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ISOTYPIC EQUIVALENCE OF ABELIAN p-GROUPS WITH SEPARABLE REDUCED PARTS

ELENA BUNINA

This paper is devoted to the great anniversary of my friend Mirjana Vuković.
I wish her many-many productive and happy years!

ABSTRACT. We prove that two Abelian p-groups with separable reduced parts are isotypically equivalent if and only if their divisible parts and their basic subgroups are elementarily equivalent. Also as a corollary we prove that any Abelian p-group with a separable reduced part is ω-strongly homogeneous.

1. INTRODUCTION

In this paper we study Abelian p-groups A with separable reduced parts. Our goal is to figure out how close are such groups in case they have the same sets of types realized in A. We show that these groups have the same types (are *isotypically equivalent*) if and only if their divisible parts and their basic subgroups are elementarily equivalent.

In this paper (Abelian) groups are our main subject, so we do not consider rings, semigroups, etc., even though most of definitions below make sense for arbitrary algebraic structures.

Definition 1.1. *Let G be a group and $(g_1,\dots,g_n)$ n-tuple of its elements. The* **type** *of this tuple in G, denoted* $\mathrm{tp}^G(g_1,\dots,g_n)$*, is the set of all first order formulas in free variables $x_1,\dots,x_n$ in the standard group theory language which are true on $(g_1,\dots,g_n)$ in G (see [6] or [8] for details).*

Definition 1.2. *The set of all types of tuples of elements of G is denoted by* $\mathrm{tp}(G)$. *Following [12], we say that two groups G and H are* **isotypic** *if* $\mathrm{tp}(G)=\mathrm{tp}(H)$, *i. e., if any type realized in G is realized in H, and vice versa.*

Isotypic groups appear naturally in logical (algebraic) geometry over groups which was developed in the works of B. I. Plotkin and his co-authors (see [10–12] for details), they play an important part in this subject. In particular, it turns out that two groups are logically equivalent if and only if they have the same sets of realizable types. So there arise two fundamental algebraic questions which are

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interesting in its own right: what are possible types of elements in a given group G and how much of the algebraic structure of G is determined by the types of its elements?

Isotypicity property of groups is related to the elementary equivalence property, though it is stronger. Indeed, two isotypic groups are elementarily equivalent, but the converse does not hold. For example, if we denote by F_n a free group of finite rank n, then groups F_n and F_m for $2 \leqslant m < n$ are elementarily equivalent [7, 15], but not isotypic [9]. Furthermore, Theorem 1 from [9] shows that two finitely generated isotypic nilpotent groups are isomorphic, but there are examples, due to Zilber, of two elementarily equivalent non-isomorphic finitely generated nilpotent of class 2 groups [16].

Isotypicity is a very strong relation on groups, which quite often implies their isomorphism. This explains the need of the following definition.

Definition 1.3. *We say that a group G is* defined by its types *if every group isotypic to G is isomorphic to G.*

It was noticed in [9] that every finitely generated group G which is defined by its types satisfies a (formally) stronger property. Namely, we say that

Definition 1.4. *A finitely generated group G is* strongly defined by types *if for any group H isotypic to G every elementary embedding $G \to H$ is an isomorphism.*

Miasnikov and Romanovsky in the paper [9] proved that

1) every virtually polycyclic group is strongly defined by its types;
2) every finitely generated metabelian group is strongly defined by its types;
3) every finitely generated rigid group is strongly defined by its types. In particular, every free solvable group of finite rank is strongly defined by its types.

R. Sclinos (unpublished) proved that finitely generated homogeneous groups are defined by types. Moreover, finitely generated co-hopfian and finitely presented hopfian groups are defined by types. Nevertheless, the main problem in the area remains widely open:

Problem 1.1. [11] *Is it true that every finitely generated group is defined by types?*

In the recent paper [2] Gvozdevsky proved that any field of finite transcendence degree over a prime subfield is defined by types. Also he gave several interesting examples of certain countable isotypic but not isomorphic structures: totally ordered sets, rings, and groups.

This paper is devoted to the following theorem:

Theorem A. For any prime p two Abelian p-groups A_1 and A_2 with separable reduced parts are isotypic if and only if their divisible parts and their basic subgroups are elementarily equivalent.

We hope that it is the first step towards the description of all groups isotypic to a given Abelian group.

2. ELEMENTARY EQUIVALENCE OF ABELIAN GROUPS

Definition 2.1. *Two groups are called* elementarily equivalent *if their first order theories coincide.*

Elementary equivalent Abelian groups were completely described in 1955 by Wanda Szmielew in [14] (see also Eclof and Fisher, [3]).

To formulate her theorem we need to introduce a set of special invariants of Abelian groups.

Let A be an Abelian group, p a prime number, $A[p]$ be the subgroup of A, containing all elements of A of the orders p or 1 (it is the so-called *p-socle* of the group A), kA be the subgroup of A, containing all elements of the form ka, $a \in A$.

The first invariant is

$$D(p;A) := \lim_{n\to\infty} \dim((p^nA)[p]) \text{ for every prime } p.$$

Note that for every $n \in \mathbb{N}$ the subgroup $(p^nA)[p]$ consists of elements which are annulated being multiplied by p, therefore it is a vector space over the field $\mathbb{Z}_p$. So $\dim(p^nA)[p]$ is for every $n \in \mathbb{N}$ a well-defined cardinal number.

Since $p^{n+1}A \subset p^nA$, then $(p^{n+1}A)[p] \subset (p^nA)[p]$, therefore

$$\dim(p^{n+1}A)[p] \leqslant \dim(p^nA)[p] \text{ for all } n \in \mathbb{N}.$$

Consequently we have a non-increasing sequence of cardinal numbers, which has a smallest element.

Thus for any Abelian group A and any prime p the invariant $D(p;A)$ is well-defined.

The second invariant is

$$Tf(p;A) := \lim_{n\to\infty} \dim(p^nA/p^{n+1}A).$$

This invariant is also well-defined for any prime p and any Abelian group A.

The third invariant is

$$U(p,n-1;A) := \dim((p^{n-1}A)[p]/(p^nA)[p]),$$

which is called the *Ulm* invariant, it defines the number of copies of $\mathbb{Z}_{p^n}$ in A.

The last invariant is $\operatorname{Exp}(A)$ which is the *exponent* of A (the smallest natural number n such that $\forall a \in A\, na = 0$).

Theorem 2.1 (Szmielew theorem on elementary classification of Abelian groups)**.** *Two Abelian groups A_1 and A_2 are elementarily equivalent if and only if their elementary invariants $D(p;\cdot)$, $Tf(p;\cdot)$, $U(p,n-1;\cdot)$ and* $\operatorname{Exp}(\cdot)$ *pairwise coincide* (*more precisely, they are either finite and coincide or simultaneously are equal to infinity*).

3. Abelian p-groups, their structure and elementary equivalence

Let p be some prime number, A be an Abelian p-group.

It is said that an element $a \in A$ is *divisible by* a positive integer n (denoted as $n \mid a$) if there is an element $x \in A$ such that $nx = a$. A group D is called *divisible* if $n \mid a$ for all $a \in D$ and all natural n. The groups $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ are examples of divisible groups. Any divisible subgroup is a direct summand of a group. A group A is called *reduced* if it has no nonzero divisible subgroups.

A subgroup G of a group A is called *pure* if the equation $nx = g \in G$ is solvable in G whenever it is solvable in the entire group A. In other words, G is pure if and only if

$$\forall n \in \mathbb{Z} \quad nG = G \cap nA.$$

A subgroup B of a group A is called a *p-basic subgroup* if it satisfies the following conditions:

(1) B is a direct sum of cyclic p-groups and infinite cyclic groups;
(2) B is pure in A;
(3) A/B is p-divisible.

Every group, for every prime p, contains p-basic subgroups [4].

Now we focus on p-groups, where p-basic subgroups are particularly important. If A is a p-group and q is a prime different from p, then evidently A has only one q-basic subgroup, namely 0. Therefore, in p-groups we may refer to the p-basic subgroups simply as *basic* subgroups, without confusion.

We need the following facts about basic subgroups.

Theorem 3.1. [13] *Assume that B is a subgroup of a p-group A, $B = \bigoplus\limits_{n=1}^{\infty} B_n$, and B_n is a direct sum of groups $\mathbb{Z}(p^n)$. Then B is a basic subgroup of A if and only if for every integer $n > 0$, the subgroup $B_1 \oplus \cdots \oplus B_n$ is a maximal p^n-bounded direct summand of A.*

Any Abelian p-group A is a direct sum of its divisible part D (isomorphic to $\bigoplus\limits_{\varkappa_0} \mathbb{Z}(p^\infty)$) and its reduced part $\overline{A}$ with a basic subgroup

$$B = \bigoplus_{n=1}^{\infty} \left(\bigoplus_{\varkappa_n} \mathbb{Z}(p^n) \right).$$

The basic subgroup B is dense in $\overline{A}$ in its p-adic topology.

An infinite system $L = \{a_i\}_{i \in I}$ of elements of the group A is called *independent* if every finite subsystem of L is independent. An independent system M of A is *maximal* if there is no independent system in A containing M properly. By the *rank* $r(A)$ of a group A we mean the cardinality of a maximal independent system containing only elements of infinite and prime power orders. The *final rank* of a basic subgroup B of a p-group A is the infimum of the cardinals $r(p^n B)$.

Definition 3.1. *Given $a \in A$, the greatest nonnegative integer r for which $p^r x = a$, is solvable for some $x \in A$, is called the* p-height $h_p(a)$ *of* a. *If $p^r x = a$ is solvable whatever r is, a is of* infinite p-height, $h_p(a) = \infty$. *If it is completely clear from the context which prime p is meant, we call $h_p(a)$ simply the* height *of a and write $h(a)$.*

Definition 3.2. *A reduced Abelian p-group A is called* separable*, if it does not contain any non-zero elements of infinite height.*

For a reduced p-group A *the first Ulm subgroup* of A is

$$A^{\mathbf{1}} = \bigcap_{n=1}^{\infty} p^n A,$$

it is the subgroup of A consisting of all elements of A of infinite height. Therefore a reduced A is separable if and only if $A^{\mathbf{1}} = 0$.

Proposition 3.1. [4] *An element a of prime power order belongs to a finite direct summand of A if and only if $\langle a \rangle$ contains no elements of infinite height.*

Theorem 3.2. [1] *Suppose that B is a subgroup of p-group A,*

$$B = B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus \dots,$$

where

$$B_n \cong \bigoplus_{\mu_n} \mathbb{Z}(p^n).$$

The subgroup B is a basic subgroup of A if and only if

$$A = B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus (B_n^* + p^n A),$$

where $n \in \mathbb{N}$,

$$B_n^* = B_{n+1} \oplus B_{n+2} \oplus \dots .$$

Since the group B has a basis, and the quotient group A/B is a direct sum of groups isomorphic to $\mathbb{Z}(p^\infty)$ (i. e. A/B also has a generating system which can be easily described) then it is natural to combine these generating systems and to obtain one for A.

We write

$$B = \bigoplus_{i \in I} \langle a_i \rangle \text{ and } A/B = \bigoplus_{j \in J} C_j^*, \text{ where } C_j^* = \mathbb{Z}(p^\infty).$$

If the direct summand C_j^* is generated by cosets $c_{j1}^*, \dots, c_{jn}^*, \dots$ modulo B with $pc_{j1}^* = 0$, $pc_{j,n+1}^* = c_{jn}^*$ ($n = 1, 2, \dots$), then, by the purity of B in A, in the group A we can pick out $c_{jn} \in c_{jn}^*$ of the same order as c_{jn}^*. Then we get the following set of relations:

$$pc_{j1} = 0,\ pc_{j,n+1} = c_{jn} = b_{jn} \quad (n \geq 1, b_{jn} \in B),$$

where b_{jn} must be of order $\leq p^n$, since $o(c_{jn}) = p^n$.

The system $\{a_i, c_{jn}\}_{i \in I, j \in J, n \in \omega}$ will be called a *quasibasis* of A.

Proposition 3.2. [5] *If $\{a_i, c_{jn}\}$ is a quasibasis of the p-group A, then every $a \in A$ can be written in the following form:*

$$a = s_1 a_{i_1} + \dots + s_m a_{i_m} + t_1 a_{j_1 n_1} + \dots + t_r a_{j_r n_r}, \tag{3.1}$$

where s_i and t_j are integers, no t_j is divisible by p, and the indices $i_1, \dots, i_m$ as well as $j_1, \dots, j_r$ are distinct. The condition (3.1) *is unique in the sense that an element a uniquely defines the terms sa_i and tc_{jn}.*

Definition 3.3. *The* final rank $\operatorname{fin} r(A)$ *of a p-group A is the minumum of all cardinal numbers* $\operatorname{rank}(p^n A)$*,* $n = 1, 2, \dots$ (see Sele, [13]).

Example 3.1. *If*

$$A = B = B_1 \oplus B_2 \oplus \dots \oplus B_n \oplus \dots, \qquad B_n = \bigoplus_{\mu_n} \mathbb{Z}(p^n),$$

then

$$\operatorname{rank}(p^n A) = \operatorname{rank}(B_{n+1} \oplus \dots) = \sum_{i=n+1}^{\infty} \mu_i$$

and therefore $\operatorname{fin} r(A) = 0$ *if and only if* $\operatorname{Exp} A = 0$*,* $\operatorname{fin} r(A) = \infty$ *if and only if* $\operatorname{Exp} A = \infty$ *(see [4], § 35).*

Let us now concentrate on elementary equivalence of two Abelian p-groups with separable reduced parts.

Suppose that $A = D \oplus \overline{A}$, where $\overline{A}$ is reduced and separable (i. e., does not contain any elements of infinite height) and has B as its basis subgroup.

Assume also that

$$D \cong \bigoplus_{\varkappa_0} \mathbb{Z}(p^\infty), \quad B_k \cong \bigoplus_{\varkappa_k} \mathbb{Z}(p^k), \quad k = 1, \dots, n, \dots.$$

1. For the first invariant $D(p;A)$ let us fix some $n \in \mathbb{N}$ and represent A as

$$A = D \oplus B_1 \oplus B_2 \oplus \dots \oplus B_n \oplus (B_n^* + p^n \overline{A})$$

as in Theorem 3.2.

Then

$$p^n A = (p^n D) \oplus p^n (B_n^* + p^n \overline{A}) = D \oplus p^n (B_n^* + p^n \overline{A}),$$

therefore

$$\dim p^n A[p] = \varkappa_0 + \dim p^n (B_n^* + p^n \overline{A})[p]$$

and

$$\lim_{n\to\infty} \dim p^n A[p] = \varkappa_0 + \operatorname{fin} r(B),$$

since $\bigcap\limits_{n=1}^{\infty} p^n \overline{A} = 0$.

2. The second invariant is

$$\begin{aligned}\lim_{n\to\infty} \dim(p^n A / p^{n+1} A) &= \lim_{n\to\infty} \dim(p^n D / p^{n+1} D \oplus p^n \overline{A} / p^{n+1} \overline{A}) \\ &= \lim_{n\to\infty} \dim(p^n \overline{A} / p^{n+1} \overline{A}) \\ &= \lim_{n\to\infty} \dim(p^n \overline{A} / p^{n+1} \overline{A})\end{aligned}$$

$$= \lim_{n\to\infty} \dim \big(p^n B_1/p^{n+1}B_1 + \cdots + p^n B_{n+1}/p^{n+1}B_{n+1} + p^n(B^*_{n+1} + p^{n+1}\overline{A})/p^{n+1}(B^*_{n+1} + p^{n+1}\overline{A})\big)$$
$$= \lim_{n\to\infty}(\varkappa_{n+1} + \varkappa_{n+2} + \cdots) = \operatorname{fin} r(B).$$

3. The third invariant is

$$U(p, n-1; A) := \dim((p^{n-1}A)[p]/(p^n A)[p]) = \varkappa_n.$$

Therefore two Abelian p-groups are elementarily equivalent if and only if their basic subgroups are elementarily equivalent and either these subgroups are both unbounded or they are bounded and in this case the divisible parts are elementarily equivalent. In its turn elementary equivalence of divisible parts means that either both of them contain a finite number of $\mathbb{Z}(p^\infty)$ in the direct sum and are isomorphic, or both of them contain an infinite number of $\mathbb{Z}(p^\infty)$ in the direct sum. The same is true for any direct summand B_i of the basic subgroups.

4. Types of elements

Suppose that the decomposition $A = D \oplus \overline{A}$ is fixed. Suppose also that we have m-tuple $(g_1, \ldots, g_m)$ of elements of A and its type $\mathrm{tp}(g_1, \ldots, g_m)$. If $g_i = d_i + a_i$, $i = 1, \ldots, m$, is a decomposition of these elements with respect to the direct summands D and $\overline{A}$, then $\mathrm{tp}(d_1, \ldots, d_m, a_1, \ldots a_m)$ contains $\mathrm{tp}(g_1, \ldots, g_m) = \mathrm{tp}(d_1 + a_1, \ldots, d_m + a_m)$. Therefore we always can assume that we study only types $\mathrm{tp}(d_1, \ldots, d_\ell, a_1, \ldots, a_m)$, $d_1, \ldots, d_\ell \in D$, $a_1, \ldots, a_m \in \overline{A}$.

Let us for some fixed decomposition $A = D \oplus \overline{A}$ have $d_1, \ldots, d_\ell \in D$ and $a_1, \ldots, a_m \in \overline{A}$. The elements $d_1, \ldots, d_\ell$ generate a direct summand

$$\langle d_1, \ldots, d_\ell \rangle = D_1 \cong \bigoplus_{n_0} \mathbb{Z}(p^\infty) \text{ in } D.$$

Knowing $\mathrm{tp}(d_1, \ldots, d_\ell)$ we can easily define n_0: it is $k-1$, where k is the minimal natural number such that there exist a subset $\{m_1, \ldots, m_k\} \subset \{1, \ldots, \ell\}$, $\alpha_1, \ldots, \alpha_k \in \mathbb{Z}$, $0 \leqslant \alpha_i < \mathrm{ord}(d_{m_i})$, $\alpha_1, \ldots, \alpha_k$ are not all zeros, such that

$$\alpha_1 d_{m_1} + \cdots + \alpha_k d_{m_k} = 0.$$

Now let us study the elements $a_1, \ldots, a_m$. All these elements have finite heights. Suppose that

$$\mathrm{ord}(a_i) = p^{t_i}, \quad h(a_i) = p^{s_i}, \quad i = 1, \ldots, \ell.$$

Let us take instead of every a_i an element b_i such that $p^{s_i} b_i = a_i$. Then every b_i has the height 0 and therefore generates a direct summand

$$a_i \in \langle b_i \rangle \cong \mathbb{Z}(p^{t_i + s_i}).$$

If we take the subgroup $\overline{B} = \langle b_1, \ldots, b_m \rangle$, it is a finite subgroup of $\overline{A}$. Since $\overline{A}$ does not contain any elements of infinite height and $\overline{B}$ is finite, then the heights of all nonzero elements of $\overline{B}$ in $\overline{A}$ are bounded, therefore $\overline{B}$ is embedded in a direct summand of $\overline{A}$. This summand is finite and is isomorphic to

$$\overline{B}_1 \oplus \cdots \oplus \overline{B}_q, \text{ where } \overline{B}_i \cong \bigoplus_{n_i} \mathbb{Z}(p^i).$$

Of course all n_i are defined by formulas with $a_1, \dots, a_m$ as parameters.

Also, we can find a basic subgroup B of $\overline{A}$ such that $\overline{B}$ is a direct summand of B.

Now we are ready to prove the main theorem (Theorem A).

Proof. Let A_1 and A_2 be two Abelian p-groups with elementarily equivalent divisible parts and elementarily equivalent basic subgroups, $A_1 = D_1 \oplus \overline{A}_1, A_2 = D_2 \oplus \overline{A}_2$ be their decompositions in the direct sum of divisible and reduced separable subgroups, $(d_1, \dots, d_\ell, a_1, \dots, a_m)$ be a tuple of element of A_1, where $d_1, \dots, d_\ell \in D_1$ and $a_1, \dots, a_m \in \overline{A}_1$. These elements generate (in the sense above) a direct summand of A_1, isomorphic to

$$C = C_d \oplus C_r \cong \bigoplus_{n_0} \mathbb{Z}(p^\infty) \oplus \bigoplus_{t=1}^{m} \left(\bigoplus_{n_t} \mathbb{Z}(p^t) \right).$$

Therefore if

$$D_1 = \bigoplus_{\varkappa_0} \mathbb{Z}(p^\infty), \quad D_2 = \bigoplus_{\varkappa'_0} \mathbb{Z}(p^\infty)$$

and

$$B_1 = \bigoplus_{t=1}^{\infty} \left(\bigoplus_{\varkappa_t} \mathbb{Z}(p^t) \right), \quad B_2 = \bigoplus_{t=1}^{\infty} \left(\bigoplus_{\varkappa'_t} \mathbb{Z}(p^t) \right),$$

then for all $i = 0, 1, \dots, m$

$$n_i \leqslant \varkappa_i = \varkappa'_i \text{ or } \varkappa_i, \varkappa'_i \text{ are both infinite.}$$

So we can denote

$$A_1 = D_1 \oplus \overline{A}_1 = (C_d \oplus \widetilde{C}_d) \oplus (C_r \oplus \widetilde{B}_1^{(1)} \oplus \cdots \oplus \widetilde{B}_1^{(m)}) \oplus (\overline{B}_1^{(m+1)} + p^m \overline{A}_1) = C \oplus \widetilde{C}_1.$$

Similarly the group A_2 can be decomposed as

$$A_2 = D_2 \oplus \overline{A}_2 \cong (C_d \oplus \widetilde{C}'_d) \oplus (C_r \oplus \widetilde{B}_2^{(1)} \oplus \cdots \oplus \widetilde{B}_2^{(m)}) \oplus (\overline{B}_2^{(m+1)} + p^m \overline{A}_2) = C \oplus \widetilde{C}_2$$

and

$$\widetilde{C}_1 \equiv \widetilde{C}_2.$$

Therefore we have $(d'_1, \dots, d'_\ell, a'_1, \dots, a'_m) \in C \in A_2$ (the same as $(d_1, \dots, d_\ell, a_1, \dots, a_m) \in C \in A_1$ such that

$$\mathrm{tp}(d'_1, \dots, d'_\ell, a'_1, \dots, a'_m) = \mathrm{tp}(d_1, \dots, d_\ell, a_1, \dots, a_m).$$

Therefore A_1 and A_2 are isotypic. □

Remark 4.1. From [4], Theorem 77.3, it follows that any countable reduced separable Abelian p-group is a direct sum of cyclic groups. Therefore for countable groups with separable reduced parts isotypicity coincides with isomorphism.

For non-countable Abelian p-groups with separable reduced parts there are many examples of isotypical and non-isomorphic groups.

5. Strong homogeneity

Definition 5.1. *A model $\mathcal{M}$ is called ω-strongly homogeneous, if for any $a_1,\dots,a_n$, $b_1,\dots,b_n \in M$ if $\operatorname{tp}(a_1,\dots,a_n) = \operatorname{tp}(b_1,\dots,b_n)$, then there exists an automorphism $\varphi \in \operatorname{Aut}\mathcal{M}$ such that $\varphi(a_i) = b_i$, $i = 1,\dots,n$.*

From the proof of Theorem A we can derive the following corollary:

Corollary 5.1. *For any prime p any Abelian p-group A with a separable reduced part is ω-strongly homogeneous.*

Proof. Let us take our group A and $a_1,\dots,a_n,b_1,\dots,b_n \in A$ with $\operatorname{tp}(a_1,\dots,a_n) = \operatorname{tp}(b_1,\dots,b_n)$. According to the previous considerations we can (without loss of generality) assume, that $a_1,\dots,a_k,b_1,\dots,b_k \in D$, where D is the divisible part of A, and $a_{k+1},\dots,a_n \in A_1$, $b_{k+1},\dots,b_n \in A_2$, where A_1 and A_2 are reduced and $A = D \oplus A_1 = D \oplus A_2$.

Since $\operatorname{tp}(a_1,\dots,a_k) = \operatorname{tp}(b_1,\dots,b_k)$, these elements are contained in isomorphic minimal direct summands of D:

$$D_1 \cong D_2 \cong \bigoplus_{\ell} \mathbb{Z}(p^\infty).$$

In these groups $a_1,\dots,a_k$ and $b_1,\dots,b_k$ are such elements that for any $\alpha_1,\dots,\alpha_k \in \mathbb{Z}$ the linear combinations $\alpha_1 a_1 + \dots + \alpha_k a_k$ and $\alpha_1 b_1 + \dots + \alpha_k b_k$ have the same orders. Therefore there exists an isomorphism $\varphi_1 : D_1 \to D_2$ such that $\varphi_1(a_i) = b_i$ for all $i = 1,\dots,k$.

Since $D = D_1 \oplus D_1' = D_2 \oplus D_2'$, where $D_1' \cong D_2'$, the isomorphism φ can be extended up to an automorphism $\varphi_2 \in \operatorname{Aut} D$.

Now let us consider the elements $a_{k+1},\dots,a_n \in A_1$ and $b_{k+1},\dots,b_n \in A_2$. Since $\operatorname{tp}(a_{k+1},\dots,a_n) = \operatorname{tp}(b_{k+1},\dots,b_n)$, these elements are contained in isomorphic minimal direct summands B_1 of A_1 and B_2 A_2, respectively.

By the same reasons, since for any $\alpha_{k+1},\dots,\alpha_n \in \mathbb{Z}$ the linear combinations $\alpha_{k+1}a_{k+1} + \dots + \alpha_n a_n$ and $\alpha_1 b_1 + \dots + \alpha_k b_k$ have the same orders and heights, there exists an isomorphism $\varphi_3 : B_1 \to B_2$ such that $\varphi_3(a_i) = b_i$ for all $i = k+1,\dots,n$.

Since $A_1 = B_1 \oplus C_1$ and $A_2 = B_2 \oplus C_2$, where $C_1 \cong C_2$, we can extend φ_3 up to an isomorphism $\varphi_4 : A_1 \to A_2$.

Since $A = D \oplus A_1 = D \oplus A_2$, the mappings φ_2 and φ_4 give us the required automorphism φ. □

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Elena Bunina
Department of Mathematics
Bar–Ilan University
5290002 Ramat Gan
Ramat Gan, ISRAEL
e-mail: *helenbunina@gmail.com*