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ON INVARIANT HYPERCOMPLEX STRUCTURES ON HOMOGENEOUS SPACES

SVJETLANA TERZIĆ

Dedicated to the 75th birthday of the dear Acc. Professor Mirjana Vuković

ABSTRACT. An existence of invariant hypercomplex structure on compact homogeneous spaces implies strong restrictions on their root structure and consequently on their characteristic Pontrjagin classes and the corresponding Chern classes. We describe these constraints by making use of Lie theory.

1. INTRODUCTION

When studying almost hypercomplex and hypercomplex structures on homogeneous spaces G/H, it naturally arises to consider those such structures which are invariant under the canonical action of a group G. In this note we analyze the obstructions for the existence of invariant almost hypercomplex structures on a compact homogeneous space G/H, where G is a compact semisimple Lie group and H is a closed connected subgroup.

An almost hypercomplex structure on a manifold M^{4n} is a field of endomorphisms I, J and K which satisfy quaternion relations $I^2 = J^2 = K^2 = -I_d$ and $IJ = -JI = K$, that is a triple of almost complex structures I, J, K such that $IJ = K$. Such a manifold is called an almost hypercomplex manifold. In this case there is the family of almost complex structures given by $aI + bJ + cK$, where $a^2 + b^2 + c^2 = 1$. If the structures I, J, K are integrable, that is they are complex structures on M^{4n}, the triple I, J, K is known as a hypercomplex structure on M^{4n} and M^{4n} is called a hypercomplex manifold. It is a classical result [9] that any hypercomplex manifold $M = M^{4n}$ admits unique torsion-free connection ∇^M, called the Obata connection such that $\nabla^M I = \nabla^M J = \nabla^M K = 0$. The vice versa is known to be true as well, that is if a manifold $M = M^{4n}$ has three almost complex structures which satisfy quaternion relations and has a torsion-free connection ∇^M such that $\nabla^M I = \nabla^M J = \nabla^M K = 0$, then M is hypercomplex, meaning that I, J, K are integrable. We mention that for a hypercomplex manifold M, the product $M \times \mathbb{C}P^1$,

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called the twistor space for M, parametrizes the induced family of complex structures at points of M, it has a structure of a complex manifold and the studying of the geometry of M can be approached through its twistor space, [8], [18].

On the other hand, an almost quaternionic manifold is a manifold $M = M^{4n}$ together with a rank-three bundle $\mathcal{G}$ of endomorphisms of the tangent bundle TM, such that $\mathcal{G}$ locally has a basis $\{I,J,K\}$ which satisfies quaternion relations. The structures I,J,K need not to be integrable nor globally defined. If there exists a torsion-free connection ∇^M such that $\nabla^M \mathcal{G} \subset T^*M \otimes \mathcal{G}$, then M is said to be a quaternionic manifold. There are many papers which study and establish relations between almost hypercomplex and almost quaternionic manifolds or hypercomplex and quaternionic manifolds, some of them are: [2], [12], [7], [5], etc.

In the seminal paper [7], the theory of invariant, that is homogeneous, hypercomplex structures on Lie groups and compact homogeneous spaces is developed. Prior to this, the invariant hypercomplex structures on Lie groups have been described in [14], but using different methods, that is those from the point of view of physics. In approaching this question Joyce in [7] appeals to the theory of homogeneous complex structures on compact manifolds given in the works [19] and [13]. Namely, in these papers it has been shown that any compact Lie group of even dimension admits an invariant almost complex structure. In [7], this result is generalized by showing that for any compact Lie group G there exits an integer k with $0 \leq k \leq \max(3, \mathrm{rk}\, G)$ such that $U(1)^k \times G$ admits an invariant hypercomplex structure. In particular, the Lie groups $U(n)$ and $SU(2n+1)$ admit invariant hypercomplex structures. As far as homogeneous spaces are concerned the result of [19] states that if H is a closed, connected subgroup of a simply-connected compact semisimple Lie group G such that G/H is even-dimensional, and semisimple part of H coincides with the semisimple part of the centralizer of a toral subgroup of K, then G/H admits an invariant complex structure. Joyce in [7] generalized this result by proving the following: let G be a compact Lie group and H an E - subgroup of length j (see [7] for definition), if K is the semisimple part of G, and F is any closed subgroup of G such that $K \subset F \subset H$, then there exists an integer k with $0 \leq k \leq \max(3, j)$ such that $U(1)^k \times G/K$ admits an invariant hypercomplex structure, that is, one that is preserved by the left action of $U(1)^k \times G$. In particular, it follows that the homogeneous spaces $U(2k+l)/U(l)$ are examples of compact homogeneois spaces which admit invariant hypercomplex structures.

We describe the constraints for the existence of invariant almost hypercomplex structure on a compact homogeneous space in terms of root theory and characteristic classes. These constraints distinguish a large class of homogeneous spaces which do not admit such structures.

2. ROOT THEORY AND INVARIANT HYPERCOMPLEX STRUCTURE ON A HOMOGENEOUS SPACE

Our main result is the following.

Theorem 2.1. *If a homogeneous space G/H, where G is a compact Lie group and H is a closed connected subgroup, admits an invariant almost hypercomplex structure, then there are no non-zero complementary roots for G related to H.*

The classical result of Hopf and Samelson [6] states that the Euler characteristic of a compact homogeneous space can not be negative. Thus, we provide the proof of the stated theorem by Proposition 2.1 and Proposition 2.2, which treat separately homogeneous spaces of non-zero and zero Euler characteristic. For the explanation of the notion of a complementary root we follow [4] and, in both considered cases of homogeneous spaces, it will be given in the proofs of these propositions.

For the background on Lie theory we use [10], [11]. Let G be a compact connected Lie group and T a maximal compact torus. Denote by $\mathfrak{g}$ the Lie algebra for G and by $\mathfrak{t}$ the Lie algebra for T. The algebra $\mathfrak{t}$ is called the Cartan algebra for $\mathfrak{g}$. The roots α_i for G with respect to T are defined to be the weights for the adjoint representation $Ad_G T$ in $\mathfrak{g}$ linearly extended to the complex vector space $\mathfrak{g}(\mathbb{C})$. The corresponding weight subspaces $\mathfrak{g}_i(\mathbb{C})$ for α_i are called the root subspaces, that is $\mathfrak{g}_i(\mathbb{C}) = \{V \in \mathfrak{g}\,|Ad_G(t)V = \alpha_i(t)(V) \text{ for all } t \in T\}$, and $\dim_{\mathbb{C}} \mathfrak{g}_i = 1$, that is $\dim_{\mathbb{R}} \mathfrak{g}_i = 2$. Then, the following root decomposition holds

$$\mathfrak{g}(\mathbb{C}) = \mathfrak{t}(\mathbb{C}) \oplus \big(\oplus\, \mathfrak{g}_{\alpha_i}(\mathbb{C})\big).$$

The roots α_i can be considered as linear forms on $\mathfrak{t}$ and they are completely defined by the Lie algebra $\mathfrak{g}$.

A similar decomposition can be done if one considers an arbitrary toral subgroup S of G. The adjoint representation $Ad_G S$ of S in $\mathfrak{g}$ is fully reducible and there is a decomposition

$$\mathfrak{g}(\mathbb{C}) = \mathfrak{s}(\mathbb{C}) \oplus \mathfrak{b}(\mathbb{C}) \oplus \big(\oplus\, \mathfrak{g}_{\beta_i}(\mathbb{C})\big)$$

into subspaces invariant for $Ad_G S$, where $\mathfrak{s}$ it the Lie algebra for S and $\dim_{\mathbb{R}} \mathfrak{g}_{\beta_i}(\mathbb{C}) = 2$. In addition $\mathfrak{s} \oplus \mathfrak{b}$ is the largest subspace in $\mathfrak{g}$ on which S acts trivially. Note that $\mathfrak{t} \subset \mathfrak{s} \oplus \mathfrak{b}$, that is $\mathfrak{s} \oplus \mathfrak{b} = \mathfrak{t} \oplus \tilde{\mathfrak{b}}$ for $\tilde{\mathfrak{b}} \subset \mathfrak{b}$. The forms β_i are non-zero integral forms on $\mathfrak{s}$, that is the weights for the representation $Ad_G S$ and they are known as the roots for G with respect to S.

2.1. Homogeneous spaces of positive Euler characteristic

Proposition 2.1. *If the Euler characteristic $\chi(G/H)$ is positive, where G/H is a homogeneous space of a compact Lie group G by a closed connected subgroup H, then G/H does not admit any invariant almost hypercomplex structure.*

Proof. Assume that $\chi(G/H) > 0$, then $\mathrm{rk}\, G = \mathrm{rk}\, H$, which implies that G and H have the common maximal torus T. The roots $\alpha_1, \ldots, \alpha_n$ for G with respect to T

can be chosen such that $\alpha_1, \dots, \alpha_k$ are the roots for H with respect to T. Then $\alpha_{k+1}, \dots, \alpha_n$ are known in the literature [4] as the complementary roots for G related to H. Note that in this case there always exists a complementary root, since otherwise, having the same root decomposition, the Lie algebras for G and H would coincide. In addition,

$$T_e(G/H) = \mathfrak{g}_{k+1} \oplus \cdots \oplus \mathfrak{g}_n$$

holds. An invariant almost complex structure J on G/H is defined by the complex structure on $T_e(G/H)$ which is invariant under the isotropy representation i_H for H in $T_e(G/H)$. This means that $J \circ i_H(h) = i_H(h) \circ J$ for any $h \in H$. The isotropy representation i_H is known to be equivalent to the representation $Ad_G H$. It implies that J commutes with the adjoint representation $Ad_G T$ of the maximal torus T. On the other hand, since any $\mathfrak{g}_i$ is an invariant subspace for the representation $Ad_G T$ and J commutes with $Ad_G T$, it follows that J induces the complex structure on a root subspace $\mathfrak{g}_i$. We know that $\dim_{\mathbb{R}} \mathfrak{g}_i = 2$, so there are only two complex structures on $\mathfrak{g}_i$ which differ by conjugation. Therefore, we deduce that if G/H would admit an invariant almost hypercomplex structure, there would exist on $\mathfrak{g}_i$ two complex structures I and J which satisfy $IJ = -JI$. This would further imply that $\dim_{\mathbb{R}} \mathfrak{g}_i = 4k$, which is not the case. □

The previous statement immediately implies:

Corollary 2.1. *If a homogeneous space G/H admits an invariant almost hypercomplex structure then $\chi(G/H) = 0$.*

Corollary 2.2. *A homogeneous space G/H with $\operatorname{rk} G = \operatorname{rk} H$ does not admit any invariant almost hypercomplex structure.*

2.2. **Homogenous spaces of zero Euler characteristic**

We consider now a homogeneous space G/H such that $\chi(G/H) = 0$ or equivalently that $\operatorname{rk} G > \operatorname{rk} H$.

Proposition 2.2. *Let G/H be a compact homogeneous space of zero Euler characteristic. If there exists a non-zero complementary root for G related to H then G/H admits no invariant almost hypercomplex structure.*

Proof. Let S be a maximal torus for H and T a maximal torus for G such that $S \subset T$, then $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras for G and H, and $\mathfrak{s}$ and $\mathfrak{t}$ the Lie algebras for S and T. Let β_i, $1 \leq i \leq m$ be the roots for G with respect to S. It is the classical result [4] that one can choose the roots α_j for G with respect to T in such a way that the roots β_i coincide with the non-zero forms on $\mathfrak{s}$ obtained by the restriction of α_j from $\mathfrak{t}$ to $\mathfrak{s}$.

The roots for H with respect to S we denote by β_i^H, $1 \leq i \leq k$. The roots β_i for G with respect to S can be chosen [4] so that they contain the roots β_i^H. Those roots among β_i which are not the roots for H are known as the complementary roots for G related to H. We denote these complementary roots by γ_i, $k+1 \leq i \leq m$. Since

$$\mathfrak{g} = \mathfrak{t} \oplus \tilde{\mathfrak{b}} \oplus (\oplus_{i=1}^{m} \mathfrak{g}_{\beta_i}), \ \ \mathfrak{h} = \mathfrak{s} \oplus (\oplus_{i=1}^{k} \mathfrak{h}_{\beta_i^H}),$$

it follows that

$$T_e(G/H) = \mathfrak{g}\,/\,\mathfrak{h} = (\mathfrak{t} \oplus \tilde{\mathfrak{b}} \oplus \left(\oplus_{i=1}^{m} \mathfrak{g}_{\beta_i}\right))/(\mathfrak{s} \oplus \left(\oplus_{i=1}^{k} \mathfrak{h}_{\beta_i^H}\right)),$$

that is

$$T_e(G/H) = \tilde{\mathfrak{t}} \oplus \tilde{\mathfrak{b}} \oplus (\oplus_{i=1}^{k+1} \mathfrak{g}_{\gamma_i}),$$

where $\tilde{\mathfrak{t}} \subset \mathfrak{t}$ is of dimension $\mathrm{rk}\, G - \mathrm{rk}\, H$. Now, an invariant almost complex structure on G/H induces complex structure on $T_e(G/H)$ which is invariant under the isotropy representation i_H for H at $T_e(G/H)$. It is then $Ad_G S$ invariant and it induces the complex structure on each $\mathfrak{g}_{\gamma_i}$. Thus, if there exists a complementary root γ_i for G related to H then any invariant almost hypercomplex structure would induce hypercomplex structure on $\mathfrak{g}_{\gamma_i}$, which is impossible. □

Corollary 2.3. *A homogeneous space G/S, where S is a toral subgroup of G does not admit any invariant almost hypercomplex structure.*

Proof. The complementary roots for G with respect to S are the roots for G with respect to S. If $\mathfrak{s}$ is the Lie algebra for S and $\mathfrak{t}$ the maximal abelian subalgebra for $\mathfrak{g}$ such that $\mathfrak{s} \subset \mathfrak{t}$, then these complementary roots are obtained by the restricting to $\mathfrak{s}$ the roots for $\mathfrak{g}$ related to $\mathfrak{t}$. Since there is no subspace in $\mathfrak{t}$ on which all roots for $\mathfrak{g}$ vanish, it follows that there exists a non-zero complementary root. Thus, Theorem 2.1 implies non-existence of an invariant almost hypercomplex structure. □

3. CHARACTERISTIC CLASSES OF HYPERCOMPLEX STRUCTURE ON A HOMOGENEOUS SPACE

The real Cartan algebra (C,d) of G/H has the form $C = H^*(B_H) \otimes H^*(G)$. We recall some facts from [3] about the structures of $H^*(B_H)$ and $H^*(G)$. The cohomology algebra of the classifying space B_H is the algebra of the Weil invariant polynomials on $\mathfrak{s}$. Also $H^*(G)$ is the exterior algebra over universal transgressive elements. The differential in the Cartan algebra is defined by $d(b \otimes 1) = 0$, $b \in H^*(B_H)$ and $d(1 \otimes z) = \rho^*(P) \otimes 1$. Here P is a polynomial from the cohomology algebra $H^*(B_G)$ which corresponds to z by the transgression in the universal bundle for G. We denote by ρ^* the restriction of the Weil invariant polynomial algebra $\mathbb{R}[\mathfrak{t}]^{W_G}$ from $\mathfrak{t}$ to $\mathfrak{s}$. The image of this restriction belongs to the Weil invariant polynomial algebra $\mathbb{R}[\mathfrak{s}]^{W_H}$, see [3].

Let γ_i be the complementary roots for G related to H. An invariant almost complex structure on G/H is completely determined by its root system, which is given by $\varepsilon_i\gamma_i$ for $\varepsilon_i = \pm 1$, see [4]. Denote

$$\bar{c} = \prod(1 + (\varepsilon_i\gamma_i)).$$

The total Chern class for G/H which corresponds to an invariant almost complex structure J can be, according to [4], computed by

$$c(G/H, J)) = (\bar{c} \bmod \rho^*(H^*(B_G))) \otimes 1. \tag{3.1}$$

Now, let

$$\bar{p} = \prod(1 + \gamma_i^2).$$

The total Pontrjagin class for G/H can be, according to [18], computed by

$$p(G/H) = (\bar{p} \bmod \rho^*(H^*(B_G))) \otimes 1. \tag{3.2}$$

Proposition 3.1. *If G/H admits an invariant almost hypercomplex structure, then:*

(1) *the total Pontrjagin class for G/H is trivial, that is $p(G/H) = 1$;*
(2) *the total Chern class is trivial, that is $c(G/H, J) = 1$ for any invariant almost complex structure J on G/H.*

Proof. Since G/H admits an invariant almost hypercomplex structure, Theorem 2.1 implies that $\gamma_i = 0$ for all i. Thus, it follows from (3.2) that $p(G/H) = 1$ and from (3.1) that $c(G/H) = 1$.

□

Remark 3.1. It follows from the proof of (3.2) given in [18] that, if $\bar{p}$ is contained in the image of the restriction $\rho^* : \mathbb{R}[\mathfrak{t}]^{W_G} \to \mathbb{R}[\mathfrak{s}]^{W_H}$ then the total Pontrjagin class of G/H is trivial. Using this it is proved in [17] that some generalized symmetric spaces as defined in [16] have trivial total Pontrjagin class. More precisely, written in the notation of the corresponding simple Lie algebras, it is proved that

$$p(A_{2n}/C_n) = p(A_{2n}/B_n) = 1,\ p(A_{2n-1}/D_n) = p(A_{2n-1}/C_n) = 1,$$
$$p(D_4/T^2) = p(D_4/G_2) = 1,\ p(E_6/T^4) = p(E_6/F_4) = 1.$$

Remark 3.2. Recall that for a homogeneous space G/H, the subgroup H is said to be totally non cohomologus to zero in the group G if the restriction $\rho^* : \mathbb{R}[\mathfrak{t}]^{W_G} \to \mathbb{R}[\mathfrak{s}]^{W_H}$ is surjective. It follows that for these spaces $\bar{p}$ is contained in the image of the map $\rho^* : \mathbb{R}[\mathfrak{t}]^{W_G} \to \mathbb{R}[\mathfrak{s}]^{W_H}$, so by Remark 3.1 the total Pontrjagin class $p(G/H)$ is trivial. In particular, it implies that $p(G/S) = 1$ for all homogeneous spaces G/S, where S it a toral subgroup in G.

Remark 3.3. The condition on vanishing of the total Pontrjagin class of a homogeneous space is not, of course, sufficient for the existence of an invariant almost hypercomplex structure. This is verified by the examples of homogeneous spaces G/S, where S is a toral subgroup for G, by making use of Corollary 2.3 and Remark 3.2.

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Svjetlana Terzić
University of Montenegro
Faculty of Science and Mathematiucs
Džordža Vašingtona bb
81000 Podgorica
Montenegro
e-mail: *sterzic@ucg.ac.me*