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$\mathcal{I}_2$-ASYMPTOTICALLY LACUNARY STATISTICAL EQUIVALENCE OF WEIGHT g OF DOUBLE SEQUENCES OF SETS

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Abstract. In this paper, our aim is to introduce new notions, namely, Wijsman asymptotically $\mathcal{I}_2$-statistical equivalence of weight g, Wijsman strongly asymptotically $\mathcal{I}_2$-lacunary equivalence of weight g and Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalence of weight g of double set sequences. We mainly investigate their relationship and also make some observations about these classes.

1. Introduction

Theory of statistical convergence was firstly originated by Fast [6]. This concept was extended to the double sequences by Mursaleen and Edely [14]. Lacunary statistical convergence was defined by Fridy and Orhan [7]. Çakan and Altay [3] presented multidimensional analogues of the results presented by Fridy and Orhan [7].

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [11] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers. Recently, Das et al. [5] introduced new notions, namely $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence by using ideal. The notion of lacunary ideal convergence of real sequences was introduced in [23].

Das, Koystrko, Wilczynski and Malik [4] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. Belen et al. [2] introduced the notion of ideal statistical convergence of double sequences, which is a new generelization of the notions of statistical convergence and usual convergence. Kumar et al. [12] introduced $\mathcal{I}$-lacunary statistical convergence of double sequences.

Nuray and Rhoades [15] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [24] defined the Wijsman lacunary statistical convergence of sequence

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of sets and considered its relation with Wijsman statistical convergence. Kişi and Nuray [8] introduced a new convergence notion, for sequences of sets, which is called Wijsman $\mathcal{I}$-convergence by using ideal. Recently, Ulusu and Dündar [26] studied the concepts of Wijsman $\mathcal{I}$-statistical convergence, Wijsman $\mathcal{I}$-lacunary statistical convergence and Wijsman strongly $\mathcal{I}$-lacunary convergence of sequences of sets.

Nuray et al. [16] studied Wijsman statistical convergence, Hausdorff statistical convergence and Wijsman statistical Cauchy double sequences of sets and investigate the relationship between them. Kişi [10] introduced the concepts of the Wijsman $\mathcal{I}_2$-statistical convergence, Wijsman $\mathcal{I}_2$-lacunary statistical convergence and Wijsman strongly $\mathcal{I}_2$-lacunary convergence of double sequences of sets and investigate the relationship between them.

Asymptotic equivalence of sequences was introduced by Pobyvanets [18]; Marouf's work [13] was extension of Pobyvanets's work. In 2003, Patterson [19] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices.

In [17] asymptotically lacunary statistical equivalent which is a natural combination of the definitions for asymptotically equivalent, statistical convergence and lacunary sequences was studied.

Also in [20], $\mathcal{I}$-asymptotically statistical equivalent and $\mathcal{I}$-asymptotically lacunary statistical equivalent sequences were examined.

The concept of asymptotically equivalence of sequences of real numbers which is defined by Marouf [13] has been extended by Ulusu and Nuray [25] to concept of Wijsman asymptotically equivalence of set sequences. In addition to these definitions, natural inclusion theorems are presented. Kişi et al. [9] introduced the concept of Wijsman $\mathcal{I}$-asymptotically equivalence of sequences of sets.

Now, we recall the basic definitions and concepts.

The upper density of weight g was defined in [1] by the formula

$$\overline{d}_g(A) = \lim_{n\to\infty} \sup \frac{A(1,n)}{g(n)}$$

for $A \subseteq \mathbb{N}$ where as before $A(1,n)$ denotes the cardinality of the set $A \cap [1,n]$. Then, the family

$$\mathcal{I}_g = \left\{ A \subseteq \mathbb{N} : \overline{d}_g(A) = 0 \right\}$$

forms an ideal. It has been observed in [1] that $\mathbb{N} \in \mathcal{I}_g$ if and only if $\frac{n}{g(n)} \to 0$ as $n \to \infty$. So we additionally assume that $\frac{n}{g(n)} \nrightarrow 0$ as $n \to \infty$ so that $\mathbb{N} \notin \mathcal{I}_g$ and $\mathcal{I}_g$ is a proper admissible ideal of $\mathbb{N}$. The collection of all such weight functions g satisfying the above properties will be denoted by G. As a natural consequence we can introduce the following definition.

Definition 1.1. ([1]) A sequence $\{x_n\}$ of real numbers is said to d_g-statistically convergent to x if for any given $\varepsilon > 0$, $\overline{d}_g(A_\varepsilon) = 0$, where A_ε is the set defined in Definition 1.3.

Savaş [21] introduced new notions, namely, $\mathcal{I}$-statistical double convergence of weight g and $\mathcal{I}$-lacunary double statistical convergence of weight g and investigated their relationship and also make some observations about these classes.

A double sequence $x = (x_{k,l})$ has a Pringsheim limit L (denoted by $P - \lim x = L$) provided that given an $\varepsilon > 0$, there exists a $n \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$, whenever $k, l > n$. We describe such an $x = (x_{k,l})$ more briefly as "P-convergent".

The double sequence $\{A_{k,l}\}$ is Wijsman convergent to A, if for each $x \in X$

$$P - \lim_{k,l\to\infty} d(x, A_{k,l}) = d(x, A) \quad \text{or} \quad \lim_{k,l\to\infty} d(x, A_{k,l}) = d(x, A).$$

In this case we write $W_2 - \lim A_{k,l} = A$.

We define $d(x; A_{kj}, B_{kj})$ as follows:

$$d(x; A_{kj}, B_{kj}) = \begin{cases} \dfrac{d(x, A_{kj})}{d(x, B_{kj})} & , \text{ if } x \notin A_{kj} \cup B_{kj} \\ L & , \text{ if } x \in A_{kj} \cup B_{kj} \end{cases}$$

The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically equivalent of multiple L if every $\varepsilon > 0$, for each $x \in X$, $\lim_{k,j\to\infty} d(x; A_{kj}, B_{kj}) = L$.

The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are said to be asymptotically statistical equivalent of multiple L if every $\varepsilon > 0$, for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \left|\{k \le m, j \le n : |d(x; A_{kj}, B_{kj}) - L| \ge \varepsilon\}\right| = 0.$$

Throughout the paper, we shall denoted by $\mathcal{I}_2$ be an admissible ideal of $\mathbb{N}$ x $\mathbb{N}$.

A double sequence $\overline{\theta} = \theta_{ru} = \{(k_r, j_u)\}$ is called double lacunary sequence if there exist two increasing sequences of integers (k_r) and (j_u) such that

$$\begin{aligned} k_0 &= 0,\ h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty \\ j_0 &= 0,\ \overline{h}_u = j_u - j_{u-1} \to \infty, \text{ as } u \to \infty. \end{aligned}$$

We will use the following notation $k_{ru} := k_r j_u$, $h_{ru} := h_r \overline{h}_u$ and θ_{ur} is determined by

$$I_{ur} := \{(k, j) : k_{r-1} < k \le k_r \text{ and } j_{u-1} < j \le j_u\},$$

$$q_r := \frac{k_r}{k_{r-1}},\ \overline{q}_u := \frac{j_u}{j_{u-1}} \text{ and } q_{ru} := q_r \overline{q}_u.$$

Throughout the paper, by $\theta_2 = \theta_{ru} = \{(k_r, j_u)\}$ we will denote a double lacunary sequence of positive real numbers, respectively, unless otherwise stated.

Let θ_{ru} be a double lacunary sequence and $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ be a non-trivial ideal.

Definition 1.2. ([27]) The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically $\mathcal{I}_2$-equivalent of multiple L if every $\varepsilon > 0$, for each $x \in X$,

$$\{(k,j) \in \mathbb{N} \times \mathbb{N} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case we write $A_{kj} \overset{(\mathcal{I}_{W2}^L)}{\sim} B_{kj}$ and simply Wijsman asymptotically $\mathcal{I}_2$-equivalent if $L = 1$.

2. Main Results

Asymptotically $\mathcal{I}_2$-lacunary statistical equivalence of double sequnces of sets was studied by Ulusu and Dündar [27]. It is natural question that whether this concept will be work for Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalence of weight g. In this paper, we gave some answers of this question and also we prove that Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalence a better tool than Wijsman asymptotically lacunary statistical equivalence.

In this section, we define the concepts of Wijsman asymptotically $\mathcal{I}_2$-statistical equivalence of weight g, Wijsman strongly asymptotically $\mathcal{I}_2$-lacunary equivalence of weight g and Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalence of weight g of double sequences of sets and investigate the relationship between them.

Definition 2.1. The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically $\mathcal{I}_2$-statistical equivalent of weight g of multiple L if for every $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{g(mn)} |\{k \leq m, j \leq n : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \overset{S(\mathcal{I}_{W2}^L)^g}{\sim} B_{kj}$ and simply Wijsman asymptotically $\mathcal{I}_2$-statistical equivalent of weight g equivalent if $L = 1$. The set of Wijsman asymptotically $\mathcal{I}_2$-statistical equivalent double sequences of weight g will be denoted $\left\{S\left(\mathcal{I}_{W2}^L\right)^g\right\}$.

For $\mathcal{I}_2 = \mathcal{I}_2^f$, Wijsman asymptotically $\mathcal{I}_2$-statistical equivalence of weight g of multiple L coincides with Wijsman asymptotically statistical equivalence of multiple L.

As an example, consider the following double sequences;

$$A_{kj} = \begin{cases} \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 + 2kjy = 0\right\}, & \text{if } k \text{ and } j \text{ are a square integer} \\ \{1,1\} & \text{,otherwise,} \end{cases}$$

and

$$B_{kj} = \begin{cases} \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - 2kjy = 0\right\}, & \text{if } k \text{ and } j \text{ are a square integer} \\ \{1,1\} & \text{,otherwise.} \end{cases}$$

If we take $\mathcal{I}_2 = \mathcal{I}_2^f$, since

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{g(mn)} \left|\{k \le m, j \le n : |d(x; A_{kj}, B_{kj}) - L| \ge \varepsilon\}\right| \ge \delta\right\} \in \mathcal{I}_2^f,$$

then the double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically $\mathcal{I}_2$-statistical equivalent.

Definition 2.2. Let $\theta_2 = \{\theta_{ru}\}$ be a double lacunary sequence. The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalent of weight g of multiple L if for $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{g\left(h_r \overline{h_u}\right)} \left|\{(k,j) \in I_{ru} : |d\left(x; A_{kj}, B_{kj}\right) - L| \ge \varepsilon\}\right| \ge \delta\right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \overset{S_\theta\left(\mathcal{I}_{W2}^L\right)^g}{\sim} B_{kj}$ and simply Wijsman $\mathcal{I}_2$-asymptotically lacunary statistical equivalent of weight g if $L = 1$. The set of Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalent double sequences of weight g will be denoted $\left\{S_\theta\left(\mathcal{I}_{W2}^L\right)^g\right\}$.

For $\mathcal{I}_2 = \mathcal{I}_2^f$, Wijsman $\mathcal{I}_2$-asymptotically lacunary statistical equivalence of weight g of multiple L coincides with Wijsman asymptotically lacunary statistical equivalence of weight g of multiple L.

Definition 2.3. Let $\theta_2 = \{\theta_{ru}\}$ be a double lacunary sequence. The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman strongly asymptotically $\mathcal{I}_2$-lacunary equivalent of weight g of multiple L provided that for every $\varepsilon > 0$, for each $x \in X$,

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{g\left(h_r \overline{h_u}\right)} \sum_{(k,j)\in I_{ru}} |d\left(x; A_{kj}, B_{kj}\right) - L| \ge \varepsilon\right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \overset{N_\theta\left[\mathcal{I}_{W2}^L\right]^g}{\sim} B_{kj}$ and simply Wijsman strongly $\mathcal{I}_2$-asymptotically lacunary equivalent of weight g if $L = 1$.

Theorem 2.4. *Let θ_2 be a lacunary sequence. Then,*

$$A_{kj} \overset{N_\theta\left[\mathcal{I}_{W2}^L\right]^g}{\sim} B_{kj} \Rightarrow A_{kj} \overset{S_\theta\left(\mathcal{I}_{W2}^L\right)^g}{\sim} B_{kj},$$

and $A_{kj} \overset{N_\theta[\mathcal{I}_{W2}^L]^g}{\sim} B_{kj}$ *is proper subset of* $A_{kj} \overset{S_\theta(\mathcal{I}_{W2}^L)^g}{\sim} B_{kj}$.

Proof. Suppose that $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman strongly asymptotically $\mathcal{I}_2$-lacunary equivalent of weight g of multiple L. Given $\varepsilon > 0$ and for each $x \in X$ we can write

$$\begin{aligned}\sum_{(k,j)\in I_{ru}} |d(x; A_{kj}, B_{kj}) - L| \geq & \sum_{\substack{(k,j)\in I_{ru} \\ |d(x;A_{kj},B_{kj})-L|\geq\varepsilon}} |d(x; A_{kj}, B_{kj}) - L| \\ \geq & \ \varepsilon \cdot |\{(k,j) \in I_{ru} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}|\end{aligned}$$

and so we get

$$\begin{aligned}&\frac{1}{\varepsilon g\left(h_r\overline{h_u}\right)} \sum_{(k,j)\in I_{ru}} |d(x; A_{kj}, B_{kj}) - L| \\ &\geq \frac{1}{g\left(h_r\overline{h_u}\right)} |\{(k,j) \in I_{ru} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}|.\end{aligned}$$

Hence, for each $x \in X$ and for any $\delta > 0$, we have

$$\begin{aligned}&\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{g\left(h_r\overline{h_u}\right)} |\{(k,j) \in I_{ru} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}| \geq \delta\right\} \\ &\subseteq \left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{g\left(h_r\overline{h_u}\right)} \sum_{(k,j)\in I_{ru}} |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon.\delta\right\} \in \mathcal{I}_2.\end{aligned}$$

Hence we have $A_{kj} \overset{S_\theta(\mathcal{I}_{W2}^L)^g}{\sim} B_{kj}$.

Now, let $\{A_{kj}\}$ be defined as follows:

$$A_{kj} := \begin{pmatrix} \{1\} & \{2\} & \{3\} & ... & \left\{\left[\sqrt[3]{h_{ru}}\right]\right\} & \{0\} & ... \\ \{2\} & \{2\} & \{3\} & ... & \left\{\left[\sqrt[3]{h_{ru}}\right]\right\} & \{0\} & ... \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \{2\} & \left[\sqrt[3]{h_{ru}}\right] & ... & ... & \left\{\left[\sqrt[3]{h_{ru}}\right]\right\} & \{0\} & ... \\ \{0\} & \{0\} & \{0\} & \{0\} & \{0\} & \{0\} & ... \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}.$$

For any $\varepsilon > 0$,

$$\frac{1}{g\left(h_r\overline{h_u}\right)} |\{(k,j) \in I_{ru} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}| \leq \frac{\left[\sqrt[3]{h_{ru}}\right]}{g\left(h_r\overline{h_u}\right)}$$

and consequently for any $\delta > 0$, we get

$$\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g\left(h_r\overline{h_u}\right)}\left|\left\{(k,j)\in I_{ru}:\left|d\left(x;A_{kj},B_{kj}\right)-0\right|\geq\varepsilon\right\}\right|\geq\delta\right\}$$
$$\subseteq\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{\left[\sqrt[3]{h_{ru}}\right]}{g\left(h_r\overline{h_u}\right)}\geq\delta\right\}.$$

Note that the set on the right hand side is a finite set and so is a member of $\mathcal{I}_2$. Thus $A_{kj}\overset{S_\theta(\mathcal{I}_{W2})^g}{\sim}B_{kj}$ for $L=0$. Again observe that

$$\frac{1}{g\left(h_r\overline{h_u}\right)}\sum_{(k,j)\in I_{ru}}\left|d\left(x;A_{kj},B_{kj}\right)-0\right|=\frac{1}{g\left(h_r\overline{h_u}\right)}\frac{\sqrt[3]{h_{ru}}\left(\left[\sqrt[3]{h_{ru}}\right]\left(\left[\sqrt[3]{h_{ru}}\right]+1\right)\right)}{2}.$$

Hence

$$\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g\left(h_r\overline{h_u}\right)}\sum_{(k,j)\in I_{ru}}\left|d\left(x;A_{kj},B_{kj}\right)-0\right|\geq\frac{1}{4}\right\}$$
$$=\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{\sqrt[3]{h_{ru}}\left(\left[\sqrt[3]{h_{ru}}\right]\left(\left[\sqrt[3]{h_{ru}}\right]+1\right)\right)}{2g\left(h_r\overline{h_u}\right)}\geq\frac{1}{2}\right\}$$

which evidently belongs to $\mathcal{F}(\mathcal{I})$ as $\mathcal{I}$ is admissible. Therefore $A_{kj}\overset{N_\theta[\mathcal{I}_{W2}]^g}{\nsim}B_{kj}$ for $L=0$. □

Theorem 2.5. *Let θ_2 be a double lacunary sequence and $d\left(x,A_{kj}\right)\ \mathcal{O}\left(d\left(x,B_{kj}\right)\right)$. Then,*

$$A_{kj}\overset{S_\theta\left(\mathcal{I}_{W2}^L\right)^g}{\sim}B_{kj}\Rightarrow A_{kj}\overset{N_\theta\left[\mathcal{I}_{W2}^L\right]^g}{\sim}B_{kj}.$$

Proof. Suppose that $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalent of weight g of multiple L and $d\left(x,A_{kj}\right)\mathcal{O}\left(d\left(x,B_{kj}\right)\right)$. Then there is a $M>0$ such that

$$\left|d\left(x;A_{kj},B_{kj}\right)-L\right|\leq M$$

for each $x\in X$ and all $k,j\in\mathbb{N}$. Given $\varepsilon>0$, for each $x\in X$ we get

$$\frac{1}{g\left(h_r\overline{h_u}\right)}\sum_{(k,j)\in I_{ru}}\left|d\left(x;A_{kj},B_{kj}\right)-L\right|=$$
$$=\frac{1}{g\left(h_r\overline{h_u}\right)}\sum_{\substack{(k,j)\in I_{ru}\\ \left|d\left(x;A_{kj},B_{kj}\right)-L\right|\geq\frac{\varepsilon}{2}}}\left|d\left(x;A_{kj},B_{kj}\right)-L\right|$$
$$+\frac{1}{g\left(h_r\overline{h_u}\right)}\sum_{\substack{(k,j)\in I_{ru}\\ \left|d\left(x;A_{kj},B_{kj}\right)-L\right|<\frac{\varepsilon}{2}}}\left|d\left(x;A_{kj},B_{kj}\right)-L\right|$$
$$\leq\frac{M}{g\left(h_r\overline{h_u}\right)}\left|\left\{(k,j)\in I_{ru}:\left|d\left(x;A_{kj},B_{kj}\right)-L\right|\geq\frac{\varepsilon}{2}\right\}\right|+\frac{\varepsilon}{2}.$$

Hence, for each $x \in X$ we have

$$\left\{ (r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{g\left(h_r\overline{h_u}\right)} \sum_{(k,j)\in I_{ru}} \left|d\left(x;A_{kj},B_{kj}\right) - L\right| \geq \varepsilon \right\}$$
$$\subseteq \left\{ (r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{g\left(h_r\overline{h_u}\right)} \left|\left\{ (k,j)\in I_{ru} : \left|d\left(x;A_{kj},B_{kj}\right) - L\right| \geq \frac{\varepsilon}{2} \right\}\right| \geq \frac{\varepsilon}{2M} \right\} \in \mathcal{I}_2.$$

Therefore, $A_{kj} \overset{N_\theta\left[\mathcal{I}_{W2}^L\right]^g}{\sim} B_{kj}$. This completes the proof. □

Theorem 2.6. *For any double lacunary sequence θ_2, Wijsman asymptotically $\mathcal{I}_2$-statistical equivalence of weight g implies Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalence of weight g if*

$$\liminf_r \frac{g\left(h_r\overline{h_u}\right)}{g\left(k_{ru}\right)} > 1.$$

Proof. Since $\liminf_r \frac{g\left(h_r\overline{h_u}\right)}{g\left(k_{ru}\right)} > 1$, so we can find a $H > 1$ such that for sufficiently large r, u we have $\frac{g\left(h_r\overline{h_u}\right)}{g\left(k_{ru}\right)} \geq H$.

Since $A_{kj} \overset{S\left(\mathcal{I}_{W2}^L\right)^g}{\sim} B_{kj}$, for every $\varepsilon > 0$ and sufficiently large r, u we have

$$\begin{aligned}
&\frac{1}{g\left(k_{ru}\right)} \left|\left\{ k \leq k_r \text{ and } j \leq j_u : \left|d\left(x;A_{kj},B_{kj}\right) - L\right| \geq \varepsilon \right\}\right| \\
&\geq \frac{1}{g\left(k_{ru}\right)} \left|\left\{ (k,j) \in I_{ru} : \left|d\left(x;A_{kj},B_{kj}\right) - L\right| \geq \varepsilon \right\}\right| \\
&\geq H \frac{1}{g\left(h_r\overline{h_u}\right)} \left|\left\{ (k,j) \in I_{ru} : \left|d\left(x;A_{kj},B_{kj}\right) - L\right| \geq \varepsilon \right\}\right|.
\end{aligned}$$

Then, for any $\delta > 0$, we get

$$\left\{ (r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{g\left(h_r\overline{h_u}\right)} \left|\left\{ (k,j) \in I_{ru} : \left|d\left(x;A_{kj},B_{kj}\right) - L\right| \geq \varepsilon \right\}\right| \geq \delta \right\}$$
$$\subseteq \left\{ (r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{g\left(k_{ru}\right)} \left|\left\{ k \leq k_r, j \leq j_u : \left|d\left(x;A_{kj},B_{kj}\right) - L\right| \geq \varepsilon \right\}\right| \geq H\delta \right\} \in \mathcal{I}_2.$$

This shows that $A_{kj} \overset{S_\theta\left(\mathcal{I}_{W2}^L\right)^g}{\sim} B_{kj}$. □

It is kwown that double lacunary statistical convergence implies double statistical convergence if and only if $\lim_{rs} \sup q_{rs} < \infty$ (see ([22])). However for arbitrary admissible ideal $\mathcal{I}_2$, this is not clear and we leave it as an open problem.

When Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalence of weight g implies Wijsman asymptotically $\mathcal{I}_2$-statistical equivalence of weight g?

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