SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.20.01.03
Vol.20 (33), No.1 (2024), 13–23

FURTHER DEVELOPMENT ON KRASNER-VUKOVIĆ PARAGRADED STRUCTURES AND p-ADIC INTERPOLATION OF YUBO JIN L-VALUES

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Dedicated to Mirjana Vuković on the occasion of her jubilee

ABSTRACT. This paper is a joint project with Siegfried Bocherer (Mannheim), developing a recent preprint of Yubo Jin (Durham UK) previous works of Anh Tuan Do (Vietnam) and Dubrovnik, IUC-2016 papers from *Sarajevo Journal of Mathematics* (Vol.12, No.2-Suppl., 2016). We wish to use paragraded structures [KrVu87], [Vu01] on differential operators and arithmetical automorphic forms on classical groups and show that these structures provide a tool to construct p-adic measures and p-adic L-functions on the corresponding non-archimedean weight spaces.

An approach to constructions of automorphic L-functions on unitary groups and their p-adic analogues is presented. For an algebraic group G over a number field K these L functions are certain Euler products $L(s,\pi,r,\chi)$. In particular, our constructions cover the L-functions in [Shi00] via the doubling method of Piatetski-Shapiro and Rallis.

A p-adic analogue of $L(s,\pi,r,\chi)$ is a p-adic analytic function $L_p(s,\pi,r,\chi)$ of p-adic arguments $s \in \mathbb{Z}_p$, χ mod p^r which interpolates algebraic numbers defined through the normalized critical values $L^*(s,\pi,r,\chi)$ of the corresponding complex analytic L-function. We present a method using arithmetic nearly-holomorphic forms and general quasi-modular forms, related to algebraic automorphic forms. It gives a technique of constructing p-adic zeta-functions via general quasi-modular forms and their Fourier coefficients.

1. INTRODUCTION

Let p be a prime number. In the present paper we wish to use graded structures [KrVu87], [Kr80], [Vu01] on the rings and modules of differential operators and

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2010 *Mathematics Subject Classification.* 11F67, 11F85, 11F33 [See also 14G20, 22E50], 16W50, 16E45.

Key words and phrases. graded structures, automorphic forms, classical groups, p-adic L-functions, differential operators, non-archimedean weight spaces, quasi-modular forms, Fourier coefficients.

This paper was presented at the *International Scientific Online Conference*, organized by Prof. Mirna Džamonja and editorial board of *Sarajevo Journal of Mathematics* in honor of their editor-in-chief Academician Mirjana Vuković and her jubilee, Sarajevo, July 21, 2023.

quasimodular forms on classical groups in order to construct p-adic measures and p-adic L-functions on certain non-archimedean weight spaces.

Arithmetical modular forms belong traditionally to the world of arithmetic, but also to the worlds of geometry, algebra and analysis. On the other hand, it is very useful, to attach zeta-functions (or L-functions) to mathematical objects of various nature as certain generating functions.

Firstly, such L-functions give a tool to link these objects to each other (expressing a general form of functoriality), and secondly, this approach allows in favorable cases to obtain answers to fundamental questions about these objects; such answers are often expressed in the form of a number (complex or p-adic).

General graded structures called paragraded [KrVu87], [Kr80], [Vu01] can be used on the rings and modules of differential operators and quasimodular forms on classical groups. Our purpose is to use them in order to construet p-adic measures and p-adic L-functions on the corresponding non-archimedean weight spaces.

In the definition of M. Krasner, without the commutativity hypothesis, the structure of a graded group is determined by the underlying structure of the abstract group, and only the homogeneous part is equipped with a partial composition, induced by that of the group.

Moreover, M. Krasner introduces useful notions of graded ring and module, and their homogeneous parts called *anneid and moduloid* where the suffixes *eid*, *oid* mean that the composition structure is partial in principle. In our situation the underlying abstract group is a certain p-adic analytic group Y of characters with values in $\mathbb{C}_p^*$ which contains a subset Y^{class} of algebraic "motivic" points. Here $\bar{\mathbb{Q}}_p$ is an algebraic closure of $\mathbb{Q}_p$ and we fix an embedding $incl_p : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$, so that elements of Y^{class} are identified with some characters with values in $\bar{\mathbb{Q}}^*$.

2. AUTOMORPHIC L-FUNCTIONS AND THEIR p-ADIC ANALOGUES

The main topics in this paper are automorphic L-functions and their p-adic analogues.

2.1. Krasner grade components for proving Kummer-type congruences for L and zeta-values

2.1.1. *Graded groups in the sense of Krasner.*

Definition 2.1. *Let G be a multiplicative group with the neutral element e. A graduation of G is a mapping $\gamma : \Delta \to Sg(G), \delta \in \Delta$ of a set Δ ("the set of grades of γ") to the set $Sg(G)$ of subgroups of G such that G is the direct sum*

$$G = \bigoplus_{\delta \in \Delta} G_\delta\,,\ g = (g_\delta)_{\delta \in \Delta}.$$

The group G endowed with such a graduation is called a graded group, and g_δ are Krasner's grade components of g with grade δ.

2.1.2. *Graded rings and modules.*

Moreover, M. Krasner introduced useful notions of graded rings and modules. Let $(A;x+y,xy)$ be a ring (not necessarily associative) and let $\gamma:\Delta\to Sg(A;x+y)$ be a graduation of its addivive group. The graduation γ will be a graduation of a ring $(A;x+y,xy)=A$ and this ring will be called graded if in addition for all $\xi,\eta\in\Delta$ there exists $\zeta\in\Delta$ such that $A_\xi A_\eta\subset A_\zeta$.

It is well known that the category of graded groups, rings, modules as well as their homogeneous parts: *homogroupoids, anneids, moduloids,* respectively is not closed with respect to the direct product and the direct sum where the homogeneous part is the direct product of the homogencous parts of the factors.

2.1.3. *Krasner–Vuković's paragraded Groups and Rings.*

Since the category of graded structures (groups, rings. modules), has no property of closure with respect to the direct sum and direct product, it was a motivation for M. Krasner and M. Vuković to focus on this interesting problem in the theory of graded structures and they were the first who overcome it. In this wav they discovered the theory of paragraded structures [KrVu87] (see also [Vu24]), which are at the same time a generalization of the classical graduation as defined by Bourbaki [Bou62] and an extension of the earlier papers done by M. Krasner and its pupils.

Definition 2.2. ([KrVu87]) *The mapping* $\pi:\Delta\to Sg(G),\pi(\delta)=G_\delta(\delta\in\Delta)$, *of a partially ordered set* $(\Delta,<)$ *that is a complete semi-lattice and inductively ordered to the set* $Sg(G)$ *of subgroups of the group G, is called a paragraduation if it satisfies the following six-axiom system:*

(i) $\pi(0)=G_0=\{e\}$, *where* $0=\inf\Delta;\delta<\delta'\Rightarrow G_\delta\subseteq G_{\delta'}$.

Remark 2.1. $H=\bigcup\limits_{\delta\in\Delta}G_\delta$ is called the *homogeneous part* of G with respect to π.

Remark 2.2. If $x\in H$, we say that $\pi(x)=\inf\{\delta\in\Delta\,|\,x\in G_\delta\}$ is the grade of x. We have $\delta(x)=0$ if and only if $x=e$. Elements $\delta(x)$, $x\in H$, are called *principal grades*, and they form a set which we will denote by Δ_P.

(ii) $\theta\subseteq\Delta\Rightarrow\bigcap\limits_{\delta\in\theta}G_\delta=G_{\inf\theta}$.

(iii) *If* $x,y\in H$ *and* $xy=zxy$, *then* $z\in H$ *and* $\delta(z)\le\inf\{\delta(x),\delta(y)\}$.

(iv) *The homogeneous part H is a generating set of G.*

(v) *Let* $A\subseteq H$ *be a subset such that for all* $x,y\in A$ *there exists an upper bound for* $\delta(x)$ *and* $\delta(y)$. *Then there exists an upper bound for all* $\delta(x),x\in A$.

(vi) *G is generated by H with the set of H-inner and left commutation relations:*
- $xy=z$ *(H-inner relations);*
- $yx=z(x,y)xy$ *(left commutation relations).*

A group with paragraduation is called a *paragraded group*.

A ring A is called paragraded if its aditive group is paragraded and if for all $\xi,\eta\in\Delta$ there exists $\zeta\in\Delta$ such that $A_\zeta A_\eta\subseteq A_\zeta$.

2.2. **Examples for p-adic groups $\mathcal{X}$, and group rings**

We use the Tate field $\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$, the completion of $\bar{\mathbb{Q}}_p$, which is a fundamental object in p-adic analysis, and thanks to Krasner we know that $\mathbb{C}_p$ is algebraically closed, (see [Am75] - Théorème 2.7.1 and *Lemme de Krasner* in [Kr74]). This famous result allows to develop analytic functions and analytic spaces over $\mathbb{C}_p$ ([Kr74], Tate, Berkovich...), and we embed $incl_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

1) Algebraically, a p-adic measure μ on $\mathcal{X}$ is an element of the completed group ring $A[[\mathcal{X}]]$, A any p-adic subring of $\mathbb{C}_p$.
2) The p-adic L-function of μ is given by the evaluation $\mathcal{L}_\mu(y) := y(\mu)$ on the group $\mathcal{Y} = \mathrm{Hom}_{cont}(\mathcal{X}, \mathbb{C}_p^*)$ of $\mathbb{C}_p^*$-valued characters of $\mathcal{X}$. *The values $\mathcal{L}_\mu(y_j)$ on algebraic characters $y_j \in \mathcal{Y}^{alg}$ determine $\mathcal{L}_\mu$ iff they satisfy Kummer-type congruences.*
3) Our setting: a p-adic torus $T = \mathcal{X}$ of a unitary group G attached to a CM field K over $\mathbb{Q}$, a quadratic extension of a totally real field F, and an n-dimensional hermitian K-vector space V. Elements of $\mathcal{Y}^{alg}$ are identified with some algebraic characters of the torus T of the unitary group.

2.3. **An extension problem.**

From a subset $J = \mathcal{Y}^{alg}$ of classical weights in $\mathcal{Y} = \mathrm{Hom}_{cont}(\mathcal{X}, \mathbb{C}_p^*)$, via the A-module $\mathcal{AM}$ of arithmetic modular forms, we wish to extend continuously a given mapping $\mathcal{L}$ to the group ring $A[\mathcal{Y}]$:

$$\mathcal{L} : \mathcal{Y}^{alg} \longrightarrow \underset{\circlearrowleft\, \mathcal{H}_A}{\overset{\curvearrowright\, \mathcal{D}_A}{\mathcal{AM}}} \longrightarrow \mathbb{C}_p,\ y_j \overset{\ell}{\mapsto} \mathcal{L}(y_j), y_j \in \mathcal{Y}^{alg}$$

(where $\mathcal{H}_A$ a Hecke algebra, $\mathcal{D}_A$ a ring of differential operators over A) in such a way that the values $\mathcal{L}(y_j)$ on $y_j \in \mathcal{Y}^{alg}$ are given by certain algebraic L-values under the embedding $incl_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$.
In the favorable (ordinary) case one extends $\mathcal{L}$ to all continuous functions $\mathcal{C}(\mathcal{X}, \mathbb{C}_p)$, or just to locally-analytic functions $\mathcal{C}^{loc-an}(\mathcal{X}, \mathbb{C}_p)$ (in the admissible case).
Advantages of the A-module $\mathcal{AM}$:

1) simpler Fourier expansions (q-expansions);
2) action of D and of the ring of differential operators $\mathcal{D}_A = A[D]$
3) action of the Hecke algebra $\mathcal{H}_A$;
4) projection $\pi_\alpha : \mathcal{AM}^\alpha \to \mathcal{AM}^\alpha$ to finite rank component ("generalized eigenvectors of Atkin's U-operator") for any non-zero Hecke eigenvalue α of level p; ℓ goes through $\mathcal{AM}^\alpha$ if $U^*(\ell) = \alpha^*\ell$.

Solution (extension of $\mathcal{L}$ to $\mathcal{C}(\mathcal{X}, \mathbb{C}_p)$) is given in the ordinary case by the abstract Kummer-type congruences:

$$\forall x \in \mathcal{X},\ \sum_j \beta_j y_j(x) \equiv 0 \pmod{p^N} \Longrightarrow \sum_j \beta_j \ell(y_j) \equiv 0 \pmod{p^N} \quad (\beta_j \in A).$$

Such congruences imply the *p-adic analytic continuation of the Riemann zeta function*.

For more general L-function $L(f,s)$, of an automorphic form f one can prove certain Kummer-type congruences using various Krasner grade components, with respect to *weights, Hecke-Dirichlet characters, eigenvalues of Hecke operators acting on spaces automorphic forms (including Atkin-type U_p-operators), and the classical Fourier coefficients of quasi-modular forms*.

It turns out that certain critical L-values $L(f,s)$, expressed through Petersson-type product $\langle f, g_s\rangle$, reduces to $\langle \pi_\alpha(f), g_s\rangle$, where g_s is an explicit arithmetical automorphic form, $\alpha \neq 0$ is a eigenvalue attached to f, and $\pi_\alpha(f)$ is the component given by the α-characteristic projection, known to be in a fixed finite dimensional space (known for Siegel modular case, and extends to the unitary case).

For an algebraic group G over a number field K these L functions are defined as certain Euler products. More precisely, we apply our constructions for the L-functions studied in Shimura's book [Shi00].

2.4. **Constructions of p-adic analogues**

The Krasner-Vuković paragraded structures are mentioned, and they play a role here similar to p-adic weights in the Kummer-Mazur and Amice-Vélu congruences. In the general case of an irreducible automorphic representation of the adelic group $G(\mathbb{A}_K)$ there is an L-function

$$L(s,\pi,r,\chi) = \prod_{\mathfrak{p}_v \, primes \, in \, K} \prod_{j=1}^{m} (1-\beta_{j,\mathfrak{p}_v} N\mathfrak{p}_v^{-s})^{-1}$$

where

$$\prod_{j=1}^{m}(1-\beta_{j,\mathfrak{p}}X) = \det(1_m - r(\mathrm{diag}(\alpha_{i,\mathfrak{p}})_i X)),$$

$\alpha_{i,\mathfrak{p}}$ are the Satake parameters of $\pi = \bigotimes_v \pi_v$ $v \in \Sigma_K$ (places in K), $\mathfrak{p} = \mathfrak{p}_v$. Here $h_v = \mathrm{diag}(\alpha_{i,\mathfrak{p}})_i$ live in the Langlands group ${}^L G(\mathbb{C})$, $r : {}^L G(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ denotes a finite dimensional representation, and $\chi : \mathbb{A}_K^*/K^* \to \mathbb{C}^*$ is a character of finite order. Constructions use the definition of Euler factors is only for unramified primes.

Although χ only implicitly appears on the right hand side, they admit extention to rather general automorphic representations on Shimura varieties via the following tools:

- ◦ Modular symbols and their higher analogues (linear forms on cohomology spaces related to automorphic forms)
- ◦ Petersson products with a fixed automorphic form, or
- ◦ linear forms coming from the Fourier coefficients (or Whittaker functions), or throught the
- ◦ *CM*-values (special points on Shimura varieties),

3. AUTOMORPHIC L-FUNCTIONS ATTACHED TO SYMPLECTIC AND UNITARY GROUPS

Let us briefly describe the L-functions attached to symplectic and unitary groups as certain Euler products in Chapter 5 of [Shi00], with critical values computed in Chapter 7, Theorem 28.8 using general nearly holomorphic arithmetical automorphic forms for the group

$$G = G(\varphi) = \{\alpha \in \mathrm{GL}_m(K) \mid \alpha\varphi^t\alpha^\rho = \nu(\alpha)\varphi\}, \nu(\alpha) \in F^*,$$

where $\varphi = \eta_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ or $\varphi = \begin{pmatrix} 1_n & 0 \\ 0 & 1_m \end{pmatrix}$, see also Ch.Skinner and E.Urban [MC] and Shimura G., [Shi00].

3.1. **The groups and automorphic forms studied by Shimura in [Shi00]**

Let F be a totally real algebraic number field, K be a totally imaginary quadratic extension of F and ρ be the generator of $\mathrm{Gal}(K/F)$. Take $\eta_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ and define

$$\begin{aligned}
&G = \mathrm{Sp}(n, F) && \text{(Case Sp)}\\
&G = \{\alpha \in \mathrm{GL}_{2n}(K) | \alpha\eta_n\alpha^* = \eta_n\} && \text{(Case UT = unitary tube)}\\
&G = \{\alpha \in \mathrm{GL}_{2n}(K) | \alpha T\alpha^* = T\} && \text{(Case UB = unitary ball)}
\end{aligned}$$

according to three cases. Assume $F = \mathbb{Q}$ for a while. The group of the real points G_∞ acts on the associated domain

$$\mathcal{H} = \begin{cases} \{z \in M(n,n,\mathbb{C}) \mid {}^tz = z, \Im(z) > 0\} & \text{(Case Sp)}\\ \{z \in M(n,n,\mathbb{C}) \mid i(z^* - z) > 0\} & \text{(Case UT)}\\ \{z \in M(p,q,\mathbb{C}) \mid 1_q - z^*z > 0\} & \text{(Case UB),} \end{cases}$$

with $(p,q), p+q = n$, being the signature of iT . Here $z^* = {}^t\bar{z}$ and ¿ means that a hermitian matrix is positive definite. In Case UB, there is the standard automorphic factor $M(g,z)$, $g \in G_\infty$, $z \in \mathcal{H}$ taking values in $\mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C})$.

3.2. **Shimura's arithmeticity in the theory of automorphic forms** [Shi00]**, p-adic zeta functions and nearly-holomorphic forms on classical groups**

3.2.1. *Automorphic L-functions via general quasi-modular forms.*

Automorphic L-functions and their p-adic versions can be obtained for quite general automorphic representations on Shimura varieties by constructing p-adic distributions out of algebraic numbers attached to automorphic forms. These numbers satisfy certain Kummer-type congruences established in different ways component-wise for various Krasner-type components.

In order to describe both algebraicity and congruences of the critical values of the zeta functions of automorphic forms on unitary and symplectic groups, we follow the review by H.Yoshida [YS] of Shimura's book "Arithmeticity in the theory of automorphic forms" [Shi00].

Also unitary Shimura varieties have recently attracted much interest (in particular by Ch. Skinner and E. Urban), see [MC], in relation with the proof of the Iwasawa Main Conjecture for GL(2).

3.3. **Work of Yubo Jin** [YJ22], [YJ23a]

Very recently Yubo Jin (Durham UK) found in [YJ22] a nice version of the doubling method of computing special L values [YJ22] for quaternionic modular forms leading to their p-adic interpolation in the ordinary case [YJ23b].

Also, he tackled [YJ23a] certain more general Classical Groups (algebracity and the p-adic interpolation) of special L-values) for certain Classical Groups (in the ordinary case) Using the techniques of Anh Tuan DO and Krasner-Vuković paragraded structures we wish to extend this result to the general admissible case (for the same Classical Groups).

4. CONSTRUCTING p-ADIC ZETA-FUNCTIOS VIA ARITHMETICAL MODULAR FORMS

Here we present a new method of constructing p-adic zeta-functios based on the use of general arithmetical modular forms on classical groups.

This method uses only algebraic numbers coming from holomorphic and nearly holomorphic modular forms, and quasi-modular forms.

A new method of constructing p-adic zeta-functios uses general quasi-modular forms and their Fourier coefficients. The symmetric space

$$\mathcal{H} = G(\mathbb{R})/((\text{maximal-compact subgroup})\mathcal{K} \times Center)$$

parametrizes certain families of abelian varieties A_z $(z \in \mathcal{H})$ so that $F \subset \mathrm{End}(A_z) \otimes \mathbb{Q}$. The CM-points z correspond to a maximal multiplication ring $\mathrm{End}(A_z)$.

For the group GL(2), N.Katz [Ka76] used arithmetical elements (real-analytic and p-adic) instead of holomorphic forms. These elements correspond also to quasi-modular forms coming from derivatives which can be defined in general using Shimura's arithmeticity and the Maass-Shimura operators. A relation real-analytic $\leftrightarrow$ p-adic modular forms comes from the notion of p-adic modular forms invented by J. P. Serre [Se73] as p-adic limits of q-expansions of modular forms with rational coefficients for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. The present method of constructing p-adic automorphic L-functions uses general quasi-modular forms, and their link to algebraic p-adic modular forms.

4.1. **Using Krasner-Vuković paragraded components for proving Amice-Vélu Kummer-type congruences for the corresponding L and zeta-values.**

Noémie Combe's project for the CIRM Conference 2024, related to her earlier joint work with Yu. I. Mannin on Frobenius varieties, used Eisenstein's series and perhaps Eisenstein's measure. A further project after the CIRM Conference 2024 extending Yuri Manin's work on δ-rings and their p-adic applications comes from the paragraded rings by Krasner-Vuković introduced in [KrVu 87] and extended by Mirjana Vuković in *Brown-McCoy and large Brown-McCoy radicals of paragraded rings* [Vu24].

Let p be a prime number. Our purpose is to indicate a link of Krasner-Vuković paragraded structures [KrVu87], [Kr80], [Vu01] and constructions of p-adic L-functions via distributions and arithmetical modular forms on classical groups.

Krasner's graded structures are flexible and well adapted to various applications, e.g. the rings and modules of differential operators on classical groups and non-archimedean weight spaces.

4.1.1. *p-adic measures via congruences.*

- Proving Kummer type congruences in the form

  Definition 4.1. *Let M be a O-module of finite rank where $O \subset \mathbb{C}_p$. For $h \geq 1$, consider the following $\mathbb{C}_p$-vector spaces of functions on $\mathbb{Z}_p^*$: $\mathcal{C} \subset \mathcal{C}^{loc-an} \subset \mathcal{C}$. Then a continuous homomorphism $\mu : \mathcal{C} \to M$ is called a (bounded) M-valued measure on $\mathbb{Z}_p^*$.*

  Let us define a measure with given integrals.

  Take a dense family of continuous functions $\{\varphi_i = \varphi_{s_i,\chi_i}\}$ in $\mathcal{C}(X_\pi, \mathbb{C}_p)$ on the p-adic space X_π. Then Kummer says:

  $$\sum_i \beta_i \varphi_i \equiv 0 \pmod{p^N} \Longrightarrow \sum_i \beta_i L^*_{geom}(\pi, s_i, \chi_i) \equiv 0 \pmod{p^N}.$$

  Each $\varphi \in \mathcal{C}(X_\pi, \mathbb{C}_p)$ can be approximated by $\{\varphi_i\}_i$, and a measure $\mu_\pi(\varphi)$ with given iz $\mu_\pi(\varphi) = L^*_{geom}(\pi, s_i, \chi_i)$ 1s a well-defined limit over all approximations of φ.

- From bounded measures on X to admissible measures using

  $$h_{\pi.p} = P_{Newton,p} d/2) - P_{Hodge}(d/2) \geq 0.$$

  Computing critical values at $s = s_*, \cdots, 8^*$ and prove admissibility congruences for them as follows.

  A $\mathbb{C}_{p^-}$ linear mapping $\mu : \mathcal{C}^h \to M$ is called an h admissible M-valued measure on $\mathbb{Z}_p^*$ if the following growth condition is satisfied

  $$\left| \int_{a+(p^v)} (x-a)^j d\mu \right|_p \leq p^{-v(h-j)}, \text{ for } j = 0, 1, \ldots, h-1.$$

  Such μ extends to $\mathcal{C}^{loc-an}$ (and to $\mathcal{Y}_p = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$, the space of definition of p-adic Mellin transform).

5. Applications to Yubo Jin L values

Although we treat here only the Siegel modular case here, the results can be extended to the general Sp- and unitary cases (UT in Shimura's terminology).

5.1. **Towards general constructions of p-adic L-functions for unitary groups.**

In the most recent version of the paper [EHLS] by Ellen Eischen, Michael Harris, Jianshu Li, Christopher Skinner, a construction of p-adic L-functions for unitary group in the ordinary case was completed using p-adic modular forms. Also, the case of Hida's families of such forms is treated using a construction of Eisenstein measures.

Acknowledgement

Many thanks to Mirna Džamonja for the invitation to give a talk on the *International Online Conference* organized by Sarajevo Journal of Mathematics in honor of their editor in chief Academician Mirjana Vuković and her jubilee.

My special thanks go to Khoai Ha Huy, Siegfried Boecherer and Vladimir Berkovich for fruitful discussions during a visit of the author to Vietnam in April 2015.

References

[Am75] Amice, Y., *Les nombres p-adiques*, 1975, PUF, Collection SUP, p.189.

[Am-V] Amice, Y. and Vélu, J., *Distributions p-adiques associées aux séries de Hecke*, Journées Arithmétiques de Bordeaux (Conf. Univ. Bordeaux, 1974), Astérisque no. 24/25, Soc. Math. France, Paris 1975, pp. 119-131

[Boe85] Böcherer, S., *Über die Funktionalgleichung automorpher L–Funktionen zur Siegelscher Modulgruppe.* J. reine angew. Math. 362 (1985) 146-168

[BoeNa13] Boecherer, Siegfried, Nagaoka, Shoyu , *On p-adic properties of Siegel modular forms*, in: Automorphic Forms. Research in Number Theory from Oman. Springer Proceedings in Mathematics and Statistics 115. Springer 2014, see also arXiv:1305.0604 [math.NT]

[Boe-Pa9] Böcherer, S., Panchishkin, A.A., *p-adic Interpolation of Triple L-functions: Analytic Aspects*. In: Automorphic Forms and L-functions II: Local Aspects – David Ginzburg, Erez Lapid, and David Soudry, Editors, AMS, BIU, 2009, 313 pp.; pp.1-41

[Boe-Pa11] Böcherer, S., Panchishkin, A.A., *Higher Twists and Higher Gauss Sums.* Vietnam Journal of Mathematics 39:3 (2011) 309-326

[Boe23] Böcherer, S., *Manuscript*, Grenoble, (December 2023).

[Bou23] Athanasis Bouganis, *Manuscript*, Grenoble, (December 2023).

[Bou62] N. Bourbaki, *Algèbre*, Chap. I, 2èm et 3èm édit., Herman, Paris, 1962.

[BS00] Böcherer, S., and Schmidt, C.-G., *p-adic measures attached to Siegel modular forms*, Ann. Inst. Fourier 50, N°5, 1375-1443 (2000).

[CourPa] Courtieu,M., Panchishkin , A.A., *Non-Archimedean L-Functions and Arithmetical Siegel Modular Forms*, Lecture Notes in Mathematics 1471, Springer-Verlag, 2004 (2nd augmented ed.)

[DoTa17] Do, Anh Tuan, *p-Adic Admissible Measures Attached to Stegel Modular Forms of Arbitrary Genus*, Vietnam Journal of Mathematics December 2017, Volume 45, Issue 4, pp 695-711.

[EE12] Eischen, Ellen E., *p-adic Differential Operators on Automorphic Forms on Unitary Groups.* Annales de l'Institut Fourier 62, No.1 (2012) 177-243.

[EHLS] Eischen Ellen E., Harris, Michael, Li, Jian-Shu, Skinner, Christopher M., *p-adic L-functions for unitary groups*, `arXiv:1602.01776v3 [math.NT]`

[GeSha88] Gelbart, S., and Shahidi, F., *Analytic Properties of Automorphic L-functions*, Academic Press, New York, 1988.

[GRPS87] Gelbart S.,Piatetski-Shapiro I.I., Rallis S. *Explicit constructions of automorphic L-functions.* Springer-Verlag, Lect. Notes in Math. N 1254 (1987) 152p.

[Ka76] Katz, N.M., *p-adic interpolation of real analytic Eisenstein series.* Ann. of Math. 104 (1976) 459–571

[KiNa16] Kikuta, Toshiyuki, Nagaoka, Shoyu, *Note on mod p property of Hermitian modular forms* arXiv:1601.03506 [math.NT]

[Ko80] Koblitz, Neal, *p-adic Analysis. A Short Course on Recent Work*, Cambridge Univ. Press, 1980

[Kr66] Krasner, M., Prolongemcnt analytique uniforme et multiforme, Collogue C.N.R.,S. n° 143, Clermont-Ferrand, 1963, Paris, Ed. C.N.R.S., 1966, p. 97-141

[Kr74] Krasner, M., *Rapport sur le prologement analytique dans les corps values complètes par la méthode des éléments analytiques quasiconnexes*, Bull. Soc. Math. France, Mem. 39-40 (1974) 131-254.

[Kr80] Krasner, M., *Anneaux gradués généraux*, Colloque d'Algèbre de Rennes (1980), 209-308.

[KrKa51] Krasner, M., Kaloujnine, L. *Produit complet des groupes de permutations et problème d'extension de groupes II* , Acta Sci. Math. Szeged , 14 (1951) p. 39-66 et 69-82.

[KrVu87] Krasner, M., Vuković, M. *Structures paragraduées (groupes, anneaux, modules)*, Queen's Papers in Pure and Applied Mathematics, 77, Queen's University, Kingston, Ontario, Canada, 1987.

[Lang95] Lang, Serge. *Introduction to modular forms. With appendixes by D. Zagier and Walter Feit.* Springer-Verlag, Berlin, 1995

[MTT86] Mazur B., Tate J., Teitelbaum J., *On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer*, Invent. Math. 84, no. 1, 1-48, 1986.

[MPIM19] Manin, Yu. I., Panchishkin, A.A., Arnab Saha, `https://www-fourier.ujf-grenoble.fr/panchish/prism/`

[MaPa05] Manin, Tu. I., Panchishkin, A.A., *Introduction to Modern Number Theory: Pundamental Problems*, Ideas and Theories (Encyclopaedia of Mathematical Sciences), Second Edition, 504 p., Springer (2005).

[Pa94] Panchishkin, A.A., *Motives over totally real fields and p-adic L-functions*, Ann. Inst. Fourier (Grenoble) 44 (1994), no. 4, 989-1023.

[Pa02] Panchishkin, A.A., *A new method of constructing p-adic L-functions associated with modular forms*, Moscow Mathematical Journal, 2 (2002), Number 2, 1-16.

[Pa03] Panchishkin, A.A., *Two variable p-adic L functions offached fo eigenfamilies of positive slope*, Invent. Math. v. 154, N3 (2003), pp. 551 - 615.

[Pa20] Panchishkin, A.A., *A motivic approach to Shimura's zeta functioms ond oftached p-adic L-functions via admissible measures*, RIMS conference "Automorphic forms, automorphic representations and related topics" (January 21-25, 2019), Kôkyûroku No. 2136 RIMS (Kyoto) 2020.

[Pa21a] Panchishkin, A.A., *Algebraic differentiol operaiors on unitary groups and their applications*. https://archive.mpimbonn.mpg.de/id/eprint/4579/1/mpim-preprint_2021-22.pdf

[Pa21b] Panchishkin, A.A., *New approaches to constructing p-adic L-functions on classical groups, algebraic differential operators, and BGG*. https://archive. mpimbonn.mpg.de/id/eprint/4580/1/mpim-preprint_2021-23.pdf

[Pa22] Panchishkin, A.A., *Algebraic differential operators on arithmetical automorphic forms, modular distributions, p-adic interpolation of their critical values via BGG modulas*, J. Math. Math. Sci. (2022, 1), 1-26.

[Se73] Serre, J.–P., *Formes modulaires et fonctions zêta p-adiques*, Lect Notes in Math. 350 (1973) 191–268 (Springer Verlag).

[Shi00] Shimura G., *Arithmeticity in the theory of automorphic forms*, Mathematical Surveys and Monographs, vol. 82 (Amer. Math. Soc., Providence, 2000).

[MC] Skinner, C. and Urban, E. *The Iwasawa Main Cconjecture for GL(2)*. http://www.math.jussieu.fr/~urban/eurp/MC.pdf

[Vu01] Vuković, M. *Structures gradués et paragradués*, Prépublication de l'Institut Fourier no 536 (2001). https://www-fourier.ujf-grenoble.fr/sites/default/files/ref_536.pdf

[Vu03] Vuković, M. *Theory of Groups and its Representations with Applications in Physics*, Sarajevo Publishing and Prirodno-matematički fakultet, Sarajevo (2003), pp. 384.

[Vu18] M.Vuković, *From Krasner's corpoid and Bourbaki's graduations to Krasner's graduations and Krasner-Vuković's paragraduations*, Sarajevo J. Math. Vol. 14 (27), No. 2 (2018), 175-190.

[Vu23] M.Vuković, *Radicals of Paragraded Rings (dedicated to A.V. Mikhalev)*, Springer's J. Math. Sci., Vol. 275, No. 4, 379-392 (2023).

[Vu24] M.Vuković, Brown-McCoy and large Brown-McCoy radicals of paragraded rings (to appear soon)

[Vu23] M.Vuković, Panoramic view of graded structures from Euler and Bourbaki-Krasner to Krasner - Vuković (in Fundam. Prikl. Mat. (in russian) 2023, T.24, No. 3, p. 23-37 it will appear in Springer's J. Math. Sci. in english).

[YS] Yoshida, H., *Review on Goro Shimura, Arithmeticity in the theory of automorphic forms [Shi00]*, Bulletin (New Series) of the AMS, vol. 39, N3 (2002), 441-448.

[YJ22] Yubo Jin, *On p-adic measures for quaternionic modular forms-1*. http://arxiv.org/abs/2209.11822v1, (23 Sep 2022)

[YJ23a] Yubo Jin, *Algebraicity and the p-adic interpolation of special L-values for certain classical groups*. http://arxiv.org/abs/2305.19113v1 (30 May 2023)

[YJ23b] Yubo Jin, *On p-adic measures for quaternionic modular forms-2*. http://arxiv.org/abs/2209.11822v2 (31 May 2023)

[YJ23] Yubo Jin, *Manuscript, Grenoble*, (December 2023).

[Z13] Zemel,S., *On quasimodular forms, almost holomorphic modular forms, and the vector-valued modular forms of Shimura.* arXiv:1307.1997 (2013)

(Received: January 30, 2024)
(Revised: May 27, 2024)

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