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DESSINS D'ENFANTS ON REDUCIBLE SURFACES

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To Mirjana Vukovič on the occasion of her 75th birthday

Abstract. In this paper we introduce dessins d'enfants on unions of surfaces, possibly glued. We show why they are natural, discuss their relations with Belyi pairs on reducible curves and provide some examples. In particular, we provide an example of a Fried pair which degenerates to a dessin on a reducible and singular curve.

1. Introduction

A *dessin d'enfant* is an embedded graph Γ on a compact smooth oriented surface M without boundary such that the complement $M \setminus \Gamma$ is homeomorphic to a disjoint union of open discs. This notion was introduced by Alexander Grothendieck in [10] who observed that these graphs have a simple (and very natural) combinatorial structure and connect various different branches of mathematics. Grothendieck put forward the systematic investigation of the correspondence between these dessins d'enfants and algebraic curves together with non-constant rational functions with at most 3 critical values on these curves, so-called *Belyi pairs*. This correspondence gives rise to plenty of intersections between different structures in category theory, algebra, algebraic geometry, complex analysis, topology, mathematical physics, etc. Since Grothendieck's time this theory was intensively developed, see for example [1, 5, 7, 8, 11, 12, 15, 20, 21, 22, 23] and references therein. The origins of the theory can be found in the special volumes [14, 16] and the modern development of the theory and its numerous applications are described in the detailed and self-contained surveys [2, 13].

The main goal of the present paper is to motivate and construct an analog of Grothendieck theory for graphs embedded into unions of surfaces, possibly glued, and to provide natural examples of such graphs and corresponding Belyi pairs. One of the ways to determine a Belyi pair is to consider it as a degeneration of a rational function with 4 critical values on the same

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algebraic curve appearing if two of the critical values converge, see [3, 13]. Although such pairs of an algebraic curve with a rational function on it with at most 4 critical values are intensively investigated, see [18, 19] and references therein, and even have a special name, *Fried pairs*, after Mike Fried, see also Def. 5.1 in this paper, their relations with Belyi pairs are far from being well-understood. Here we propagate the systematic approach to this subject based on the investigation of generalized dessins d'enfants and Belyi pairs on reducible curves over the field of complex numbers $\mathbb{C}$.

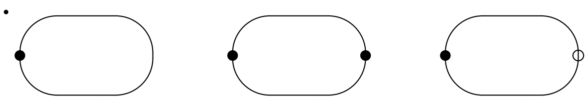
The paper is organized as follows. In Section 2 we introduce dessins on unions of surfaces and give several examples. In Section 3 we introduce Belyi pairs on reducible curves and recall some results concerning their relations with dessins on unions of surfaces. In Section 4 following [4, Chapter 1] we introduce corresponding categories and establish the functors between them. In Section 5 we consider a family of Fried pairs and show how this family can degenerate either to an ordinary Belyi pair or to a Belyi pair on a reducible curve.

2. Generalized dessins d'enfants

In this work we consider finite graphs possibly with multiple edges and loops. We denote by V and E the sets of vertices and edges of a graph.

Definition 2.1. *A graph is called* bicolored *if all its vertices are colored in two different colors, say black and white, in such a way that any two different vertices of an edge are colored differently.*

Example 2.1.



a*)* **b***)* **c***)*

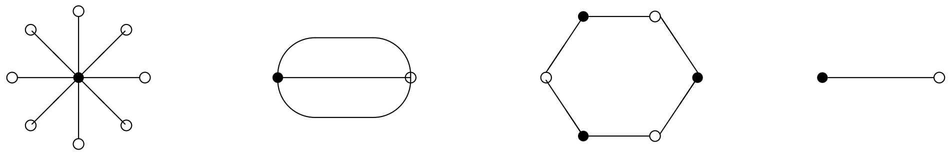
Figure 1. *Graphs and bicolored graphs.*

At Fig. 1 a) we see a graph which can not be transformed to a bicolored graph; at Fig. 1 b) — a graph which is not bicolored, but admits a bicolored structure, and at Fig. 1 c) — a bicolored graph.

Remark 2.1. *Note that a bicolored graph is the same as a bipartite graph with the chosen coloring. More precisely, each bipartite graph provides exactly 2 bicolored graphs, that are different dessins d'enfants in general.*

Definition 2.2. *A* dessin d'enfant D *is a compact connected smooth oriented surface* M *together with a bicolored graph* Γ *embedded into* M *such that the complement* $M \setminus \Gamma$ *is homeomorphic to a disjoint union of open discs. Such a disk is called a* face *of the dessin.* Vertices *and* edges *of the dessin are vertices and edges of the corresponding graph.*

Example 2.2.



a) **b**) **c**) **d**)

Figure 2. *Dessins d'enfants on the sphere.*

All four dessins at Fig. 2 are spherical dessins, however if we consider a graph at Fig. 2 b) just as an abstract graph, it can be embedded in a torus as well. The corresponding dessin (the torus with the graph on it) can be obtained just by the gluing of opposite sides of the hexagon at Fig. 2, c).

The dessin at Fig. 2 d) is minimal, both in the common sense and in the sense that there is a unique morphism from each bicolored dessin to this dessin, the exact meaning of the last assertion will be clarified later in the paper. It is really useful and therefore has a special name.

Definition 2.3. *We call the dessin d'enfant at Fig. 2 d) the* Belyi sphere.

In order to introduce dessins d'enfants on the unions of surfaces, which we call *generalized dessins d'enfants*, we need the following notions:

Definition 2.4. *Let M be a disjoint union of compact smooth oriented surfaces without boundary. We define an equivalence relation $\sim$ on M as follows: there exists a finite set $S \subset M$ such that $P \sim Q$ implies that either $P = Q$ or both P, Q are in S. A* generalized surface *is $X := M/\sim$. We denote by $\pi : M \to X$ a projection transformation.*

Remark 2.2. *The equivalence relation $\sim$ is a formalization of the gluing of surfaces in the finite number of points. Note that a surface can be glued to itself and several surfaces can be glued in one point. Also a generalized surface can be disconnected.*

Definition 2.5. *Let $O = O_1 \coprod O_2 \coprod \ldots \coprod O_N$ be a disjoint union of open disks O_k. Points $P_k \in O_k$, $k = 1, \ldots, N$, are fixed. Let $\sim_1$ be an equivalence relation defined on O in such a way that if $P \sim_1 Q$ then either $P = Q$ or $P, Q \in \{P_1, \ldots, P_N\}$. Then $O/\sim_1$ is called the* admissible disk union. *The points $P_k \in O_k$, $k = 1, \ldots, N$, are called* centers *of discs.*

The admissible disk union is a union of disks such that some of them are glued; at most one point of gluing can be specified in each disk, however, in this specified point more than two disks can be glued. The admissible disk union can be disconnected.

Now generalized dessins, or dessins on unions of surfaces, can be defined as follows:

Definition 2.6. *Let X be a generalized surface, Γ be a bicolored graph embedded in X, possibly disconnected. The pair $D = (X, \Gamma)$ is called a* generalized dessin d'enfant *if three conditions below are satisfied:*

(1) *If $x \in \Gamma$ and $|\{\pi^{-1}(x)\}| > 1$ then x is a vertex of Γ.*
(2) *Let A be a vertex of Γ. For any punctured neighborhood $\dot{U}(A) \subseteq X$ of A there exists an edge e_A of Γ such that A is incident to e_A and $e_A \cap U(A) \neq \emptyset$,*
(3) *The complement $X \setminus \Gamma$ is homeomorphic to an admissible disk union.*

Remark 2.3. *Note that generalized dessins d'enfants correspond to neither stable nor semi-stable curves introduced by Deligne and Mumford, see* [9]. *The main reason is that Def. 2.6 admits that more than two surfaces are glued in one point and does not admit surfaces without edges of the graph.*

Thus, a generalized dessin is a disjoint union of Grothendieck dessins such that some of the dessins may be glued either in the vertices of the same color or in the centers of faces.

Definition 2.7. Vertices *and* edges *of a generalized dessin are vertices and edges of the corresponding graph;* faces *of a generalized dessin are the connected components of the complement of the graph to the generalized surface, i.e., the connected components of the admissible disk union.*

Each face of a generalized dessin is homeomorphic either to an open disk or to several open disks glued in a point.

Example 2.3. *We provide examples of generalized dessins. The big circles and ovals in Figures 3 and 4 depict real 2-spheres (complex lines):*

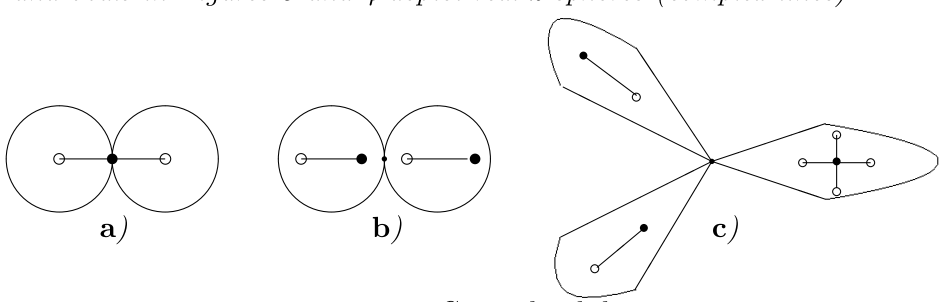


Figure 3. *Generalized dessins*

Note that there are 7 generalized dessins in Fig. 3. Namely, a), b), c), a)∪b), a)∪c), b)∪c), a)∪b)∪c).

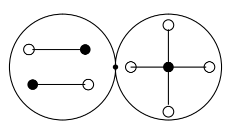


Figure 4. *An embedded graph, which is not a generalized dessin*

Remark 2.4. *If a generalized dessin is connected and the set S from Def. 2.4 is empty, then it is a Grothendieck dessin. Also any connected component of a generalized dessin, which is not self-glued, together with a graph on it, is a Grothendieck dessin d'enfant.*

Below we will provide some definitions and notation which are useful both for dessins and for generalized dessins. We are not going to specify whether D is a dessin d'enfant or a generalized dessin if it is not necessary. We will use the term dessin for the both of them.

We denote by $\alpha(D), \omega(D), n(D), \gamma(D)$ the number of black vertices, white vertices, edges, and faces of D, correspondingly. We will write just $\alpha, \omega, n, \gamma$ if the dessin D is clear from the context.

Definition 2.8. *A* valency of a vertex *of a dessin is a number of edges incident to this vertex. A* valency of a face *is defined to be the number of edges incident to this face divided by 2, note that if both sides of an edge are incident to a given face then this edge should be counted twice.*

Definition 2.9. *Any sequence of numbers $\langle a_1, \dots, a_\alpha | w_1, \dots, w_\omega | c_1, \dots, c_\gamma \rangle$ is called* a combinatorial type. *A combinatorial type is called* realizable *if there is a dessin d'enfant with α white vertices of the valencies $\{a_1, \dots, a_\alpha\}$, ω black vertices of the valencies $\{w_1, \dots, w_\omega\}$, and γ faces of the valencies $\{c_1, \dots, c_\gamma\}$. A combinatorial type is called* generally realizable *if there is a generalized dessin d'enfant satisfying the above conditions.*

Example 2.4. *The generalized dessins in Fig. 3 have the following combinatorial types:*

a): $\langle 2|1,1|1,1\rangle$,

b): $\langle 1,1|1,1|2\rangle$,

c): $\langle 4,1,1|1,1,1,1,1,1|6\rangle$,

a) ∪ b) ∪ c): $\langle 4,2,1,1,1,1|1^{10}|4,2,1,1,1,1\rangle$, *here 1^{10} denotes the sequence of 10 elements 1.*

Remark 2.5. *Although a graph on Fig. 4 is not a dessin, we can define a valency of its unique face (complement to the graph), which is not an admissible disk union in this case, but a cylinder. Namely, the* valency *is again a half of the number of all edges incident to this face. Thus the corresponding combinatorial type can be considered: $\langle 4,1,1|1^6|6\rangle$. It is straightforward to see that this combinatorial type is generally realizable. Namely, its realization is the generalized dessin on Fig. 3 c).*

We are going to generalize this example as follows:

Definition 2.10. *A graph Γ is* tamely *embedded into a generalized surface X, if it is embedded and conditions 1, 2 from Definition 2.6 are satisfied. A* face *of a tamely embedded graph is any connected component of* $X \setminus \Gamma$. *A* valency *of a face is the half of the number of edges incident to this face.*

This definition means that two components can not be glued beyond faces and vertices, in particular, this means that face can not be glued to a vertex. Thus faces and their valencies are well-defined. The following proposition generalizes Remark 2.5.

Proposition 2.1. *A combinatorial type of any graph which is tamely embedded into a generalized surface is generally realizable.*

Proof. Consider the complement of the surface to the graph. It is the union of disks, admissible disk unions, and admissible unions of certain surfaces with cuts. By definition, the valency of a face does not depend on the genus of a surface. So, we can change all surfaces by either spheres or glued spheres without changing the combinatorial type. Now, each sphere with k cuts has the same list of valencies as an admissible union of k spheres, such that the boundary of i-th disk is homeomorphic to i-th cut under the corresponding reordering, $i = 1, \dots, k$. Also if two or more such spheres are glued to another sphere in more than 1 point, then we can move all such points to a certain point without changing valencies. Thus we constructed a generalized dessin with the same list of valencies. □

Definition 2.11. *Two dessins are said to be* isomorphic *if there exists an orientation preserving homeomorphism between corresponding generalized surfaces under which one dessin is transformed into another.*

Remark 2.6. *It is straightforward to see that isomorphic dessins have the same combinatorial types. However, combinatorial type does not determine a dessin up to isomorphism, even on the sphere, see for example Fig. 5.*

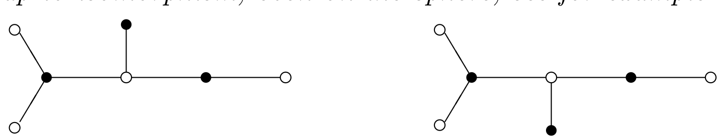


Figure 5. *Non-isomorphic dessins with the same combinatorial type* $\langle 3, 2, 1 | 3, 1, 1, 1 | 6 \rangle$.

Remark 2.7. *In some cases it is more convenient to consider non-bicolored graphs embedded into surfaces in such a way that the complement is homeomerphic to a disjoint union of open discs, see* [21, 23]. *In this case we will speak about* non-bicolored dessins. *It is easy to see that non-bicolored dessins can be obtained from bicolored ones by forgetting the coloring. Conversely, for any non-bicolored dessin d'enfant we can add a vertex of the other color in the middle of each edge to get a dessin d'enfant in the sense of Definition 2.2. All definitions concerning generalized dessins remain the *same. The only difference in definitions is that in the non-bicolored context we say that a valency of a face is the number of edges incident to this face (without dividing by 2).*

3. Belyi pairs

Definition 3.1. *A* Belyi pair $(\mathcal{X}, \beta)$ *is a smooth irreducible algebraic curve* $\mathcal{X}$ *together with a non-constant rational function* $\beta : \mathcal{X} \to \mathbf{P}^1(\mathbb{C})$*, which has at most three critical values. The function* β *is usually called a* Belyi function*.*

Remark 3.1. *Up to the linear-fractional transformation of* $\mathbf{P}^1(\mathbb{C})$ *we may and we do fix the critical values of* β*, crit*$(\beta) \subseteq \{0, 1, \infty\}$*.*

The curves carrying a Belyi pair are widely spread objects among algebraic curves:

Theorem 3.1. [6] *Let* $\mathcal{X}$ *be a smooth complete irreducible algebraic curve over* $\mathbb{C}$*. Then the following statements are equivalent:*
(1) $\mathcal{X}$ *is isomorphic to the complexification of a curve defined over* $\overline{\mathbb{Q}}$*;*
(2) *There exists a Belyi function on* $\mathcal{X}$*.*

If $(\mathcal{X}, \beta)$ is a Belyi pair, then $\beta^{-1}([0, 1])$ is a graph on the topological model X of $\mathcal{X}$ whose edges are $\{\beta^{-1}((0, 1))\}$, black vertices are $\{\beta^{-1}(0)\}$, and white vertices are $\{\beta^{-1}(1)\}$. It appears that $(X, \beta^{-1}([0, 1]))$ is a Grothendieck dessin d'enfant.

Moreover, the following result holds.

Theorem 3.2. ([10],[21]) *Any Grothendieck dessin d'enfant can be obtained by the above construction from some Belyi pair. This pair is defined uniquely up to an isomorphism.*

However, further we would like to consider possibly reducible curves and curves with self-intersections in order to obtain the parallel statements for generalized dessins.

Definition 3.2. *Let* $\mathcal{X}$ *be a complete algebraic curve, possibly, reducible and singular. Let* $\mathcal{S}$ *be the set of all singularities of* $\mathcal{X}$*,* $\beta : \mathcal{X} \to \mathbf{P}^1(\mathbb{C})$ *be a rational function, non-constant on any irreducible component of* $\mathcal{X}$*. A point* $z \in \mathbf{P}^1(\mathbb{C})$ *is called a* critical value *of* β *if one of the following two conditions is satisfied:*
(1) *there exists* $P \in \mathcal{S}$ *such that* $\beta(P) = z$*;*
(2) *there exists* $P \in \mathcal{X} \setminus \mathcal{S}$ *such that* $d\beta(P) = 0$ *and* $\beta(P) = z$*.*

Definition 3.3. *Let* $\mathcal{X}$ *be a complete algebraic curve, possibly, reducible and singular. A function* $\beta : \mathcal{X} \to \mathbf{P}^1(\mathbb{C})$ *is called a* Belyi function *if it is a rational function, non-constant on any irreducible component of* $\mathcal{X}$ *and all* *its critical values lie in the set $\{0,1,\infty\}$. The pair $(\mathcal{X},\beta)$ is called a Belyi pair in this case.*

Example 3.1. *Let us consider a complete curve $\mathcal{X}$ in $\mathbb{C}^2$ defined by the equation $xy=0$ and a function $\beta=x+y$ on this curve. Then $(\mathcal{X},\beta)$ is a Belyi pair. We may say that the "corresponding" generalized dessin is the dessin on Fig. 3 a). Also the dessin in Fig. 3 b) "corresponds" to the function $\frac{1}{x}+\frac{1}{y}$ on the same curve.*

4. Categories and their equivalence

In this section following [4, Chapter 1] we provide a brief review of category theory approach to generalized dessins d'enfants. We introduce categories of generalized dessins d'enfants and Belyi pairs on reducible curves, establish functors between them, and discuss their properties.

4.1. **The category of generalized dessins d'enfant**

Definition 4.1. *Let $X_1=M_1/\sim_1$ and $X_2=M_2/\sim_2$ be generalized surfaces, $\pi_i:M_i\to X_i$, $i=1,2$, be corresponding projection transformations. Homogeneous surjective transformation $f:X_1\to X_2$ is called a* branched covering of generalized surfaces *if there exists a branched covering $\varphi:M_1\to M_2$ such that for any $P\in M_1$ it holds that $f(\pi_1(P))=\pi_2(\varphi(P))$.*

Definition 4.2. *Let $D_1=(X_1,\Gamma_1)$, $D_2=(X_2,\Gamma_2)$ be generalized dessins d'enfants. We say that $f:D_1\to D_2$ is an* admissible *map of dessins if it satisfies $f^{-1}(\Gamma_2)=\Gamma_1$ and the full preimages of the sets of the edges, the black vertices and the white vertices of D_2 are the sets of of the edges, the black vertices and the white vertices of D_1, correspondingly.*

It is straightforward to check (see also [4, Proposition 1.1.57]) that a composition of admissible maps of generalized dessins is an admissible map of generalized dessins.

Definition 4.3. *Let D_1,D_2 be generalized dessins. Let $f_1,f_2:D_1\to D_2$ be admissible maps. We say that f_1 is* equivalent *to f_2 (denote $f_1\sim f_2$) if f_1 and f_2 are homotopic in the class of admissible maps.*

Definition 4.4. *A* morphism *of generalized dessins is an equivalence class of admissible maps of these dessins.*

It is proved in [4, Proposition 1.1.62] that the composition of morphisms is well-defined as an equivalence class of a composition of two representatives.

Definition 4.5. *The* category of generalized dessins d'enfants $\widehat{\mathcal{DESS}}$ *is defined to be a category with the objects determined by Def. 2.6 and morphisms determined by Def. 4.4.*

4.2. **The category of Belyi pairs**

Definition 4.6. *A* morphism *of Belyi pairs* $(\mathcal{X}_1, \beta_1)$, $(\mathcal{X}_2, \beta_2)$ *is a morphism of algebraic curves* $\mu : \mathcal{X}_1 \to \mathcal{X}_2$ *such that* $\beta_1 = \mu \circ \beta_2$.

Definition 4.7. *The* category of Belyi pairs, *consists of objects determined by Def. 3.3 and morphisms determined by Def. 4.6.*

We deal with the following two subcategories of this category.

Definition 4.8. *A* category of Belyi pairs on smooth irreducible curves *is the full subcategory of the category introduced in Def. 4.7, whose objects are Belyi pairs on smooth irreducible curves.*

This is the category introduced and investigated in [21]. It is equivalent to the category of Grothendieck dessins d'enfants, see [17].

Below in this paper an algebraic curve is called *generic* if it is smooth, possibly reducible, and its only singularities are transverse multiple points.

Definition 4.9. *A* category of Belyi pairs on generic curves *is the full subcategory of the category introduced in Def. 4.7, whose objects are Belyi pairs on generic curves. It is denoted* $\mathcal{BE}\widehat{\mathcal{LPAI}}\mathcal{R}$.

Lemma 4.1. [4, Proposition 1.1.83] *For any generalized dessin d'enfant* D *there exists a morphism from* D *to the Belyi sphere.*

4.3. **Functors**

Definition 4.10. *Let* $\mathcal{X}$ *be a generic curve,* β *be a Belyi function on* $\mathcal{X}$. *We define a pair* (X, Γ) *related to* $(\mathcal{X}, \beta)$ *by*

- X *is a topological model of the curve* $\mathcal{X}$,
- $\Gamma = \beta^{-1}([0; 1])$,
- $\beta^{-1}(0)$ *are black and* $\beta^{-1}(1)$ *are white.*

Note that Γ is a bicolored embedded graph in X by the construction.

Lemma 4.2. [4, Proposition 1.6.5] *Let* $\mathcal{X}$ *be a generic algebraic curve. Let* $\beta : \mathcal{X} \to \mathbf{P}^1(\mathbb{C})$ *be a Belyi function. Then the pair* (X, Γ) *constructed in Def. 4.10 is a generalized dessin d'enfant.*

Definition 4.11. *The* functor **draw** : $\mathcal{BE}\widehat{\mathcal{LPAI}}\mathcal{R} \to \widehat{\mathcal{DESS}}$ on objects *in* $\mathcal{BE}\widehat{\mathcal{LPAI}}\mathcal{R}$ *is defined by* $\mathbf{draw}(\mathcal{X}, \beta) = (X, \Gamma)$.

Definition 4.12. *Let* $(\mathcal{X}_1, \beta_1)$, $(\mathcal{X}_2, \beta_2)$ *be Belyi pairs, where* $\mathcal{X}_1, \mathcal{X}_2$ *are generic,* $\varphi : \mathcal{X}_1 \to \mathcal{X}_2$ *be a morphism of algebraic curves. We denote by* X_1, X_2 *the topological models of* $\mathcal{X}_1$, $\mathcal{X}_2$, *correspondingly, and define the transformation* $\phi : X_1 \to X_2$ *as a map between surfaces, pointwise coinciding with the map* φ.

Lemma 4.3. [4, Proposition 1.6.8, 1.6.9] *Let* $\mathbf{draw}(\mathcal{X}_1, \beta_1)$, $\mathbf{draw}(\mathcal{X}_2, \beta_2)$ *be generalized dessins corresponding to Belyi pairs* $(\mathcal{X}_1, \beta_1)$, $(\mathcal{X}_2, \beta_2)$,

$$\varphi : (\mathcal{X}_1, \beta_1) \to (\mathcal{X}_2, \beta_2)$$

be a morphism of these pairs. Then the transformation ϕ *constructed in Def. 4.12 is an admissible map of dessins.*

Definition 4.13. *Let* $(\mathcal{X}_1, \beta_1)$, $(\mathcal{X}_2, \beta_2)$ *be Belyi pairs,* $\varphi : (\mathcal{X}_1, \beta_1) \to (\mathcal{X}_2, \beta_2)$ *be a morphism of these pairs. We define the action of the* functor $\mathbf{draw} : \mathcal{BEL}\widehat{\mathcal{PAI}}\mathcal{R} \to \widehat{\mathcal{DESS}}$ on morphisms $\varphi \in \mathcal{BEL}\widehat{\mathcal{PAI}}\mathcal{R}$ *by*

$$\mathbf{draw}(\varphi) : \mathbf{draw}(\mathcal{X}_1, \beta_1) \to \mathbf{draw}(\mathcal{X}_2, \beta_2)$$

is the equivalence class of ϕ *constructed in Definition 4.12 with respect to a homotopy in the class of admissible maps.*

Let us consider an example illustrating this definition.

Example 4.1. *Let us consider the Belyi pair* $(\mathbf{P}^1(\mathbb{C}), x)$*, i.e., complex sphere with the function* x *on it. Then* $\mathbf{draw}(\mathbf{P}^1(\mathbb{C}), x)$ *is the Belyi sphere, described in Def. 2.3, see also Fig. 2 d).*

So, now the functor $\mathbf{draw} : \mathcal{BEL}\widehat{\mathcal{PAI}}\mathcal{R} \to \widehat{\mathcal{DESS}}$ is defined. We are going to define the functor $\mathbf{groth} : \widehat{\mathcal{DESS}} \to \mathcal{BEL}\widehat{\mathcal{PAI}}\mathcal{R}$.

Definition 4.14. *Let* $(S^2, [0,1])$ *be the Belyi sphere (introduced in Def. 2.3). We define*

$$\mathbf{groth}(S^2, [0,1]) := (\mathbf{P}^1(\mathbb{C}), x).$$

Then according to Ex. 4.1 we have $(S^2, [0,1]) = \mathbf{draw}(\mathbf{groth}(S^2, [0,1])$.

Definition 4.15. *Let* $D_1 = (X, \Gamma)$ *be a generalized dessin d'enfant and* $f_{D_1} : D_1 \to (S^2, [0,1])$ *be an admissible map of generalized dessins. Let* $\mathcal{X}$ *be a pullback of the complex structure defined on* $S^2 = \mathbf{P}^1(\mathbb{C})$ *under the map* f_{D_1}*, and* $\beta : \mathcal{X} \to \mathbf{P}^1(\mathbb{C})$ *be a function pointwise determined by* $\beta(P) = x(f_{D_1}(P))$ *for each* $P \in X$*. Then* $\mathbf{groth}(D_1) := (\mathcal{X}, \beta)$.

It follows directly from the definition that the function β introduced above is a Belyi function. It is straightforward, but see also [4, Proposition 1.7.3], that if $f_{D_1} : D_1 \to (S^2, [0,1])$, $g_{D_1} : D_1 \to (S^2, [0,1])$ are different admissible maps, then the corresponding Belyi pairs $(\mathcal{X}_f, \beta_f)$ and $(\mathcal{X}_g, \beta_g)$ are isomorphic. This shows that the introduced definition is consistent.

Lemma 4.4. *Let* D_1, D_2 *be generalized dessins d'enfants, and*

$$\psi : D_1 \to D_2$$

be a morphism between them. Denote $(\mathcal{X}_1, \beta_1) = \mathbf{groth}(D_1)$, $(\mathcal{X}_2, \beta_2) = \mathbf{groth}(D_2)$*. Then there exists an admissible map* f *such that the diagram at Fig. 6 is commutative.*

$$
\begin{tikzcd}
\mathcal{X}_1 \arrow[rr, "f"] \arrow[dr, "\beta_1"'] & & \mathcal{X}_2 \arrow[dl, "\beta_2"] \\
& \mathbf{P}^1(\mathbb{C}) &
\end{tikzcd}
$$

Figure 6. *Definition of the morphism* $\mathbf{groth}(\psi)$.

Proof. Follows directly from the definitions, see also [4, Prop. 1.7.3]. □

Definition 4.16. *Let D_1, D_2 be generalized dessins d'enfants, and $\psi : D_1 \to D_2$ be their morphism, $(\mathcal{X}_1, \beta_1) = \mathbf{groth}(D_1)$, $(\mathcal{X}_2, \beta_2) = \mathbf{groth}(D_2)$, and the representative f of the morphism ψ is chosen as in Lemma 4.4. Then we define $\mathbf{groth}(\psi) = f$.*

Theorem 4.1. *The categories $\widehat{\mathcal{BELPAIR}}$ and $\widehat{\mathcal{DESS}}$ are equivalent.*

Proof. Follows directly by the construction, namely, the functors $\mathbf{groth} \circ \mathbf{draw}$ and $\mathbf{draw} \circ \mathbf{groth}$ are homotopic to the identity. See [4, Theorem 1.9.3]. □

5. Why do we need generalized dessins d'enfants?

Belyi pairs are discrete points in Hurwitz or moduli spaces. However they are particular cases as well as the minimal points of a stratification of a more general class of functions with bounded number of critical values. The previous step of the stratification is represented by the Fried pairs:

Definition 5.1. [18, 19] *A pair $(\mathcal{X}, F : \mathcal{X} \to \mathbf{P}^1(\mathbb{C}))$ is called a* Fried pair *if $\mathcal{X}$ is a complete smooth irreducible curve, and F is a non-constant rational function on this curve with at most 4 critical values. The function F is called the* Fried function.

Directly by definition Fried pairs constitute families continuously depending on a parameter while all four critical values are different. If for some value of the parameter critical values coincide, then either the curve $\mathcal{X}$ becomes singular, or the function F becomes a Belyi function. Moreover, it is proved in [13] that each Belyi pair can be obtained as a degeneration of a certain Fried pair. Below we provide an example showing that degeneration of the curve possessing a Fried function provides a Belyi function on reducible curve and a generalized dessin d'enfant.

Example 5.1. *Let*

$$y = \frac{x^3 + x^2}{tx + 1}, \tag{5.1}$$

be a family of rational functions on the complex sphere parametrized by x, the functions depend on the parameter t. It is straightforward to see that for each value of $t \notin \{1, \infty\}$ the function $y(x)$ has degree 3, and hence $y(x)$ is a Fried function.

The behavior of Fried pairs can be demonstrated by the preimage of the triangle between three of four critical values. For the function (5.1) let the value of the parameter t be chosen in such a way that all four critical values are different. Then this preimage consists of three triangles. If $t = \frac{19}{20}$ then critical values are 0, C_1, approximately 0.47, and C_2, approximately 2.87. All three critical values are real. At Fig. 7 we draw the preimage of the curvilinear triangle with the vertices $0, C_1, C_2$, two real sides, and the side between 0 and C_2 being the semi-circle in the upper half-plane.

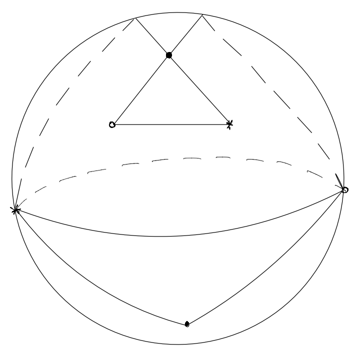


Figure 7. *Preimage of the critical value triangle for* (5.1).

We denote by the black circle the preimage of 0, by the white circle the preimage of C_1, and by $*$ the preimage of C_2.

Theorem 5.1. *The family of Fried functions (5.1) degenerates if and only if $t \in \{0, 1, 9, \infty\}$. If either $t = 0$ or $t = 9$, then Grothendieck dessins d'enfants appear. If $t = 1$ then the degree of the function decreases, and the generalized dessin d'enfant appears naturally.*

Proof. 1. Let us first consider $t \in \{0, 9\}$.

Note that if $t = 0$ then (5.1) degenerates to the Belyi function $y = x^3 + x^2$ on the sphere $\mathbf{P}^1(\mathbb{C})$), and the corresponding Grothendieck dessin d'enfant is depicted in Fig 8.

Figure 8. Dessin d'enfant corresponding to (5.1) if $t = 0$.

2. If $t = 9$ then

$$y = \frac{x^3 + x^2}{9x + 1}$$

is a Belyi function with the critical values $\{-\frac{1}{27}, 0, \infty\}$ and the corresponding dessin is shown in Fig 9.

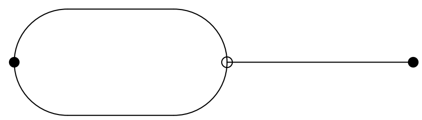


Figure 9. *Dessin d'enfant corresponding to* (5.1) *if* $t = 9$.

3. If $t \notin \{0, 1, 9, \infty\}$, then the function (5.1) has exactly 4 critical values.

4. To investigate how this Fried pair behaves if $t \to 1$ we first rewrite (5.1) in the projective plane $(x : y : z)$ and in the form

$$(tx + z)zy = x^3 + x^2z. \tag{5.2}$$

However, in this form we have a *family* of *algebraic curves* depending on the parameter t and the Fried function $(y : z)$ on each of these curves.

If $z = 0$ then the curve (5.2) is singular, so we consider first the case $z \neq 0$. Then taking $z = 1$ and dividing (5.2) by $(tx + 1)$ with a reminder we obtain

$$(tx + 1)(\frac{1}{t}x^2 + \frac{t-1}{t^2}x + \frac{1-t}{t^3} - y) = \frac{1-t}{t^3}. \tag{5.3}$$

Although the curve is complex and it is impossible to visualize it, we draw the corresponding real pictures in Fig. 10.

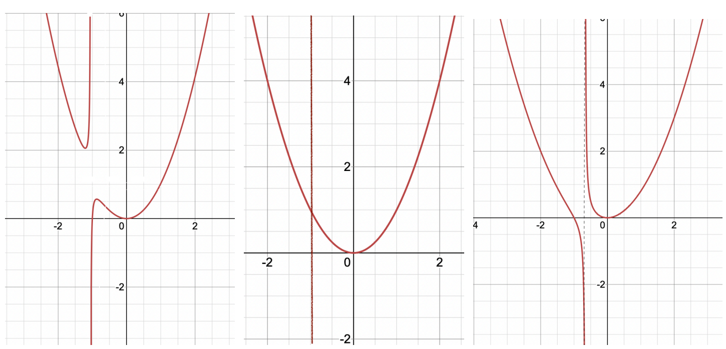


Figure 10. *Real parts of the curve* (5.3) *for* $t < 1$, $t = 1$, *and* $t > 1$.

The left graphic corresponds to the value $t < 1$, the right graphic corresponds to $t > 1$, and the middle one corresponds to $t = 1$. It can be seen even in these pictures that the curve changes its behavior if t goes through the point $t = 1$. Substituting $t = 1$ in the equality (5.3) we obtain the reducible curve $(x + 1)(x^2 - y) = 0$. It is straightforward to see that y has three critical values on this curve: 0 (the vertex of the parabola), 1 (the intersection point of two irreducible components), and ∞. Since ∞ corresponds to $z = 0$ in the projective coordinates, we have to consider this case in details. Re-writing (5.2) for $y = 1$ we get $(tx + z)z = x^3 + x^2 z$ which is a square polynomial in z, namely $z^2 + (tx - x^2)z - x^3 = 0$. The roots are $\frac{1}{2}x(t - x \pm \sqrt{t^2 - 2xt + x^2 + 4x})$. So, each curve from the family (5.2) has a transversal self-intersection point at $(0 : 1 : 0)$. Hence the generic curve from the family (5.2) looks like it is shown in Fig. 11, namely the complex line from Fig. 7 intersects itself. We draw the preimage of the critical value triangle on the same picture.

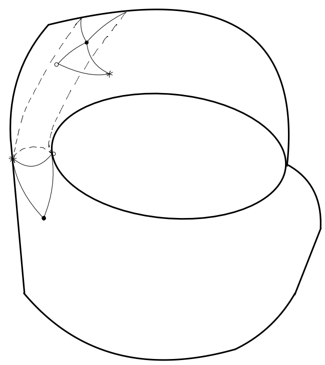


Figure 11. Preimage of the critical value triangle for (5.2).

Letting $t \to 1$ we transform (5.2) into

$$(x + z)(zy - x^2) = 0,$$

so the curve (5.2) becomes reducible and it is the intersection of a line and a conic. This degeneration corresponds to the contraction of the "meridian" in Fig. 11 caused by convergence of the vertices marked by the star and the white circle to each other. All 3 triangles degenerate to the edges of a dessin d'enfant. This dessin on the union of two curves is depicted in Fig. 12.

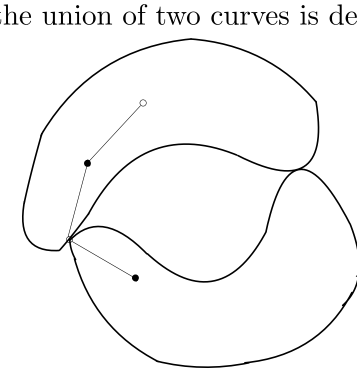


Figure 12. *The dessin d'enfant of the Fried pair* (5.2) *if* $t \to 1$.

On the complex line, which is the lower sphere in Fig. 12, we have the dessin which is the 1-edge tree, and on the conic, which is shown as the upper sphere, we have the dessin which is the 2-edge tree. The line and the conic intersect each other at the points $(-1 : 1 : 1)$ and $(0 : 1 : 0)$. □

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