and}\quad\int\limits_0^\infty\frac{\tau\sin(\tau t)}{\gamma^2+\tau^2}\,d\tau = \frac{\pi}{2}e^{-\gamma t}.$$

Therefore,

$$\begin{aligned}I_1 &= e^{\gamma t}\int\limits_0^T\left(1-\frac{\tau}{T}\right)^\beta\frac{2\gamma\cos(\tau t)+2\tau\sin(\tau t)}{\gamma^2+\tau^2}\,d\tau\\ &\to 2e^{\gamma t}\left[\int\limits_0^\infty\frac{\gamma\cos(\tau t)}{\gamma^2+\tau^2}d\tau+\int\limits_0^\infty\frac{\tau\sin(\tau t)}{\gamma^2+\tau^2}d\tau\right] = 2\pi\end{aligned}$$

as $T\to\infty$. Now we will show that

$$I_2 = \int\limits_{-T}^T\left(1-\frac{|\tau|}{T}\right)^\beta e^{(\gamma+i\tau)t}d\tau\int\limits_0^\infty e^{-(\gamma+i\tau)s}\,(S(s)x-S(t)x)\,ds \to 0 \text{ as } T\to\infty.$$

We interchange the order of integration and obtain

$$I_2 = \int\limits_0^\infty e^{\gamma(t-s)}\,(S(s)x-S(t)x)\int\limits_{-T}^T\left(1-\frac{|\tau|}{T}\right)^\beta e^{i\tau(t-s)}d\tau\,ds.$$

For any $\varepsilon > 0$ we can find $\delta=\delta(\varepsilon)$, $0<\delta<1$ and $0<\delta<t$, such that $\|S(s)x-S(t)x\| < \varepsilon$ for all $s\in[t-\delta,\ t+\delta]$. Now, $I_2 = J_1(T)+J_2(T)+J_3(T)$, where

$$J_1(T) = \int\limits_0^{t-\delta} e^{\gamma(t-s)}\,(S(s)x-S(t)x)\,ds\int\limits_{-T}^T\left(1-\frac{|\tau|}{T}\right)^\beta e^{i\tau(t-s)}d\tau,$$

$$J_2(T) = \int_{t-\delta}^{t+\delta} e^{\gamma(t-s)} \left(S(s)x - S(t)x\right) ds \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right)^{\beta} e^{i\tau(t-s)} d\tau$$

$$J_3(T) = \int_{t+\delta}^{\infty} e^{\gamma(t-s)} \left(S(s)x - S(t)x\right) ds \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right)^{\beta} e^{i\tau(t-s)} d\tau.$$

It is straightforward to see that

$$J_1(T) = \int_{\delta}^{t} e^{\gamma\sigma} \left[S(t-\sigma)x - S(t)x\right] 2T \int_{0}^{1} (1-u)^{\beta} \cos(\sigma T u) du\, d\sigma,$$

and

$$J_1(T) = 2 \int_{\delta T}^{tT} e^{\frac{\gamma s}{T}} \left[S\left(t - \frac{s}{T}\right) x - S(t)x\right] \int_{0}^{1} (1-u)^{\beta} \cos(su) du\, ds.$$

Use integration by parts to obtain $\int_0^1 (1-u)^{\beta} \cos(su) du$. We obtain

$$J_1(T) = 2\beta \int_{\delta T}^{tT} e^{\frac{\gamma s}{T}} \frac{S\left(t - \frac{s}{T}\right) x - S(t)x}{s} \int_{0}^{1} (1-u)^{\beta-1} \sin(su) du\, ds.$$

Now Lemma 3.3 gives $|J_1(T)| \leq LM_1 \int_{\delta T}^{tT} \frac{ds}{s^{1+\beta}}$, for some constants L and M_1. From here it directly follows that $J_1(T) \to 0$ as $T \to \infty$. Let us estimate

$$J_2(T) = \int_{t-\delta}^{t+\delta} e^{\gamma(t-s)} \left(S(s)x - S(t)x\right) ds \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right)^{\beta} e^{i\tau(t-s)} d\tau.$$

Obviously,

$$J_2(T) = \int_{-\delta}^{\delta} e^{\gamma\sigma} \left[S(t-\sigma)x - S(t)x\right] 2T \int_{0}^{1} (1-u)^{\beta} \cos(\sigma T u) du\, d\sigma,$$

or $J_2(T) = \overline{J_2(T)} + \overline{\overline{J_2(T)}}$, where

$$\overline{J_2(T)} = \int_{0}^{\delta} e^{\gamma\sigma} \left[S(t-\sigma)x - S(t)x\right] 2T \int_{0}^{1} (1-u)^{\beta} \cos(\sigma T u) du\, d\sigma$$

$$\overline{\overline{J_2(T)}} = \int_{0}^{\delta} e^{-\gamma\sigma} \left[S(t+\sigma)x - S(t)x\right] 2T \int_{0}^{1} (1-u)^{\beta} \cos(\sigma T u) du\, d\sigma.$$

Further, we have

$$\overline{J_2(T)} = 2\int_0^{\delta T} e^{\frac{\gamma s}{T}}\left[S\left(t-\frac{s}{T}\right)x - S(t)x\right]ds\int_0^1 (1-u)^\beta\cos(su)du$$
$$= 2\beta\int_0^{\delta T} e^{\frac{\gamma s}{T}}\frac{S\left(t-\frac{s}{T}\right)x - S(t)x}{s}ds\int_0^1 (1-u)^{\beta-1}\sin(su)du$$
$$= 2\beta\int_0^{1} e^{\frac{\gamma s}{T}}\frac{S\left(t-\frac{s}{T}\right)x - S(t)x}{s}ds\int_0^1 (1-u)^{\beta-1}\sin(su)du$$
$$+ 2\beta\int_1^{\delta T} e^{\frac{\gamma s}{T}}\frac{S\left(t-\frac{s}{T}\right)x - S(t)x}{s}ds\int_0^1 (1-u)^{\beta-1}\sin(su)du$$

and

$$\left\|\int_0^{1} e^{\frac{\gamma s}{T}}\frac{S\left(t-\frac{s}{T}\right)x - S(t)x}{s}ds\int_0^1 (1-u)^{\beta-1}\sin(su)du\right\|$$
$$\leq \int_0^{1} e^{\frac{\gamma s}{T}}\frac{\left\|S\left(t-\frac{s}{T}\right)x - S(t)x\right\|}{s}ds\int_0^1 (1-u)^{\beta-1}\sin(su)du \leq \varepsilon\cdot K_1,$$

where K_1 is a suitable constant independent of ε. Namely, the last expression can be bounded above by $e^{\frac{\gamma}{T}}\varepsilon\int_0^1 ds\int_0^1 (1-u)^{\beta-1}udu$, while $\|S\left(t-\frac{s}{T}\right)x - S(t)x\| \leq \varepsilon$ and $\left|\frac{\sin(su)}{s}\right| \leq u$.

Using Lemma 3.3, we obtain

$$\left\|\int_1^{\delta T} e^{\frac{\gamma s}{T}}\frac{S\left(t-\frac{s}{T}\right)x - S(t)x}{s}ds\int_0^1 (1-u)^{\beta-1}\sin(su)du\right\|$$
$$\leq \int_1^{\delta T} e^{\frac{\gamma s}{T}}\frac{\left\|S\left(t-\frac{s}{T}\right)x - S(t)x\right\|}{s}\frac{M_1}{s^\beta}ds \leq \varepsilon\cdot M_1\cdot\max_{\sigma\in[0,1]} e^{\gamma\sigma}\int_1^{\delta T}\frac{ds}{s^{1+\beta}} \leq \varepsilon\cdot K_2,$$

where K_2 is a constant independent of ε.

Similarly, it can be proved that $\left\|\overline{\overline{J_2(T)}}\right\| \leq \varepsilon\cdot K_3$, where K_3 is a constant independent of ε. Hence, $\|J_2(T)\| \leq \varepsilon\cdot K$, where K is a constant independent of ε.

Furthermore,

$$J_3(T) = \int\limits_{t+\delta}^{\infty} e^{\gamma(t-s)} \left(S(s)x - S(t)x\right) ds \int\limits_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right)^{\beta} e^{i\tau(t-s)} d\tau$$
$$= \int\limits_{\delta}^{\infty} e^{-\gamma\sigma} \left[S(t+\sigma)x - S(t)x\right] 2T d\sigma \int\limits_{0}^{1} (1-u)^{\beta} \cos(\sigma T u) du$$
$$= 2 \int\limits_{\delta T}^{\infty} e^{-\frac{\gamma s}{T}} \left[S\left(t + \frac{s}{T}\right) x - S(t)x\right] ds \int\limits_{0}^{1} (1-u)^{\beta} \cos(su) du$$
$$= 2\beta \int\limits_{\delta T}^{\infty} e^{-\frac{\gamma s}{T}} \frac{S\left(t + \frac{s}{T}\right) x - S(t)x}{s} ds \int\limits_{0}^{1} (1-u)^{\beta-1} \sin(su) du.$$

Then Lemma 3.3 implies

$$\|J_3(T)\| \le 2\beta \int\limits_{\delta T}^{\infty} e^{-\frac{\gamma s}{T}} 2M e^{\omega_0\left(t+\frac{s}{T}\right)} \frac{M_1}{s^{1+\beta}} ds \le S M_1 \int\limits_{\delta T}^{\infty} \frac{ds}{s^{1+\beta}}$$

(for some constants S and M_1). Now we see that $J_3(T) \to 0$ as $T \to \infty$. Hence, $I_2 \to 0$ as $T \to \infty$, and the proof is completed. From the proof of the theorem one can see that the limit is uniform in t on any bounded interval $[a, b] \subset [0, \infty)$. □

Theorem 3.2 and Theorem 3.3 imply

Corollary 3.1. *Let $(S(t))_{t\ge 0}$ be an $\alpha-$ times integrated, exponentially bounded semigroup on a Banach space X ($\alpha \in \mathbb{R}^+$) . Then, for every $\beta > 0$, $\gamma > \max(\omega_0, 0)$, $x \in X$ and $t \ge 0$:*

$$S(t)x = \frac{1}{2\pi i}(C, \beta) - \lim_{\omega\to\infty} \int\limits_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t} \frac{R(\lambda, A)x}{\lambda^{\alpha}} d\lambda.$$

References

[1] W. Arendt, *Resolvent positive operators and integrated semigroups*, Proc. London Math. Soc., (3) 54 (1987), 321–349.

[2] H. Kellermann and M. Hieber, *Integrated semigroups*, J. Funct. Anal., 84 (1989), 160–180.

[3] H. Kellermann, *Integrated semigroups*, Dissertation, Universitat Tubingen, 1986.

[4] F. Neubrander, *Integrated semigroups and their applications to the abstract Cauchy problem*, Pacific J. Math., 135 (1988), 111–155.
[5] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York-Berlin, 1983.
[6] E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloq. Publ., Vol. 31. Providence, Rhode Island, 1957.
[7] H. Thieme, *Integrated semigroups and integrated solutions to abstract Cauchy problems*, J. Math. Anal. Appl., 152 (1990), 416–447.
[8] S. Kalabušić and F. Vajzović, *Exponential formula for one-time integrated semigroups*, Novi Sad J. Math., 33 (2003), 1–12.
[9] M. Mijatović, S. Pilipović and F. Vajzović, $\alpha-$ *times integrated semigroups* ($\alpha \in \mathbb{R}^+$), J. Math. Anal. Appl., 210 (1997), 790–803.

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