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EQUIVALENCE OF K-FUNCTIONALS AND MODULUS OF SMOOTHNESS GENERATED BY THE q-RUBIN OPERATOR

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Abstract. In this paper, the equivalence between K-functionals and modulus of smoothness tied to a q-Rubin operator was studied.

1. Introduction

Given a positive real number r and a positive integer m, the classical modulus of smoothness is defined, for a function $f \in L^2(\mathbb{R})$, by

$$w_m(f,r) = \sup_{0<h\leq r} \parallel \triangle_h^m \parallel_2,$$

where

$$\triangle_h^m = (\tau^h - I)^m f.$$

I is the unit operator and τ^h stands for the usual translation operator, given by $\tau^h f(x) = f(x+h)$.
The classical K-functional, introduced in [6], is defined by

$$K_m(f,r) = \inf\{\parallel f-g \parallel_2 + r \parallel D^m g \parallel_2; g \in W_2^m\},$$

where W_2^m is the Sobolev space defined by

$$W_2^m = \{f \in L^2(\mathbb{R}) : D^j f \in L^2(\mathbb{R}), j = 1, \ldots\ldots, m\}.$$

Note that the operator $D = \frac{d}{dx}$. An oustanding result of the theory of approximation of functions on $\mathbb{R}$, which gives the equivalence between modulus of smoothness and $K-$functionals, can be formulated as follows :

Theorem 1.1. *There are two positive constants c_1 and c_2 such that, for all $f \in L^2(\mathbb{R})$ and $r > 0$, we have*

$$c_1 w_m(f,r) \leq K_m(f,r^m) \leq c_2 w_m(f,r).$$

For more details, one can see page 111 in the book of Paul L.Bulzer and Hubert Berens (see [2]).

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2. Preliminaries

Throughout this paper, we assume that $0 < q < 1$.
In this section , we provide some facts about harmonic analysis related to the q-Rubin's Operator δ_q. We will show briefly only properties required for the discussion. For more details, we refer to [3].
We put $\mathbb{R}_q = \{\pm q^n : n \in Z\}$, $\widehat{\mathbb{R}_q} = \mathbb{R}_q \cup \{0\}$.
For a $\in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1; (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n = 1,2,....; (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

We also denote

$$[x]_q = \frac{1-q^x}{1-q}, \ \ x \in \mathbb{C}, \ \text{ and } [n]_{q!} = \frac{(q;q)_n}{(1-q)^n}, \ \ n \in \mathbb{N}.$$

A q-analogue of the classical exponential function is given by (see [8, 9])

$$e(z;q^2) = cos(-iz;q^2) + isin(-iz;q^2),$$

where

$$\cos(x;q^2) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\frac{x^{2n}}{[2n]_{q!}}, \ \ \sin(x;q^2) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\frac{x^{2n+1}}{[2n+1]_{q!}}$$

These three functions are entire on $\mathbb{C}$. Remark that, when q tends to 1, they tend to the corresponding classical ones pointwise and uniformly on compacts.

Lemma 2.1. *We have*

(1) *For all* $x \in \mathbb{R}_q$,
$|cos(x;q^2)| \le \frac{1}{(q;q)_\infty}$, $|sin(x;q^2)| \le \frac{1}{(q;q)_\infty}$ *and* $|e(ix;q^2)| \le \frac{2}{(q;q)_\infty}$.
(2) $\lim_{x\longrightarrow\pm\infty} x^{\frac{1}{2}}\sin(x;q^2) = \lim_{x\longrightarrow\pm\infty} x^{\frac{1}{2}}\cos(x;q^2) = 0.$
(3) *There is* $c_1 > 0$ *such that* $|1 - e(ix;q^2)| \ge c_1$ *with* $|x| \ge 1$.

Proof. To prove 1., we can see [8, Lemma 6].
We can find a proof of assertion 2. in [9, lemma 1].
Let us check assertion 3. Using the fact that $\lim_{x\longrightarrow+\infty} x^{\frac{1}{2}}e(ix;q^2) = 0$, there exist two numbers $M > 0$ and $x_0 > 0$ such that $x \ge x_0$ and $|e(ix;q^2)| \le \frac{M}{\sqrt{x_0}}$.
Let $m = \min_{x\in[1,x_0]} |1 - e(ix;q^2)|$. Since $x \ge 1$, we get the inequality

$$|1 - e(ix;q^2)| \ge c_1, \text{ where } c_1 = \min\left\{m, \frac{M}{\sqrt{x_0}}\right\}. \qquad \square$$

The q^2-analogue differential operator is defined by

$$\delta_q(f)(z) = \begin{cases} \dfrac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z} & \text{if } z \neq 0, \\ \lim\limits_{x \longrightarrow 0} \delta_q(f)(x) \ (in \mathbb{R}_q) & \text{if } z = 0. \end{cases} \tag{2.1}$$

A repeated application of the q^2-analogue differential operator n times is denoted by:

$$\delta_q^0 f = f, \ \ \delta_q^{n+1} f = \delta_q(\delta_q^n f).$$

The q-Jackson integrals,(see [5]), are defined by

$$\int_a^b f(x) d_q x = (1-q) b \sum_{n=0}^{\infty} q^n f(bq^n) - (1-q) a \sum_{n=0}^{\infty} q^n f(aq^n))$$

and

$$\int_{-\infty}^{+\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n) + (1-q) \sum_{n=-\infty}^{\infty} q^n f(-q^n).$$

Notation 2.1. We denote by

• $S_q(\mathbb{R}_q)$ the space of functions f defined on $\mathbb{R}_q$ satisfying, for all m, n non negative integers,

$$P_{m,n,q}(f) = \sup_{x \in \mathbb{R}_q} \mid x^m \delta_q^n f(x) \mid < \infty$$

and

$$\lim_{x \longrightarrow 0} \delta_q^n f(x) \ \ (\text{in } \mathbb{R}_q) \ \text{ exists}$$

• $S_q^{'}(\mathbb{R}_q)$ the space of tempered distributions on $\mathbb{R}_q$. It is the topological dual space of $S_q(\mathbb{R}_q)$.

• $L_q^p(\mathbb{R}_q) = \{f : \|f\|_{p,q} = (\int_{-\infty}^{+\infty} |f(x)|^p d_q x)^{\frac{1}{p}} < \infty\}$.

• $L_q^\infty(\mathbb{R}_q) = \{f : \|f\|_{\infty,q} = \sup_{x \in \mathbb{R}_q} \mid f(x) \mid < \infty\}$.

Definition 2.1. In [9], R. L. Rubin defined the q-Rubin-Fourier transform by

$$\mathcal{F}_q(f)(x) = K \int_{-\infty}^{+\infty} f(t) e(-itx; q^2) d_q t, \tag{2.2}$$

where $K = \frac{(1+q)^{\frac{1}{2}}}{2\Gamma_{q^2}(\frac{1}{2})}$ and $\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}$ is the q-Gamma function.

Theorem 2.2. *$\mathcal{F}_q$ is an isomorphism from $L_q^2(\mathbb{R}_q)$ onto it self. Furthermore, for $f \in L_q^2(\mathbb{R}_q)$, we have*

$$\|\mathcal{F}_q(f)\|_{2,q} = \|f\|_{2,q} \tag{2.3}$$

and

$$\forall t \in \mathbb{R}_q, \ f(t) = K \int_{-\infty}^{+\infty} \mathcal{F}_q(f)(x) e(itx; q^2) d_q t.$$

Definition 2.2. (1) For $u \in S_q^{'}(\mathbb{R}_q)$, we define the distribution $\delta_q u$, by

$$< \delta_q u, \psi >= - < u, \delta_q \psi >, \ \ \psi \in S_q(\mathbb{R}_q)$$

(2) The q-Rubin transform of a q-distribution $u \in S_q^{'}(\mathbb{R}_q)$ is defined by

$$< \mathcal{F}_q(u), \psi >=< u, \mathcal{F}_q(\psi) >, \quad where \ \ u \in S_q^{'}(\mathbb{R}_q), \ and \ \psi \in S_q(\mathbb{R}_q).$$

Theorem 2.3. *The q-Rubin transform $\mathcal{F}_q$ is an isomorphism from $S_q^{'}(\mathbb{R}_q)$ onto itself.*

Lemma 2.4. *Let $u \in S_q^{'}(\mathbb{R}_q)$ and $f \in S_q(\mathbb{R}_q)$. Then for $j = 1, 2, ...$ we have*

$$\mathcal{F}_q(\delta_q^j f)(\lambda) = (i\lambda)^j \mathcal{F}_q(f)(\lambda) \tag{2.4}$$

$$\mathcal{F}_q(\delta_q^j u) = (-i\lambda)^j \mathcal{F}_q(u). \tag{2.5}$$

Definition 2.3. For $x \in \widehat{\mathbb{R}_q}$, the q-translation operator $\tau_{q,x}$ is defined, as in [8], on $L_q^1(\mathbb{R}_q)$ by

$$\tau_{q,x}(f)(y) = K \int_{-\infty}^{+\infty} \mathcal{F}_q(f)(t) e(itx; q^2) e(ity; q^2) d_q t, \ y \in \mathbb{R}_q,$$

$$\tau_{q,0}(f)(y) = f(y).$$

It is shown in [8] that the q-translation operator can be extended to $L_q^2(\mathbb{R}_q)$ and we have the following result

Proposition 2.5. *For all $f \in L_q^2(\mathbb{R}_q)$, we have $\tau_{q,x} f \in L_q^2(\mathbb{R}_q)$ and*

$$\|\tau_{q,x} f\|_{2,q} = \frac{2}{(q;q)_\infty} \leq \|f\|_{2,q}, \ x \in \widehat{\mathbb{R}_q}. \tag{2.6}$$

Furthermore,

$$\mathcal{F}_q(\tau_{q,x} f)(\lambda) = e(i\lambda x; q^2) \mathcal{F}_q(\lambda), x \in \widehat{\mathbb{R}_q}. \tag{2.7}$$

Notation 2.2. Consider $m = 1, 2, ...$ Let $W_{2,q}^m$ be the Sobolev type space constructed from the $q-$Rubin operator δ_q, by

$$W_{2,q}^m = \{f \in L_q^2(\mathbb{R}_q) \ : \delta_q^j f \ \in L_{\alpha,q}^2(\mathbb{R}_q), \ j = 1, 2..., m\}$$

More explicitly, $f \in W_{2,q}^m$ if and only if for each $j = 1, 2..., m$, there is a function in $L_q^2(\mathbb{R}_q)$ abusively denoted by $\delta_q^j f$, such that $\delta_q^j T_f^{\alpha;q} = T_{\delta_q^j f}^{\alpha;q}$.

Proposition 2.6. *For $f \in W_{2,q}^m$, we have*

$$\mathcal{F}_q(\delta_q^m f)(\lambda) = (-i\lambda)^m \mathcal{F}_q(f)(\lambda). \tag{2.8}$$

Proof. We deduce the wanted result using proposition 5.7 in [3] . $\square$

3. Equivalence of K-Functionnals and Modulus of Smoothness

Definition 3.1. Let $f \in L_q^2(\mathbb{R}_q)$ and $r > 0$. Then
• The generalized modulus of smoothness is defined by

$$w_m(f,r)_{2,q} = \sup_{0<h\leq r} \| \triangle_h^m f \|_{2,q},$$

where

$$\triangle_h^m f = (\tau_{q,h} - I)^m f,$$

I being the unit operator.
• The generalized K-functional is defined by

$$K_m(f,r)_{2,q} = \inf\{\| f - g \|_{2,q} + r \| \delta_q^m g \|_{2,q}; g \in W_{2,q}^m\}.$$

The following theorem establishes the equivalence of the modulus of Smoothness and the K-functional.

Theorem 3.1. *There are two positive constants c_1 and c_2 such that*

$$c_1 w_m(f,r)_{2,q} \leq K_m(f,r^m)_{2,q} \leq c_2 w_m(f,r)_{2,q},$$

for all $f \in L_q^2(\mathbb{R}_q)$ and $r > 0$.

In order to prove Theorem 3.1, we need some preliminary results.

Lemma 3.2. *Let $f \in L_q^2(\mathbb{R}_q)$ and $h > 0$. Then*

$$\| \triangle_h^m f\|_{2,q} \leq c^m \|f\|_{2,q} \tag{3.1}$$

$$\mathcal{F}_q(\triangle_h^m f)(\lambda) = (e(i\lambda h; q^2) - 1)^m \mathcal{F}_q(f)(\lambda), \tag{3.2}$$

where c is a constant verified $c \geq \frac{2}{(q;q)_\infty}$.

Proof. The result follows easily by using $(2.6), (2.7)$, and an induction on m. □

Lemma 3.3. *Let $f \in W_{2,q}^m$, $r > 0$. The following inequality is true.*

$$w_m(f,r)_{2,q} \leq c_2 r^m \|\delta^m f\|_{2,q}.$$

Proof. Assume that $h \in]0,r]$. By $(2.4), (3.2)$ and theorem 2.2 we have

$$\begin{aligned} \| \triangle_h^m f\|_{2,q}^2 &= ||\mathcal{F}_q(\triangle_h^m f)(\lambda)||_{2,q}^2 \\ &= ||(1 - e(i\lambda h; q^2))^m \mathcal{F}_q(f)(\lambda)||_{2,q}^2 \\ &= h^{2m} ||\tfrac{(1-e(i\lambda h;q^2))^m}{(hi\lambda)^m}(i\lambda)^m \mathcal{F}_q(f)(\lambda)||_{2,q}^2 \end{aligned}$$

According to Lemma 2.1, with all $l \in \widehat{\mathbb{R}_q}$ we have the inequality
$|\frac{(1-e(il;q^2)^m}{l^m}| \leq \beta^m$, where $\beta \in \mathbb{R}$. Then we deduce
$\| \triangle_h^m f\|_{2,q}^2 \leq h^{2m}\beta^{2m}||(i\lambda)^m \mathcal{F}_q(f)(\lambda)||_{2,q}^2 = h^{2m}\beta^{2m}||\mathcal{F}_q(\delta_q^m f)(\lambda)||_{2,q}^2 =$
$= h^{2m}\beta^{2m}||\delta_q^m f||_{2,q}^2$.
Calculating the supremum with respect to all $h \in]0,r]$, we obtain
$w_m(f,r)_{2,q} \leq c_2 r^m \|\delta_q^m f\|_{2,q}$, where $c_2 = \beta^m$. □

Definition 3.2. For any function $f \in L_q^2(\mathbb{R}_q)$ and any number $\nu > 0$, let us define the function

$$P_\nu(f)(x) := K \int_{-\nu}^{\nu} \mathcal{F}_q(f)(\lambda) e(i\lambda x; q^2) d_q\lambda = \mathcal{F}_q^{-1}(\mathcal{F}_q(f)\chi_\nu(x)),$$

where $\chi_\nu(\lambda)$ is the characteristic function of the segment $[-\nu, \nu], \mathcal{F}_q^{-1}$ is the inverse q-Rubin-Fourier transform.

Remark 3.1. *One can easily prove that the function $P_\nu(f)$ is infinitely q-differentiable and belongs to all classes $W_{2,q}^m, m \in \mathbb{N}$.*

Lemma 3.4. *There is a positive constant c_3 such that*

$$\|f - P_\nu(f)\|_{2,q} \leq c_3 \| \triangle_{1/\nu}^m f\|_{2,q},$$

for any $f \in L_q^2(\mathbb{R}_q)$ and $\nu > 0$.

Proof. Using the Plancherel formula, we obtain

$$\begin{aligned} \|f - P_\nu(f)\|_{2,q}^2 &= \|\mathcal{F}_q(f - P_\nu f)\|_{2,q}^2 \\ &= \|(1 - \chi_\nu(\lambda))\mathcal{F}_q(f)(\lambda)\|_{2,q}^2 \\ &= \int_{\mathbb{R}} | 1 - \chi_\nu(\lambda) |^2 | \mathcal{F}_q(f)(\lambda) |^2 \, d_q\lambda. \end{aligned}$$

By lemma 2.1 there is a constant $c_1 > 0$ such that

$$| 1 - e(i\lambda/\nu; q^2) | \geq c_1,$$

where $\lambda \in \mathbb{R}$; $| \lambda | \geq \nu$. From this and (3.2), we get

$$\begin{aligned} \|f - P_\nu(f)\|_{2,q}^2 &\leq c_1^{-2m} \int_{|\lambda| \geq \nu} | 1 - e(i\lambda/\nu; q^2) |^{2m} | \mathcal{F}_q(f)(\lambda) |^2 \, d_q\lambda \\ &\leq c_1^{-2m} \int_{\mathbb{R}} | \mathcal{F}_q(\triangle_{1/\nu}^m f)(\lambda) |^2 \, d_q\lambda \\ &= c_1^{-2m} \| \triangle_{1/\nu}^m f\|_{2,q}^2. \end{aligned}$$

Thus, we get the inequality $\|f - P_\nu(f)\|_{2,q} \leq c_3 \| \triangle_{1/\nu}^m f\|_{2,q}$, where $c_3 = \frac{1}{(c_1)^m}$. □

Corollary 3.5. *There is a positive constant c_3 such that*

$$\|f - P_\nu(f)\|_{2,q} \leq c_3 w_m(f, 1/\nu)_{2,q},$$

where $f \in L_q^2(\mathbb{R}_q)$ and $\nu > 0$.

Lemma 3.6. *There is a positive constant c_4 such that*

$$\|\delta^m(P_\nu(f))\|_{2,q} \leq c_4 \nu^m \| \triangle_{1/\nu}^m f\|_{2,q},$$

where $f \in L_q^2(\mathbb{R}_q)$ and $\nu > 0$.

Proof.

$$\begin{aligned}\|\delta^m(P_\nu(f))\|_{2,q} &= \|\mathcal{F}_q(\delta^m(P_\nu(f)))\|_{2,q} \\ &= \|(i\lambda)^m\chi_\nu(\lambda)\mathcal{F}_q(f)(\lambda)\|_{2,q} \\ &= \|\lambda^m\chi_\nu(\lambda)\mathcal{F}_q(f)(\lambda)\|_{2,q} \\ &= \|\tfrac{\lambda^m\chi_\nu(\lambda)}{(1-e(i\lambda/\nu;q^2))^m}(1-e(i\lambda/\nu;q^2))^m\mathcal{F}_q(f)(\lambda)\|_{2,q}\end{aligned}$$

Note that

$$\begin{aligned}\sup_{|\lambda|\in\mathbb{R}}\frac{|\lambda|^m\chi_\nu(\lambda)}{|\ 1-e(i|\lambda|/\nu;q^2)\ |^m} &= \nu^m\sup_{|\lambda|\leq\nu}\frac{(|\lambda|/\nu)^m}{|\ 1-e(i|\lambda|/\nu;q^2)\ |^m} \\ &= \nu^m\sup_{|t|\leq 1}\frac{t^m}{|\ 1-e(it;q^2)\ |^m}\end{aligned}$$

The result is obtained by considering $c_4=\sup_{|t|\leq 1}\dfrac{t^m}{|\ 1-e(it;q^2)\ |^m}$. □

Corollary 3.7. *There is a positive constant c_4 such that*

$$\|\delta_q^m(P_\nu(f))\|_{2,q}\leq c_4\nu^m w_m(f,1/\nu)_{2,q},$$

where $f\in L_q^2(\mathbb{R}_q)$ and $\nu>0$.

Proof of Theorem 3.1. (1) Let $h\in]0,r], g\in W_{2,q}^m$. Using Lemmas 3.3 and 3.2, we have $\|\triangle_h^m f\|_{2,q}=\|\triangle_h^m(f-g+g)\|_{2,q}\leq\|\triangle_h^m(f-g)\|_{2,q}+\|\triangle_h^m(g)\|_{2,q}\leq c^m\|f-g\|_{2,q}+c_2h^m\|\delta_q^m(g)\|_{2,q}\ \leq\ c_5(\|f-g\|_{2,q}+r^m\|\delta_q^m(g)\|_{2,q})$, where $c_5=max\{c^m,c_2\}$. Calculating the supremum with respect to $h\in]0,r]$ and the infimum with respect to all possible functions $g\in W_{2,q}^m$, we obtain $w_m(f,r)_{2,q}\leq c_5K_m(f,r^m)_{2,q}$.

(2) We have $P_\nu(f)\in W_{2,q}^m$. Using the definition of a K-functional we get $K_m(f,r^m)_{2,q}\leq\|f-P_\nu(f)\|_{2,q}+r^m\|\delta_q^m(P_\nu(f))\|_{2,q}$. Corollaries 3.5 and 3.7 yield to $K_m(f,r^m)_{2,q}\leq c_3w_m(f,1/\nu)_{2,q}+c_4(r\nu)^m w_m(f,1/\nu)_{2,q}$. Since ν is an arbitrary positive value, choosing $r=\frac{1}{\nu}$, we obtain $K_m(f,r^m)_{2,q}\leq c_6w_m(f,1/\nu)_{2,q}$, where $c_6=c_3+c_4$. This completes the proof. □

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