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A GENERALIZATION OF MEIR-KEELER TYPE COMMON FIXED POINT THEOREM FOR FOUR NONCONTINUOUS MAPPINGS

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Abstract. In this paper, using a combination of methods used in [1], [20] and [22] the results from [3, Theorem 1], [14, Theorem 1] and [15, Theorem 1] are improved removing the assumption of continuity, relaxing compatibility to the weak compatibility property and replacing the completeness of the space with a set of four alternative conditions for four functions satisfying an implicit relation.

1. Introduction

Let S and T be self mappings of a metric space (X, d). Jungck [4] defines S and T to be compatible if $\lim d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim Sx_n = \lim Tx_n = x$ for some $x \in X$. In 1993, Jungck, Murthy and Cho [6] defines S and T to be compatible of type (A) if $\lim d(TSx_n, S^2x_n) = 0$ and $\lim (STx_n, T^2x_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim Sx_n = \lim Tx_n = x$ for some $x \in X$.

By [6, Ex.2.1 and Ex.2.2] it follows that the notions of compatible mappings and compatible mappings of type (A) are independent. Recently, Pathak and Khan [17] introduced a new concept of compatible mappings of type (B) as a generalization of compatible mappings of type (A). S and T is said to be compatible of type (B) if

$$d(STx_n, T^2x_n) \leq \frac{1}{2}\left[\lim d(STx_n, St) + \lim d(St, S^2x_n)\right]$$

$$d(TSx_n, S^2x_n) \leq \frac{1}{2}\left[\lim d(TSx_n, Tt) + \lim d(Tt, T^2x_n)\right]$$

whenever $\{x_n\}$ is a sequence in X such that $\lim Sx_n = \lim Tx_n = t$ for some $t \in X$.

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Clearly, compatible mappings of type (A) are compatible of type (B) . By [17, Ex.2.4] it follows that the implication is not reversible. In [18] the concept of compatible mappings of type (P) was introduced and compared with compatible mappings and compatible mappings of type (A). S and T are compatible of type (P) if lim $d(S^2x_n, T^2x_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that lim $Sx_n =$ lim $Tx_n = t$ for some $t \in X$.

Lemma 1. [4] (resp. [6], [17], [18]). *Let S and T be compatible (resp. compatible of type* (A)*, compatible of type* (B)*, compatible of type* (P)*) self mappings of a metric space (X, d). If $Sx = Tx$ for some $x \in X$, then $STx = TSx$.*

In 1994, Pant [11] introduced the notion of pointwise R-weakly commuting mappings. It is proved in [12] that the notion of pointwise R-weakly commuting is equivalent to commutativity in coincidence points.

Jungck [5] defines S and T to be weakly compatible if $Sx = Tx$ implies $STx = TSx$.

Thus S and T are weakly compatible if and only if S and T are pointwise R-weakly commuting mappings.

Remark 1. By Lemma 1 it follows that every compatible (resp. compatible of type (A), compatible of type (B), compatible of type (P)) pair of mappings is weakly compatible.

The following example is an example of weakly compatible mappings which is not compatible (resp. compatible of type (A), compatible of type (P)).

Let X=[2,20] with usual metric. Define

$$Tx = \begin{cases} 2 & \text{if} \quad x = 2 \\ 12 + x & \text{if} \quad 2 < x \le 5; \\ x - 3 & \text{if} \quad 5 < x \le 20 \end{cases} \quad S_x = \begin{cases} 2 & \text{if} \quad x \in \{2\} \cup (5, 20] \\ 8 & \text{if} \quad 2 < x \le 5. \end{cases}$$

S and T are weakly compatible since they commute at their coincidence point [12]. By [19] S and T are not compatible of type (A) and not compatible of type (P). S and T are not compatible of type (B). Indeed, let us consider a decreasing sequence $\{x_n\}$ such that $\lim x_n = 5$. Then $\lim Tx_n = 2$, $\lim Sx_n = 2$, $\lim STx_n = 8$, $\lim T^2x_n = 14$, $\lim S^2x_n = 2$.

Then $\lim d(STx_n, T^2x_n) = 6 > \frac{1}{2}[\lim d(STx_n, St) + \lim d(St, S^2x_n)] = \frac{1}{2}(6 + 0) = 3$

2. Preliminaries

In 1969, Meir and Keeler [8] established a fixed point theorem for self mappings of a metric space (X, d) satisfying the following condition:

For every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\epsilon < d(x,y) < \epsilon + \delta \text{ implies } d(fx,fy) < \epsilon. \tag{2.1}$$

There exists a vast literature which generalizes the result of Meir and Keeler.

In [7], Maiti and Pal proved a fixed point theorem for a self mapping f of a metric space (X,d) satisfying the following condition, which is a generalization of (2.1) :

For every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\epsilon \le \max\{d(x,y),\, d(x,fx), d(y,fy)\} < \epsilon + \delta \text{ implies } d(fx,fy) < \epsilon. \tag{2.2}$$

In [16] and [21], Park-Rhoades, respectively, Rao-Rao extend this result for two mappings f and g of a metric space (X,d) satisfying the following condition:

$$\epsilon < \max\{d(fx,fy), d(fx,gx), d(fy,gy), \frac{1}{2}[d(fx,gy) + d(fy,gx)]\} < \epsilon + \delta \text{ implies } d(gx,gy) < \epsilon. \tag{2.3}$$

In 1986, Jungck [4] and Pant [9] extend these results for four mappings. It is know from Jungck [4], Pant [10], [12], [13] and other papers that in the case of theorems for four mappings $A, B, S, T : (X,d) \to (X,d)$, a condition of type Meir-Keeler does not assure the existence of a fixed point.

The following theorem was recently proved in [3].

Theorem 1. [3]. *Let (A,S) and (B,T) be the compatible pairs of self mappings of a complete metric space (X,d) such that*

(i) $A(X) \subset T(X)$, $B(X) \subset S(X)$,

(ii) *given $\epsilon > 0$ there exists a $\delta > 0$ such that for all x, y in X, $\epsilon \le \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty); \frac{1}{2}[d(Sx,By) + d(Ax,Ty)]\} < \epsilon + \delta$ implies $d(Ax,By) < \epsilon$ and*

(iii) $d(Ax,By) < k[d(Sx,Ty) + d(Ax,Sx) + d(By,Ty) + d(Sx,By) + d(Ax,Ty)]$, *for every* $0 \le k \le \frac{1}{3}$.

If one of mappings A, B, S and T is continuous then A, B, S and T have a unique common fixed point.

The following two theorems appear in [14], resp. [15].

Theorem 2. [14]. *Let A, B, S and T be mappings as in Theorem* 1 *satisfying* (i) *and* (ii) *and*

(iv) $d(Ax,By) < \max\{k_1 d(Sx,Ty), k_2[d(Ax,Sx) + d(By,Ty)]/2, [d(Sx, By + d(Ax,Ty)]/2)\}$ *for* $k_1 \ge 0$ *and* $1 \le k_2 < 2$.

If one of the mappings A, B, S and T is continuous then A, B, S and T have a unique common fixed point.

Theorem 3. [15]. *Let A, B, S and T be mappings as in Theorem 1 satisfying* (i) *and* (ii) *and*

(v) $d(Ax, By) < \max\{d(Sx, Ty), [d(Ax, Tx)+d(By, Ty)]/2, k[d(Sx, By) +d(Ax, Ty)]/2\}$ *for* $1 \le k \le 2$.

If one of the mappings A, B, S and T is continuous then A, B, S and T have a unique common fixed point.

3. Implicit relations

Let $\mathcal{F}_4$ the set of all continuous functions $F(t_1, \dots t_4) : R_+^4 \to R$ satisfying the following condition:
(F_1): If $F(u, 0, u, u) \le 0$ then $u = 0$.
The function $F(t_1, \dots t_4)$ satisfies condition (F_u) if $F(u, u, 0, 2u) \ge 0; \forall u > 0$.

Example 1. $F(t_1, \dots, t_4) = t_1 - at_2 - bt_3 - ct_4$, where $a, b, c \ge 0$, $0 < b+c < 1$, $0 \le a + 2c \le 1$.
$(F_1) : F(u, 0, u, u) = u(1 - b - c) \le 0$ implies $u = 0$.
$(F_u) : F(u, u, 0, 2u) = u(1 - a - 2c) \ge 0$; $\forall u > 0$.

If $a = b = c = 1$ we have the following example:

Example 2. $F(t_1, \dots, t_4) = t_1 - k(t_2 + t_3 + t_4)$, where $0 \le k \le \frac{1}{3}$.

Example 3. $F(t_1, \dots, t_4) = t_1^2 - k(t_2^2 + t_3^2 + t_4^2)$, where $0 \le k \le \frac{1}{3}$.

The proof is similar to the proof of Example 1.

Example 4. $F(t_1, \dots, t_4) = t_1 - \max\{t_2, \frac{t_3}{2}, \frac{kt_4}{2}\}$, where $0 \le k \le 1$.
(F_1): $F(u, 0, u, u) = u(1 - \frac{1}{2}) \le 0$ implies $u = 0$.
$(F_u) : F(u, u, 0, 2u) = u - \max\{u, ku\} = 0$; $\forall u \ge 0$.

Example 5. $F(t_1, \dots, t_4) = t_1 - \max\{k_1 t_2, \frac{k_2}{2} t_3, \frac{t_4}{2}\}$ where $0 \le k_1 \le 1$; $1 \le k_2 < 2$.
$(F_1) : F(u, 0, u, u) = u(1 - \frac{k_2}{2}) \le 0$ implies $u = 0$.
$(F_u) : F(u, u, 0, 2u) = 0$; $\forall u > 0$.

Example 6. $F(t_1, \dots t_6) = t_1^2 - t_2^2 - \frac{bt_3t_4}{1+t_2+t_3}$, where $0 \le b < 1$.
(F_1): If $F(u, 0, u, u) = u^2 - \frac{bu^2}{1+u} \le 0$, then $u^2(1-b) \le 0$ which implies $u = 0$.
$(F_u) : F(u, u, 0, 2u) = 0$, $\forall u > 0$.

Theorem 4. *Let (X, d) be a metric space and $S, T, I, J : (X, d) \to (X, d)$ four mappings satisfying the inequality*

$$F(d(Sx, Ty), d(Ix, Jy), d(Ix, Sx) + d(Jy, Ty), d(Ix, Ty) + d(Jy, Sx)) < 0 \qquad (3.1)$$

for all x, y in X, where F satisfies property (F_u). Then S, T, I and J have at most one common fixed point.

Proof. Suppose that S, T, I, J have two common fixed points z and v. Then by (3.1) we have successively

$$F(d(Sz, Tu), d(Iz, Ju), d(Iz, Sz) + d(Ju, Tu), d(Iz, Tu) + d(Ju, Sz)) < 0,$$
$$F(d(z, u), d(z, u), 0, 2d(z, u)) < 0,$$

a contradiction of (F_u). □

In this paper, using a combination of methods used in [1], [20] and [22] the results from Theorems 1-3 are improved by removing the assumption of continuity, relaxing compatibility to weak compatibility property and replacing the completeness of the space with a set of four alternative conditions for four functions satisfying a implicit relation.

4. Main result

Theorem 5. *Let S, T, I and J be self mappings of a metric space (X, d) such that*

a) *$S(X) \subset J(X)$ and $T(X) \subset I(X)$,*
b) *given $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\epsilon \leq \max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), \frac{1}{2}[d(Ix, Sy) + d(Jy, Sx)]\} < \epsilon + \delta \text{ implies } d(Sx, Ty) < \epsilon$$

c) *there exists $F \in \mathcal{F}_4$ such that the inequality (3.1) holds for all x, y in X.*

If one of $S(X), T(X), I(X)$ and $J(X)$ is a complete subspace of X, then

d) *S and I have a coincidence point,*
e) *T and J have a coincidence point.*

Moreover, if the pairs (S, I) and (T, J) are weakly compatible, then S, T, I and J have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Then, since (a) holds, we can define inductively a sequence

$$\{Sx_0, Tx_1, Sx_2, Tx_3, \dots, Sx_{2n}, Tx_{2n+1}, \dots\}$$

such that

$$y_{2n} = Sx_{2n} = Jx_{2n+1}, y_{2n+1} = Tx_{2n+1} = Ix_{2n+2}$$

for $n = 0, 1, 2, \dots$

By [2, Lemma 2.2] it follows that $\{y_n\}$ is a Cauchy sequence in X.

Now suppose that $J(X)$ is a complete subspace of X, then the subsequence $y_{2n} = Jx_{2n+1}$ is a Cauchy sequence in $J(X)$ and hence has a limit u.

Let $v \in J^{-1}u$, then $Jv = u$. Since y_{2n} is convergent, then y_n is convergent to u and y_{2n+1} also converges to u. Setting $x = x_{2n}$ and $y = v$ in (3.1) we have

$$F(d(Sx_{2n}, Tv), d(Ix_{2n}, Jv), d(Ix_{2n}), Sx_{2n}) \\ + d(Jv, Tv), d(Ix_{2n}, Tv) + d(Jv, Sx_{2n})) < 0.$$

Letting n tend to infinity we obtain

$$F(d(u, Tv), 0, d(u, Tv), d(u, Tv) \leq 0$$

By (F_1) we have $u = Tv$. Hence J and T have a coincidence point. Since $T(X) \subset I(X)$, $u = Tv$ implies that $u \in I(X)$.

Let $w \in I^{-1}u$, then $Iw = u$. Setting $x = w$ and $y = x_{2n+1}$ we obtain by (F_1) that $Sw = u$. Thus S and I have a coincidence point. If one assumes that $I(X)$ is complete, then analogous arguments establish the existence of a coincidence point.

The remaining two cases are essentially the same as the previous cases. Indeed, if S(X) is complete then by (a) $u \in S(X) \subset I(X)$. Then (d) and (e) are completely established .

By $u = Jv = Tv$ and by the weak compatibility of (J, T) we have

$$Tu = TJv = JTv = Ju$$

By $u = Iw = Sw$ and by the weak compatibility of (I, S) we have

$$Su = SIw = ISw = Iu$$

By (3.1) we have successively

$$F(d(Sw, Tu), d(Iw, Jv), d(Iw, Sw) + d(Ju, Tu), d(Iw, Tu) + d(Ju, Sw)) < 0$$
$$F(d(u, Tu), d(u, Tu), 0, 2d(u, Tu)) < 0$$

a contradiction of (F_u) if $d(u, Tu) \neq 0$. Therefore, $u = Tu$.

Similarly one can show that $Su = u$. Thus,

$$u = Tu = Ju = Su = Iu$$

□

The uniqueness of the common fixed point follows from Theorem 4.

Corollary 1. *Let S, T, I and J be the self mappings of a complete metric space satisfying conditions* (a), (b), (c) *of Theorem* 5. *Then conditions* (d) *and* (e) *of Theorem* 5 *hold.*

Moreover, if the pairs (S, I) and (T, J) are compatible (resp.compatible of type (A)*, compatible of type* (B)*, compatible of type* (P)*) then S, T, I and J have a unique common fixed point.*

Proof. The proof follows by Theorem 5 and Remark 1. □

Corollary 2. *Theorem* 1.

Proof. The proof follows by Corollary 1 and Example 2. □

Remark 2. By Corollary 1 and Example 4 we obtain Theorem 3 for $0 \leq k \leq 1$. By Corollary 1 and Example 5 we obtain Theorem 2 for $0 \leq k_1 \leq 1$ and $1 \leq k_2 < 2$.

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