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LINEAR MAPS PRESERVING THE CULLIS DETERMINANT OF $(n+1)\times n$ MATRICES

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To Mirjana Vuković on the occasion of her birthday

ABSTRACT. In this paper we give an explicit description of linear maps preserving the Cullis determinant of rectangular matrices of the size $(n+1)\times n$. Unlike the result about the ordinary determinant, it appears that linear preservers of Cullis determinant can be singular. We provide the corresponding examples and characterize the case when these maps are non-singular.

1. INTRODUCTION

The determinant of a matrix is a classical object of investigations in Linear Algebra and its applications. Usually only the determinant of square matrices is considered, but different attempts to generalize the notion of determinant to the set of rectangular matrices have been done for a long time. Cullis introduced the concept of determinant (he called it *determinoid*) of a rectangular matrix in his monograph [2] and it is presumably the first published generalization of the determinant to the rectangular case. Several properties known for the classical determinant are studied and shown to be valid for the Cullis determinant in [2, §5, §27, §32], and we recall some of them below. Algebraic characterization of the Cullis determinant can be found in [1, 7].

In 1966 Radić [11] independently proposed a definition of the determinant of a rectangular matrix, which is equivalent to the Cullis definition, and since that, in some papers it is called *Radić's determinant* [1] or *Cullis-Radić determinant* [6]. After that there were several other generalizations of the determinant of a square matrix to rectangular matrices given, for example, in [10, 12, 13].

The notion of determinant has been studied in many contexts and one of them is the investigation of linear maps preserving the determinant. The first result in this direction dates back to 1897 and is due to Frobenius [4].

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Theorem 1.1 (Frobenius, [4, §7, Theorem I])**.** *Let $S\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a bijective linear map satisfying* $\det(S(X)) = \det(X)$ *for all $X \in M_n(\mathbb{C})$. Then there exist matrices $M,N \in M_n(\mathbb{C})$ with* $\det(MN) = 1$ *such that*

$$S(X) = MXN \text{ for all } X \in M_n(\mathbb{C}) \quad \text{or} \quad S(X) = MX^tN \text{ for all } X \in M_n(\mathbb{C}).$$

This result by Frobenius prompted the investigation of so-called linear preserver problems concerning the characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. The research started by Frobenius was continued in the works by Schur, Dieudonné, Dynkin and others. One may see [9] for an extensive survey.

In this paper we provide an analog of Frobenius theorem for the Cullis determinant in the case of $(n+1)\times n$ matrices. We would like to draw the attention to the following three important features of the Cullis determinant. Unlike the result by Frobenius, it appears that linear preservers of the Cullis determinant can be singular (see Example 4.2). Moreover, our characterization theorem is valid both in singular and non-singular cases, and we underline that the form of the maps in the characterization result is the same in singular and non-singular cases. Moreover, we obtain the complete characterization of non-singular linear preservers of the Cullis determinant.

Our paper is organized as follows. Section 2 contains basic definitions and notations used in the paper as well as the main properties of Cullis determinants. In Section 3 we formulate and prove the main result, namely the complete characterization of all linear maps preserving the Cullis determinant over an arbitrary field is provided. Section 4 contains various natural examples and counter-examples of such maps including the examples of singular maps preserving the Cullis determinant as well as the characterization of non-singular maps preserving the Cullis determinant.

2. Preliminaries

We denote by $M_{nk}(\mathbb{F})$ the set of all $n\times k$ matrices with the entries from a certain field $\mathbb{F}$ and write $M_n(\mathbb{F}) = M_{nn}(\mathbb{F})$. $O_{nk} \in M_{nk}(\mathbb{F})$ denotes the matrix with all entries equal zero. By $X_{(i,j)}$ we denote the element of a matrix X lying on the intersection of its i-th row and j-th column. We omit the subscripts if this cannot lead to a misunderstanding. Let us denote by $E_{ij} \in M_{n+1}(\mathbb{F})$ a matrix, whose entries are all equal to zero besides the entry on the intersection of the i-th row and the j-th column, which is equal to one.

We introduce the basic definitions and concepts following [5].

By I we denote the set $\{1,\dots,n\}$ of indices, and by K we denote its subset $\{1,\dots,k\} \subseteq I$ $(k \le n)$. By S_K^I we denote the set of injections from K to I which has the cardinality $|S_K^I| = \frac{n!}{(n-k)!}$.

Definition 2.1. *Given an injection $\sigma \in S_K^I$, we denote by* $\operatorname{sgn}_{nk}(\sigma)$ *the product*

$$s\cdot(-1)^{\sum_{l=1}^{k}(\sigma(l)-l)},$$

where $s=\operatorname{sgn}(\pi)$ *is the sign of the permutation*

$$\pi=\begin{pmatrix} i_1 & \dots & i_k \\ \sigma(1) & \dots & \sigma(k)\end{pmatrix},$$

here $\sigma(K)=\{i_1,\dots,i_k\}$*, and* $i_1<i_2<\dots<i_k$*.*

Definition 2.2 ([5], Theorem 13)**.** *Let* $n\geq k$*,* $X\in M_{nk}(\mathbb{F})$*. Then Cullis determinant* $\det_{nk}(X)$ *of* X *is defined to be the function:*

$$\det_{nk}(X)=\sum_{\sigma\in S_k^I}\operatorname{sgn}_{nk}(\sigma)X_{(\sigma(1),1)}X_{(\sigma(2),2)}\dots X_{(\sigma(k),k)}.$$

Recall properties of $\det_{nk}$ which will be used in this paper (see [2, §5, §27, §32] or [5] for detailed proofs):

Theorem 2.3 ([5, Theorem 13, Theorem 16])**.**

(1) *For* $X\in M_n(\mathbb{F})$, $\det_{nn}(X)=\det(X)$.
(2) *For* $X\in M_{nk}(\mathbb{F})$, $\det_{nk}(X)$ *is a linear function of columns of* X*.*
(3) *If a matrix* $X\in M_{nk}(\mathbb{F})$ *has two identical columns or one of its columns is a linear combination of other columns, then* $\det_{nk}(X)$ *is equal to zero.*
(4) *For* $X\in M_{nk}(\mathbb{F})$*, interchanging any two columns of* X *changes the sign of* $\det_{nk}(X)$.
(5) *Adding a linear combination of columns to another column does not change* $\det_{nk}$*.*
(6) *For* $X\in M_{nk}(\mathbb{F})$, $\det_{nk}(X)$ *can be calculated using the Laplace expansion along a column of* X*.*

If $X\in M_{n+1\,n}(\mathbb{F})$, then we denote its Cullis determinant $\det_{n+1\,n}(X)$ by $\mathrm{d_C}(X)$.

3. LINEAR MAPS PRESERVING $\mathrm{d_C}$

In this section we obtain the explicit description of linear maps preserving the Cullis determinant. We start with the formulation of the main result of our paper for which we need the following definitions.

Definition 3.1. *For every matrix* $X\in M_n(\mathbb{F})$ *by* $R^+(X)$ *we denote the matrix* $R^+(X)=\left(\begin{smallmatrix}O_{1n}\\ X\end{smallmatrix}\right)\in M_{n+1\,n}(\mathbb{F})$*, obtained by adjoining to* X *the vector* $(0,\dots,0)$ *as a first row.*

Definition 3.2. *For every matrix* $X\in M_{n+1\,n}(\mathbb{F})$ *by* $R^-(X)\in M_n(\mathbb{F})$ *we denote the matrix obtained from* X *by deleting of the first row.*

Definition 3.3. *For every vector* $x\in\mathbb{F}^n$ *we denote by* $F(x)=\left(\begin{smallmatrix}x^t\\ O_n\end{smallmatrix}\right)\in M_{n+1\,n}(\mathbb{F})$ *the matrix obtained by adjoining the row vector* x^t *to the square zero matrix as a first row.*

Definition 3.4. *By* $\Phi \in M_{n+1}(\mathbb{F})$ *we denote the matrix* $\Phi = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 1 & 0 & \cdots & 1 \end{pmatrix}$.

Now we are ready to formulate our main result.

Theorem 3.5. *Let* $\mathbb{F}$ *be an arbitrary field. Then a linear map*

$$T\colon M_{n+1\,n}(\mathbb{F}) \to M_{n+1\,n}(\mathbb{F})$$

satisfies $\mathrm{d_C}(T(X)) = \mathrm{d_C}(X)$ *for all* $X \in M_{n+1\,n}(\mathbb{F})$ *if and only if there exist matrices* $M, N \in M_n(\mathbb{F})$ *with* $\det(MN) = 1$ *and a linear map* $\alpha\colon M_{n+1\,n}(\mathbb{F}) \to M_{1\,n}(\mathbb{F})$ *such that*

$$T(X) = \Phi \cdot F(\alpha(\Phi^{-1}X)) + \Phi \cdot R^{+}(M(R^{-}(\Phi^{-1}X))N) \ \ \forall X \in M_{n+1\,n}(\mathbb{F}) \tag{3.1}$$

or

$$T(X) = \Phi \cdot F(\alpha(\Phi^{-1}X)) + \Phi \cdot R^{+}(M(R^{-}(\Phi^{-1}X))^{t}N) \ \ \forall X \in M_{n+1\,n}(\mathbb{F}). \tag{3.2}$$

In order to prove this theorem we establish the following relation connecting the usual determinant and the $(n+1)\times n$ Cullis determinant.

Definition 3.6. *Let* $X \in M_{n+1\,n}(\mathbb{F})$. *Then* $C^{+}(X) := (R^{+}(X^{t}))^{t}$, *namely*

$$C^{+}(X) = \begin{pmatrix} O_{n+1\,1} & X \end{pmatrix} \in M_{n+1}(\mathbb{F}).$$

Theorem 3.7. *Suppose that* $X \in M_{n+1\,n}(\mathbb{F})$. *Then*

$$\mathrm{d_C}(X) = (-1)^{n} \det\left(\sum_{i=1}^{n+1} E_{i1} + C^{+}(X)\right). \tag{3.3}$$

Proof. For any matrix of the form $Y = \begin{pmatrix} X & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \end{pmatrix} \in M_{n+1}(\mathbb{F})$ by Laplace expansion along the last column we get

$$\det(Y) = (-1)^{n} \det\left(\sum_{i=1}^{n+1} E_{i1} + C^{+}(X)\right).$$

For every $\sigma \in S_{\{1,\ldots,n\}}^{\{1,\ldots,n+1\}}$ consider $\tilde{\sigma} \in S^{n+1}$ defined as follows: $\tilde{\sigma}(i) = \sigma(i)$ for every $1 \le i \le n$ and $\tilde{\sigma}(n+1) = j$, where j is the unique element of $\{1,\ldots,n+1\} \setminus \sigma(\{1,\ldots,n\})$. Then $\mathrm{sgn}(\tilde{\sigma}) = \mathrm{sgn}_{n+1\,n}(\sigma)$, because $\sum_{l}(\sigma(l) - l) = n+1-j$, which is equal to the difference between the number of inversions in $\begin{pmatrix} i_1 & \ldots & i_k \\ \sigma(1) & \ldots & \sigma(k) \end{pmatrix}$ from the definition of $\mathrm{sgn}_{n+1\,n}(\sigma)$ and the number of inversions of the permutation $\tilde{\sigma}$.

Therefore by the definitions of the Cullis determinant of X and the determinant of Y we get

$$\begin{aligned}
\mathrm{d_C}(X) &= \sum_{\sigma\in S^{\{1,\dots,n+1\}}_{\{1,\dots,n\}}} \mathrm{sgn}_{n+1\,n}(\sigma) X_{(\sigma(1),1)}X_{(\sigma(2),2)}\dots X_{(\sigma(n),n)}\\
&= \sum_{\sigma\in S^{\{1,\dots,n+1\}}_{\{1,\dots,n\}}} \mathrm{sgn}(\tilde\sigma) Y_{(\tilde\sigma(1),1)}\dots Y_{(\tilde\sigma(n),n)}Y_{(\tilde\sigma(n+1),n+1)}\\
&= \sum_{\tau\in S_{n+1}} \mathrm{sgn}(\tau) Y_{(\tau(1),1)}\dots Y_{(\tau(n),n)}Y_{(\tau(n+1),n+1)}\\
&= \det(Y) = (-1)^n \det\left(\sum_{i=1}^{n+1} E_{i1} + C^+(X)\right). \qquad \square
\end{aligned}$$

Observe that for the matrix Φ introduced in Definition 3.4

$$\det(\Phi)=1,\quad \Phi\begin{pmatrix}1\\0\\ \vdots\\ 0\end{pmatrix} = \begin{pmatrix}1\\1\\ \vdots\\ 1\end{pmatrix},\ \text{and } \Phi^{-1} = \begin{pmatrix}1 & 0 & \cdots & 0\\ -1 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ -1 & 0 & \cdots & 1\end{pmatrix}.$$

Let $Y=(y_{ij})\in M_{n+1}(\mathbb{F})$ satisfy $y_{i1}=1$ for all $i=1,\dots,n+1$. Then the first column of the matrix $\Phi^{-1}Y$ is equal to the vector $(1,0,\dots,0)^t$.

Theorem 3.8. *Let $T: M_{n+1\,n}(\mathbb{F})\to M_{n+1\,n}(\mathbb{F})$ be a linear map. Then T preserves the Cullis determinant if and only if*

$$\det(E_{11}+C^+(U(X))) = \det(E_{11}+C^+(X)). \tag{3.4}$$

for all $X\in M_{n+1\,n}(\mathbb{F})$, where $U: M_{n+1\,n}(\mathbb{F})\to M_{n+1\,n}(\mathbb{F})$ is the linear map defined by

$$U(X) = \Phi^{-1}\cdot T(\Phi X). \tag{3.5}$$

Proof. It is straightforward to see that U is linear.

To prove necessity, suppose that T preserves $\mathrm{d_C}$. Then consider the following sequence of equalities: $\det(E_{11}+C^+(U(X))) =$

$$\begin{aligned}
&= \det(E_{11}+C^+(\Phi^{-1}T(\Phi X)))\\
&= \det(\Phi)\cdot\det(E_{11}+C^+(\Phi^{-1}\cdot T(\Phi X))) && (\text{since } \det(\Phi)=1)\\
&= \det(\Phi(E_{11}+C^+(\Phi^{-1}\cdot T(\Phi X))))\\
&= \det(\Phi E_{11}+\Phi\cdot C^+(\Phi^{-1}\cdot T(\Phi X)))\\
&= \det(\Phi E_{11}+C^+(T(\Phi X))) && (\text{since } C^+(\Phi^{-1}\cdot T(\Phi X)) = \Phi^{-1}\cdot C^+(T(\Phi X)))\\
&= \det(\sum_{i=1}^{n+1} E_{i1}+C^+(T(\Phi X))) && (\text{since } \Phi\cdot E_{11} = \sum_{i=1}^{n+1} E_{i1})\\
&= (-1)^n\,\mathrm{d_C}(T(\Phi X)) && (\text{by the equality (3.3)})\\
&= (-1)^n\,\mathrm{d_C}(\Phi X) && (\text{since } T \text{ preserves } \mathrm{d_C})
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^n \cdot (-1)^n \det(\sum_{i=1}^{n+1} E_{i1} + C^+(\Phi X)) && \text{(by the equality (3.3))}\\
&= \det(\Phi^{-1}) \det((\sum_{i=1}^{n+1} E_{i1} + C^+(\Phi X))) && \text{(since } \det(\Phi^{-1}) = 1)\\
&= \det(\Phi^{-1}(\sum_{i=1}^{n+1} E_{i1} + C^+(\Phi X))) &&\\
&= \det(E_{11} + \Phi^{-1} C^+(\Phi X)) && \text{(since } \Phi^{-1}\sum_{i=1}^{n+1} E_{i1}\cdot = E_{11})\\
&= \det(E_{11} + \Phi^{-1}\Phi C^+(X)) && \text{(since } C^+(\Phi\cdot X) = \Phi\cdot C^+(X))\\
&= \det(E_{11} + C^+(X)).
\end{aligned}
$$

To prove sufficiency, suppose that condition (3.4) holds for the map U. Then consider the following sequence of equalities, which is similar to the previous one

$$
\begin{aligned}
\mathrm{d_C}(T(X)) &= (-1)^n \det(\sum_i E_{i1} + C^+(\Phi\cdot U(\Phi^{-1}X)))\\
&= (-1)^n \det(\Phi^{-1}) \cdot \det(\sum_i E_{i1} + C^+(\Phi\cdot U(\Phi^{-1}X)))\\
&= (-1)^n \det(\Phi^{-1}(\sum_i E_{i1} + C^+(\Phi\cdot U(\Phi^{-1}X))))\\
&= (-1)^n \det(E_{11} + C^+(U(\Phi^{-1}X))) = (-1)^n \det(E_{11} + \Phi^{-1}X)\\
&= (-1)^n \det(\Phi)\cdot\det(E_{11} + \Phi^{-1}X) = (-1)^n \det(\Phi(E_{11} + \Phi^{-1}X))\\
&= (-1)^n \det(\sum_i E_{i1} + X) = (-1)^n \cdot (-1)^n \mathrm{d_C}(X) = \mathrm{d_C}(X). \qquad \square
\end{aligned}
$$

Now we describe all linear maps acting on $M_{n+1\,n}(\mathbb{F})$ and satisfying the condition (3.4). For this, we need again Definitions 3.1 and 3.2.

Lemma 3.9. *Suppose that $S\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is a linear map such that $\det(S(X)) = \det(X)$ for all $X \in M_n(\mathbb{F})$ and $\alpha\colon M_{n+1\,n}(\mathbb{F}) \to M_{1\,n}(\mathbb{F})$ is the linear map. Then the linear map $T : M_{n+1\,n}(\mathbb{F}) \to M_{n+1\,n}(\mathbb{F})$ defined by*

$$T(X) = F(\alpha(X)) + R^+(S(R^-(X))),$$

satisfies the condition (3.4).

Proof. For any $Y \in M_n(\mathbb{F})$ by applying the Laplace expansion of $\det(E_{11} + C^+(X))$ along the first column we obtain

$$\det(E_{11} + C^+(Y)) = \det(R^-(Y)) \tag{3.6}$$

and hence it does not depend on the first row of Y. Then by applying the equality (3.6) to $Y = T(X)$ and X we get

$$\begin{aligned}\det(E_{11}+C^{+}(T(X))) &= \det(R^{-}(T(X))) = \det R^{-}((F(\alpha(X))+R^{+}(S(R^{-}(X)))))\\ &= \det(R^{-}(R^{+}(S(R^{-}(X))))) = \det(S(R^{-}(X))) = \det(R^{-}(X))\\ &= \det(E_{11}+C^{+}(X)).\end{aligned}$$

Here the 3-rd equality follows from the definition of R^{-} and α. Therefore, $T(X)$ satisfies the condition (3.4). ☐

We can also immediately obtain a similar result concerning linear maps preserving $\mathrm{d_C}$.

Lemma 3.10. *Suppose that* $S\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ *is a linear map such that* $\det(S(X)) = \det(X)$ *for all* $X \in M_n(\mathbb{F})$ *and* $\alpha\colon M_{n+1\,n}(\mathbb{F}) \to M_{1\,n}(\mathbb{F})$ *is the linear map. Then the linear map* $T : M_{n+1\,n}(\mathbb{F}) \to M_{n+1\,n}(\mathbb{F})$ *defined by*

$$T(X) = \Phi\cdot F(\alpha(\Phi^{-1}X)) + \Phi\cdot R^{+}((R^{-}(S(\Phi^{-1}X))))$$

preserves $\mathrm{d_C}$.

Proof. Define $U(X)$ by

$$U(X) = \Phi^{-1}\cdot T(\Phi X).$$

Then we get

$$U(X) = F(\alpha(X)) + R^{+}(S(R^{-}(X))).$$

Hence $U(X)$ satisfies the condition (3.4) by Theorem 3.9. Therefore $T(X)$ preserves $\mathrm{d_C}$ by Theorem 3.8. ☐

Lemma 3.11. *Suppose that* $M \in M_k(\mathbb{F})$ *and* $\det(M) = 0$. *Then there exists a nonzero matrix* $N \in M_k(\mathbb{F})$ *with* $\det(N) = 0$ *such that* $\det(M+N) \neq 0$.

Proof. Denote $\mathrm{rk}(M) = r$. Since $\det(M) = 0$, we have $r < n$. Then there exists N with $\mathrm{rk}(N) = n-r$ such that $M+N$ is invertible. Following that, $\det(M+N) \neq 0$ and $\det(N) = 0$ because $\mathrm{rk}(N) < n$. Therefore N satisfies the required conditions. ☐

Lemma 3.12. *Suppose that* $A \in M_{n+1\,n}(\mathbb{F})$ *and*

$$\det(E_{11}+C^{+}(A+X)) = \det(E_{11}+C^{+}(X)) \tag{3.7}$$

for all $X \in M_{n+1\,n}(\mathbb{F})$. *Then* $R^{-}(A)$ *is the zero matrix.*

Proof. We prove by contradiction. Suppose that there exists $A \in M_{n+1\,n}(\mathbb{F})$ which satisfies the conditions of the lemma and $R^{-}(A) \neq 0$. By substitution of the zero matrix instead of X to (3.7) we get that $\det(E_{11}+C^{+}(A)) = \det(E_{11}) = 0$ and therefore $\det(R^{-}(A)) = 0$ by the equality (3.6). By Lemma 3.11 applied to $R^{-}(A)$, we can find a matrix N with $\det(N) = 0$ such that $\det(N+R^{-}(A)) \neq 0$ and therefore

$$\det(E_{11}+C^{+}(A+R^{+}(N))) = \det(N+R^{-}(A)) \neq 0.$$

On other hand, due to the conditions of the lemma and Laplace expansion of the matrix $(E_{11}+C^{+}(R^{+}(N)))$ along the first column,

$$\det(E_{11}+C^{+}(A+R^{+}(N))) = \det(E_{11}+C^{+}(R^{+}(N))) = \det(N) = 0,$$

which is a contradiction. □

Lemma 3.13. *Suppose that $T: M_{n+1\,n}(\mathbb{F}) \to M_{n+1\,n}(\mathbb{F})$ is the map satisfying condition (3.4). Then for every $X \in M_{n+1\,n}(\mathbb{F})$ with nonzero $R^{-}(X)$ the matrix $R^{-}(T(X))$ is also nonzero.*

Proof. Suppose that $T: M_{n+1\,n}(\mathbb{F}) \to M_{n+1\,n}(\mathbb{F})$ is a map satisfying condition (3.4) and $A \in M_{n+1\,n}(\mathbb{F})$ is a matrix such that $R^{-}(T(A)) = 0$. Then the following sequence of equalities holds:

$$\begin{aligned}\det(E_{11}+C^{+}(A+X)) &= \det(E_{11}+C^{+}(T(A+X))) \\ &= \det(E_{11}+C^{+}(0+T(X))) = \det(E_{11}+C^{+}(X))\end{aligned}$$

for every $X \in M_{n+1\,n}(\mathbb{F})$. Hence by Lemma 3.12 we get that $R^{-}(A) = 0$. □

Lemma 3.14. *Suppose that $T: M_{n+1\,n}(\mathbb{F}) \to M_{n+1\,n}(\mathbb{F})$ is the map satisfying condition (3.4). Then for every $Y \in M_n(\mathbb{F})$ there exists $X \in M_n(\mathbb{F})$ such that*

$$R^{-}(T(R^{+}(X))) = Y.$$

Proof. By Lemma 3.13 the linear map $S: M_n(\mathbb{F}) \to M_n(\mathbb{F})$ defined by

$$S(X) = R^{-}(T(R^{+}(X)))$$

is injective. Hence it is a bijection, and for every $Y \in M_n(\mathbb{F})$ there exists a matrix $Z \in M_n(\mathbb{F})$ such that $R^{-}(T(R^{+}(Z))) = Y$. Hence we can set $X = R^{+}(Z)$. □

Definition 3.15. *By $\mathcal{Z} \subset M_{n+1n}(\mathbb{F})$ we denote the set of the matrices of the form $\begin{pmatrix} x^t \\ O_{nn} \end{pmatrix}$, where $x \in \mathbb{F}^n$.*

Lemma 3.16. *Suppose that $T: M_{n+1\,n}(\mathbb{F}) \to M_{n+1\,n}(\mathbb{F})$ is a linear map satisfying condition (3.4). Then $T(\mathcal{Z}) \subset \mathcal{Z}$.*

Proof. We prove by contradiction. Suppose there exists a nonzero matrix $Z \in \mathcal{Z}$ such that $R^{-}(T(Z))$ is nonzero. Hence by Lemma 3.14 there exists $B \in M_n(\mathbb{F})$ such that $R^{-}(T(R^{+}(B))) = -R^{-}(T(Z))$. Observe that $B \neq 0$ since $R^{-}(T(Z)) \neq 0$. Let us consider

$$C = R^{+}(B) + Z \in M_{n+1\,n}(\mathbb{F}).$$

Then $R^{-}(C) \neq 0$, but $R^{-}(T(C)) = 0$, which contradicts Lemma 3.13. □

Theorem 3.17. *Suppose that T is a linear map satisfying condition (3.4). Then there exists a linear map $S: M_n(\mathbb{F}) \to M_n(\mathbb{F})$ preserving the determinant and a linear map $\alpha: M_{n+1\,n}(\mathbb{F}) \to M_{1\,n}(\mathbb{F})$ such that*

$$T(X) = F(\alpha(X)) + R^{+}(S(R^{-}(X)))$$

for all $X \in M_{n+1\,n}(\mathbb{F})$.

Proof. From the condition (3.4) by expanding $\det(E_{11}+C^{+}(X))$ along the first column we have that $\det(R^{-}(T(X)))=\det(R^{-}(X))$ for all $X\in M_{n+1\,n}$. Therefore, the linear map $S\colon M_n(\mathbb{F})\to M_n(\mathbb{F})$ defined by

$$S(X)=R^{-}(T(R^{+}(X)))$$

preserves the determinant.

Suppose that $X\in M_{n+1\,n}(\mathbb{F})$. Consider the following equality:

$$X=\begin{pmatrix} X_{(1,1)} & \dots & X_{(1,n)} \\ & O_n & \end{pmatrix}+\begin{pmatrix} & O_{1\,n} & \\ X_{(2,1)} & \cdots & X_{(2,n)} \\ X_{(n+1,1)} & \dots & X_{(n+1,n)} \end{pmatrix}.$$

This equality is equivalent to the following equality using R^{+} and R^{-} :

$$X=\begin{pmatrix} X_{(1,1)} & \dots & X_{(1,n)} \\ & O_n & \end{pmatrix}+R^{+}(R^{-}(X)). \tag{3.8}$$

Apply T to both sides of the equality (3.8):

$$T(X)=T\left(\begin{pmatrix} X_{(1,1)} & \dots & X_{(1,n)} \\ & O_n & \end{pmatrix}\right)+T(R^{+}(R^{-}(X))).$$

Then use the equality (3.8) for $T(R^{+}(R^{-}(X)))$

$$T(X)=T\left(\begin{pmatrix} X_{(1,1)} & \dots & X_{(1,n)} \\ & O_n & \end{pmatrix}\right)+\begin{pmatrix} T(R^{+}(R^{-}(X)))_{(1,1)} & \dots & T(R^{+}(R^{-}(X)))_{(1,n)} \\ & O_n & \end{pmatrix}$$
$$+R^{+}(R^{-}(T(R^{+}(R^{-}(X))))).$$

Then substitute $S(X)$ instead of $R^{-}(T(R^{+}(X)))$

$$T(X)=T\left(\begin{pmatrix} X_{(1,1)} & \dots & X_{(1,n)} \\ & O_n & \end{pmatrix}\right)+\begin{pmatrix} T(R^{+}(R^{-}(X)))_{(1,1)} & \dots & T(R^{+}(R^{-}(X)))_{(1,n)} \\ & O_n & \end{pmatrix}$$
$$+R^{+}(S(R^{-}(X))). \tag{3.9}$$

If we define $\beta\colon M_{n+1\,n}(\mathbb{F})\to M_{n+1\,n}(\mathbb{F})$ by

$$\beta(X)=T\left(\begin{pmatrix} X_{(1,1)} & \dots & X_{(1,n)} \\ & O_n & \end{pmatrix}\right)+\begin{pmatrix} T(R^{+}(R^{-}(X)))_{(1,1)} & \dots & T(R^{+}(R^{-}(X)))_{(1,n)} \\ & O_n & \end{pmatrix},$$

then from the equality (3.9) we get

$$T(X)=\beta(X)+R^{+}(S(R^{-}(X))).$$

β is a linear map, because it is represented as a sum of a projection and a linear map. By Lemma 3.16, $T\left(\begin{pmatrix} X_{(1,1)} & \dots & X_{(1,n)} \\ & O_n & \end{pmatrix}\right)\in\mathcal{Z}$ and therefore $\beta(X)\in\mathcal{Z}$. Hence there exists a linear map $\alpha\colon M_{n+1\,n}(\mathbb{F})\to M_{1\,n}(\mathbb{F})$ such that

$$\beta(X)=F(\alpha(X))$$

for all $X\in M_{n+1\,n}(\mathbb{F})$, which finishes the proof. $\square$

Now, we are ready to prove the main result.

Proof of Theorem 3.5. From Lemma 3.10 it follows that every linear map of the form (3.1) or (3.2) preserves $\mathrm{d_C}$.

Consider the map $T\colon M_{n+1\,n}(\mathbb{F}) \to M_{n+1\,n}(\mathbb{F})$ which is linear and satisfies $\mathrm{d_C}(T(X)) = \mathrm{d_C}(X)$ for all $X \in M_{n+1\,n}(\mathbb{F})$. By Theorem 3.17 there exists a linear map $S\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ preserving the determinant and there exists a linear map $\alpha\colon M_{n+1\,n}(\mathbb{F}) \to M_{1\,n}(\mathbb{F})$ such that

$$T(X) = F(\alpha(X)) + R^{+}(S(R^{-}(X)))$$

for all $X \in M_{n+1\,n}(\mathbb{F})$. Now apply Frobenius theorem (Theorem 1.1) to the map S to obtain the required result. Note that although Theorem 1.1 was originally proved only for the field $\mathbb{C}$ of complex numbers, its statement is true for any field. Indeed, if the map preserves the determinant, then it is bijective by [8, Lemma 7] and it preserves the set of matrices with the zero determinant, i.e., the set of singular matrices. Then one can apply the Dieudonné theorem, see [3, Theorem 3], and observe that only multiplication $X \mapsto AXB$ with square matrices A,B such that $\det(AB) = 1$ preserves the determinant. $\square$

4. Examples of linear maps preserving $\mathrm{d_C}$

Example 4.1. *Suppose $N \in M_n(\mathbb{F})$, $\det(N) = 1$. Then $T\colon M_{n+1\,n}(\mathbb{F}) \to M_{n+1\,n}(\mathbb{F})$ defined by $T(X) = XN$ preserves $\mathrm{d_C}$.*

Proof. It is possible to represent the right multiplication by N according to the formula (3.1). Indeed, let us consider

$$\alpha(X) = \begin{pmatrix} (XN)_{(1,1)} & \dots & (XN)_{(1,n)} \end{pmatrix}$$

and M is equal to the unit matrix. We get

$$\begin{aligned}
T(X) &= \Phi(F(\alpha(\Phi^{-1}X)) + R^{+}((R^{-}(\Phi^{-1}X))N)) \\
&= \Phi\left(\begin{pmatrix} (XN)_{(1,1)} \ \dots \ (XN)_{(1,n)} \\ O_n \end{pmatrix} + \begin{pmatrix} 0 & \dots & 0 \\ (XN)_{(2,1)} - (XN)_{(1,1)} & \dots & (XN)_{(2,n)} - (XN)_{(1,n)} \\ \vdots & \vdots & \vdots \\ (XN)_{(n+1,1)} & \dots & (XN)_{(n+1,n)} - (XN)_{(1,n)} \end{pmatrix}\right) \\
&= \Phi\begin{pmatrix} (XN)_{(1,1)} & \dots & (XN)_{(1,n)} \\ (XN)_{(2,1)} - (XN)_{(1,1)} & \dots & (XN)_{(2,n)} - (XN)_{(1,n)} \\ \vdots & \vdots & \vdots \\ (XN)_{(n+1,1)} & \dots & (XN)_{(n+1,n)} - (XN)_{(1,n)} \end{pmatrix} = XN.
\end{aligned}$$

$\square$

In contrast, the left multiplication does not necessary preserve $\mathrm{d_C}$. For example, if we take the matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \in M_{n+1}(\mathbb{F}),$$

then for the matrix

$$X = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \in M_{n+1\,n}(\mathbb{F})$$

we have that $\mathrm{d_C}(X) = (-1)^n \cdot 2$ and $\mathrm{d_C}(MX) = (-1)^n$.

Although linear preserver T in the statement of the Frobenius theorem (Theorem 1.1) is required to be non-singular, this requirement can be omitted. As it was shown in [8, Lemma 7], every linear map preserving the determinant function is automatically non-singular.

On the contrary, a linear map preserving $\mathrm{d_C}$ can be singular, which could be shown using our description (Theorem 3.5).

Example 4.2. *Suppose that in the setting of Theorem 3.5* $\alpha(X) = -X_{(1,1)}$, *and* $M = N$ *are identity matrices. Then the map* T *defined by the formula (3.1) is singular. Indeed,*

$$\begin{aligned} T(X) &= \Phi(F(\alpha(\Phi^{-1}X))) + R^{+}((R^{-}(\Phi^{-1}X)))) \\ &= \Phi(F(\alpha(\Phi^{-1}X))) + X = \begin{pmatrix} -X_{(1,1)} & \\ \vdots & O_{n+1\,n-1} \\ -X_{(1,1)} & \end{pmatrix} + X, \end{aligned}$$

is singular, because $T(M) = 0$, *where* $M = \big((1,\dots,1)^t \quad O_{n+1\,n-1}\big)$.

We can also characterize all non-singular preservers (Theorem 4.7).

Recall that by $\mathcal{Z} \subset M_{n+1n}(\mathbb{F})$ we denote the set of the matrices of the form $\left(\begin{smallmatrix} x^t \\ O_{nn} \end{smallmatrix}\right)$, where $x \in \mathbb{F}^n$.

By $f|_M$ we denote a restriction of a map f on a set M.

Lemma 4.3. *Suppose that* $U\colon M_{n+1\,n}(\mathbb{F}) \to M_{n+1\,n}(\mathbb{F})$ *is linear map satisfying Equation 3.4. Then* U *is non-singular if and only if* $U|_{\mathcal{Z}}$ *is non-singular.*

Proof. The necessity is clear. To prove sufficiency, suppose that U satisfies conditions of the theorem and $U(X) = 0$ for some $0 \neq X \in M_{n+1n}(\mathbb{F})$. Therefore $R^{-}(U(X)) = 0$ and by Lemma 3.13, $R^{-}((X)) = 0$. Hence $X \in \mathcal{Z}$. Finally, since $U|_{\mathcal{Z}}$ is non-singular and $U(X) = 0$, we get that $X = 0$ and therefore U is non-singular. □

By correspondence defined in Equation 3.5 we can transfer results of Lemma 3.16 and Lemma 4.3 to linear maps preserving $\mathrm{d_C}$.

Definition 4.4. *By* $\mathcal{E} \subset M_{n+1n}(\mathbb{F})$ *we denote the set of the matrices such that all their rows are equal.*

Lemma 4.5. *Consider the matrix* Φ, *defined in Definition 3.4. Then* $\Phi Z \in \mathcal{E}$ *for all* $Z \in \mathcal{E}$ *and* $\Phi^{-1}E \in \mathcal{Z}$ *for all* $E \in \mathcal{E}$.

Proof. Direct computation. □

Theorem 4.6. *Suppose that T preserves $\mathrm{d_C}$. Then $T(E) \in \mathcal{E}$ for all $E \in \mathcal{E}$.*

Proof. By Theorem 3.8 the map U, defined by

$$U(X) = \Phi^{-1}(T(\Phi X)),$$

satisfies the equation (3.4). Hence, by Lemma 4.3, U preserves $\mathcal{Z}$. Observe that T can be obtained from U by equality

$$T(X) = \Phi(U(\Phi^{-1}X)).$$

Suppose that $E \in \mathcal{E}$. Then $\Phi^{-1}E \in \mathcal{Z}$, by Lemma 4.5 and hence $U(\Phi^{-1}E) \in \mathcal{Z}$ by Lemma 3.16. Therefore, by applying Lemma 4.5 we obtain

$$T(E) = \Phi(U(\Phi^{-1}E)) \in \mathcal{E}. \qquad \square$$

Theorem 4.7. *Suppose that $T : M_{n+1\,n}(\mathbb{F}) \to M_{n+1\,n}(\mathbb{F})$ is a linear map satisfying Equation 3.4. Then T is non-singular if and only if $T|_{\mathcal{E}}$ is non-singular.*

Proof. Sufficiency follows from Theorem 4.6. Now suppose that $T|_{\mathcal{E}}$ is non-singular. By Theorem 3.8, the map U defined by

$$U(X) = \Phi^{-1}(T(\Phi X))$$

satisfies Equation 3.4 and is non-singular on $\mathcal{Z}$. Hence, by Lemma 4.3, the map U is non-singular and therefore T is non-singular. $\square$

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