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ALMOST DIAGONALIZATION OF ΨDO'S OVER VARIOUS GENERALIZED FUNCTION SPACES

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This article is dedicated to Academician Mirjana Vuković on the occasion of her 75th birthday

ABSTRACT. Inductive and projective type sequence spaces of sub- and super-exponential growth, and the corresponding inductive and projective limits of modulation spaces are considered as a framework for almost diagonalization of pseudo-differential operators. Moreover, recent results of the first author and B. Prangoski related to the almost diagonalization of pseudo-differential operators in the context of Hörmander metrics are reviewed.

1. INTRODUCTION

The main goal of this paper is to offer a brief review of some recent results on the almost diagonalization of pseudo-differential operators with symbols in various projective and inductive limits of modulation spaces, and spaces of generalized functions. The results were presented at the conference in Sarajevo dedicated to the 75. anniversary of academician Mirjana Vuković.

Properties of pseudo-differential operators depend on the assigned classes of symbols. Here we consider the Weyl correspondence between operators and symbols, see (1.3). Apart from the classical Hörmander classes (cf. [19]), certain modulation spaces are recognized to be useful symbol classes, see [13] where the tools of time-frequency analysis are used in approximate diagonalization of related operators. This approach is thereafter developed and successfully used in different contexts, see [5] and the references given there. Let us just mention sparse decompositions for Schrödinger-type propagators given in [4], and diagonalization in the framework of tempered ultra-distributions, [24].

General results from [14] are recently extended to Hörmander metrics by the first author and B. Prangoski in [23]. It turns out that the class of weights used in [23] could be extended to the class of moderate weights (see subsection 2.2). The main aim of this paper is to provide necessary background material for investigations in that direction. This includes the introduction of new symbol classes as well as exposition of results for approximate diagonalization in the context of Gelfand-Shilov spaces.

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We end this section with an explanation of the idea behind the notion of approximate diagonalization of operators.

1.1. **Motivation**

Let us start with a general and simple example of a matrix type operator on a Hilbert space. Let ψ_n, $n \in \mathbb{N}$, be a basis for a separable Hilbert space $\mathcal{H}$, $f = \sum_{n\in\mathbb{N}} a_n\psi_n \in \mathcal{H}$, and $A : \mathcal{H} \to \mathcal{H}$ be linear and continuous. Then

$$Af = \sum_{n\in\mathbb{N}} a_n A\psi_n = \sum_{n\in\mathbb{N}} a_n \sum_{m\in\mathbb{N}} b_{n,m}\psi_m = \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}} b_{n,m}a_n\psi_m.$$

So, A can be viewed as the action of an infinite matrix on a space of sequences:

$$(a_n)_{n\in\mathbb{N}} \to (Af)_{m\in\mathbb{N}} = \big(\sum_{n\in\mathbb{N}} b_{n,m}a_n\big)_{m\in\mathbb{N}},$$

More generally, instead of $\mathbb{N}\times\mathbb{N}$ one can observe indices in Λ, a discrete subgroup of $\mathbb{R}^{2d}$. Such group is often represented as $A\,\mathbb{Z}^{2d}$ ($\mathbb{Z}^{2d}$ is the set of integer points in $\mathbb{R}^{2d}$), where A is a $2d$-dimensional, regular matrix with determinant $\det A < 1$. We will also use the term lattice for such Λ.

For the sake of simplicity, consider the lattice points of the form

$$\lambda = (\alpha k, \beta i) \in \Lambda, \qquad k, i \in \mathbb{Z}^d.$$

i.e. $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$. Then the time-frequency shifts of $g \in L^2(\mathbb{R}^d)$ are given by

$$\pi(\lambda)g = \pi_{\alpha k,\beta i}g = e^{2\pi\alpha k\cdot t}g(t-\beta i),\ \ \lambda = (\alpha k, \beta i),\ \ k\cdot t = \langle k, t\rangle = \sum_{j}^{d} k_j t_j.$$

Recall that the set $\mathcal{G}(g,\Lambda) = \{\pi(\lambda)g; \lambda \in \Lambda\}$ is a Gabor frame in $L^2(\mathbb{R}^d)$ if for every $f \in L^2(\mathbb{R}^d)$ there exist $c_1, c_2 > 0$ such that

$$c_1 \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \le ||f||^2_{L^2} \le\ c_2 \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \tag{1.1}$$

($\langle\cdot,\cdot\rangle$ here denotes the scalar product in $L^2(\mathbb{R}^d)$). If $\mathcal{G}(g,\Lambda)$ is a frame, then there exists a dual window $\gamma \in L^2(\mathbb{R}^d)$ such that

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)\gamma, \qquad f \in L^2(\mathbb{R}^d). \tag{1.2}$$

If $c_1 = c_2$ then $\mathcal{G}(g,\Lambda)$ is called a tight frame and $\gamma = Cg$ for some constant $C > 0$.

If $H_0(x) = e^{-a|x|^2}$, $x \in \mathbb{R}^d$, $a > 0$, and $\lambda \in \Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, $\alpha\beta < 1$, then $\mathcal{G}(H_0,\lambda)$ is a frame in $L^2(\mathbb{R}^d)$, and its dual frame γ satisfies

$$|\gamma(x)| + |\widehat{\gamma}(x)| \le Ce^{-c|x|^2}, \quad x \in \mathbb{R}^d.$$

Such frames are also referred as superframes. For the details we refer to [15,21,25]. Moreover, it is well known (see [5, Theorem 3.2.21]) that for $g(x) = e^{-\pi|x|^2}$, $x \in \mathbb{R}^d$, the set

$$\mathcal{G}(g,\Lambda) = \mathcal{G}(g,\alpha,\beta) = \{\pi(\lambda)g; \lambda \in \Lambda\}$$

is a Gabor frame for $L^2(\mathbb{R}^d)$ if and only if $\alpha\beta < 1$. We also mention that if $\alpha > 0$ and $g \in W(\mathbb{R}^d)$ are such that

$$a \leq \sum_{k\in\mathbf{Z}^d} |g(x-\alpha k)|^2 \leq b, \quad x \in \mathbb{R}^d,$$

then there exists $\beta_0 = \beta_0(\alpha)$ such that $\mathcal{G}(g,\alpha,\beta)$ is the frame for all $\beta \leq \beta_0$. Here, $W(\mathbb{R}^d)$ is the Wiener space which consists of function that are locally bounded, and globally in $L^1(\mathbb{R}^d)$. For details we refer to [5] (see Section 3).

Next we introduce pseudo-differential operators. Let $\pi(\lambda)g$, $\lambda \in \Lambda \subset \mathbb{R}^{2d}$, be a tight frame, and for $f, \phi \in L^2(\mathbb{R}^d)$ we consider expansions

$$f(t) = \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda) g(t), \qquad \phi(t) = \sum_{\lambda\in\Lambda} b_\lambda \pi(\lambda) g(t), \quad t \in \mathbb{R}^d.$$

The Weyl-Hörmander pseudodifferential operator (or the Weyl transform) a^w with the symbol $a \in \mathcal{S}'\left(\mathbb{R}^{2d}\right)$ (see subsection 2.1 for the notation) is defined by

$$a^w f(x) = \int_{\mathbb{R}^{2d}} a\left(\frac{x+y}{2}, \xi\right) e^{2\pi i(x-y)\cdot\xi} f(y)\,dy\,d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d). \tag{1.3}$$

Then we have

$$\begin{aligned}\langle a^w(t,D)f(t), \phi(t)\rangle &= \langle \sum_{(k,i)\in\mathbb{Z}^{2d}} a_{k,j} a^w(t,D)\pi_{\alpha k,\beta i} g(t), \sum_{(p,q)\in\mathbb{Z}^{2d}} b_{p,q}\pi_{\alpha p,\beta q} g(t)\rangle \\ &= \sum_{k,i}\sum_{(p,q)\in\mathbb{Z}} a_{k,i} b_{p,q} \langle a^w(t,D)\pi_{\alpha k,\beta i} g(t), \pi_{\alpha p,\beta q} g(t)\rangle.\end{aligned}$$

If we denote by $A = A_{\mathbb{Z}^{2d}\times\mathbb{Z}^{2d}}$ the infinite dimensional matrix of the dimension $\mathbb{Z}^{2d}\times\mathbb{Z}^{2d}$ with elements

$$\langle a^w(t,D)\pi_{\alpha k,\beta i} g(t), \pi_{\alpha p,\beta q} g(t)\rangle, \qquad (k,i),(p,q) \in \mathbb{Z}^{2d},$$

then

$$\langle a^w(t,D)f(t), \phi(t)\rangle = (a_{k,i})_{1\times\mathbb{Z}^{2d}} A\,(b_{p,q})_{\mathbb{Z}^{2d}\times 1}, \qquad (k,i),(p,q) \in \mathbb{Z}^{2d},$$

where the expansion of f and ϕ is clear.

In such a way we may represent a^w as a matrix type operator whose properties are determined by the matrix elements. Asymptotic decay estimates of these elements away from diagonal are related to mapping properties of the corresponding operator. For that reason the term approximate diagonalization is used to describe techniques based on these observations.

Thus, the goal is to characterize the decrease of the matrix A far from its diagonal. In the continuous case, instead of a matrix, we use an integral representation, namely the short-time Fourier transform (STFT), as it will be explained in Section 2.

2. Preliminaries

In this section we fix general notation, and then proceed with basic facts on weight functions, short-time Fourier transform, modulation spaces, and pseudo-differential operators.

2.1. **Notation**

By $\mathbb{Z},\mathbb{N},\mathbb{N}_0,\mathbb{R}$ and $\mathbb{C}$ we denote the sets of all integers, positive integers, non-negative integers, reals, and complex numbers, respectively. For $x=(x_1,\dots,x_d)\in\mathbb{R}^d$, and a multi-index $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{N}_0^d$, we use the notation: $|x|:=(x_1^2+\dots+x_d^2)^{1/2}$, $x^\alpha:=\prod_{j=1}^d x_j^{\alpha_j}$, $|\alpha|:=\alpha_1+\dots+\alpha_d$, $\alpha!:=\alpha_1!\cdots\alpha_d!$, $D^\alpha=D_x^\alpha:=D_1^{\alpha_1}\cdots D_d^{\alpha_d}$, where $D_j^{\alpha_j}:=(-i\partial/\partial x_j)^{\alpha_j}$ $(j=1,\dots,d)$. We write capital letters $X,Y,Z,..$ for elements in $\mathbb{R}^{2d}$, and $A\lesssim B$ means $A\le cB$ for a suitable constant $c>0$.

The symbol $K\subset\subset V$ for an open $V\subset\mathbb{R}^d$ means that K is a compact subset of V. By $\hookrightarrow$ we denote continuous embeddings between two Banach spaces. The norm in $L^p\left(\mathbb{R}^d\right)$ is denoted by $\|\cdot\|_p$, $1\le p\le\infty$, and the corresponding sequence spaces will be denoted by l^p. The Fourier transform $\mathcal{F}$ of a function $f\in L^1$ will be denoted by

$$\hat f(\xi)=\int_{\mathbb{R}^d} f(x)e^{-2\pi i x\xi}dx \qquad (\mathcal{F}^{-1}f(\xi)=\mathcal{F}f(-\xi)).$$

$\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz space of infinitely smooth ($C^\infty(\mathbb{R}^d)$) functions which, together with their derivatives, decay at infinity faster than any inverse polynomial. Its dual space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$.

The function f belongs to the weighted space $L^p_v(\mathbb{R}^d)$ if $fv\in L^p(\mathbb{R}^d)$, $1\le p\le\infty$, where v is a weight function, see below.

2.2. **Weights**

A function v is called a weight function, or simply a weight, if it is locally bounded, non-negative, even and continuous on $\mathbb{R}^d$. Recall,

a) v is submultiplicative if, $v(x+y)\lesssim v(x)v(y)$, $x,y\in\mathbb{R}^d$,
b) v is subconvolutive if $v^{-1}\in L^1$ and $(v^{-1}*v^{-1})(x)\lesssim v^{-1}(x)$, $x\in\mathbb{R}^d$.
c) m is moderate if there exists submultiplicative weight v such that,

$$m(x+y)\lesssim v(x)m(y), \qquad x,y\in\mathbb{R}^d.$$

Clearly, if m is submultiplicative than it is also moderate. By $\mathcal{M}_v$ we denote the set of all $v-$ moderate weights.

Example 2.1. *Standard examples of weight functions are given by*

$$m_{a,b,c,t}(x)=e^{a|x|^b}(1+|x|)^c(\log(e+|x|))^t,\quad a,c,t\in\mathbb{R},\ b\ge 0,\ x\in\mathbb{R}^d. \tag{2.1}$$

Properties of $m_{a,b,c,t}$ are given in the following Lemma from [16] (see also [7]).

Lemma 2.1. *If $m = m_{a,b,c,t}$ is given by* (2.1) *then*

a) *m is submultiplicative if $a,c,t \geq 0$, and $0 \leq b \leq 1$,*
b) *m is subconvolutive if $a > 0, c,t \in \mathbb{R}$, and $0 < b < 1$,*
c) *m is moderate if $a,c,t \in \mathbb{R}$, and $0 \leq b \leq 1$.*
In addition, if v is arbitrary submultiplicative weight then there exists $r > 0$ such that $v(x) \lesssim e^{r|x|}$, $x \in \mathbb{R}^d$.

In this paper we consider weights of the form

$$m_r^s(x) = e^{r|x|^{1/s}}, \quad r > 0, \quad s > 0 \quad x \in \mathbb{R}^d. \tag{2.2}$$

By Lemma 2.1 it follows that $m_r^s = m_{r,1/s,0,0}$ is submultiplicative and subconvolutive if $r > 0$ and $s > 1$.

Let us briefly discuss subconvolutivity of m_r^s, since it will be used later on. Clearly, $(m_r^s)^{-1} = m_{-r}^s \in L^1(\mathbb{R}^d)$. For the case $r > 0$ and $0 < s \leq 1$, [5, Lemma 1.3.5] gives

$$m_{-r}^s * m_{-r}^s(x) \leq C m_{-2^{-1/s}r}^s(x), \quad x \in \mathbb{R}^d. \tag{2.3}$$

Indeed, note that simple inequality $|x+y|^{1/s} \leq 2^{\frac{1-s}{s}}(|x|^{1/s} + |y|^{1/s})$, $x,y \in \mathbb{R}^d$, implies that

$$m_r^s(x+y) \leq m^s_{2^{\frac{1-s}{s}}r}(x) m^s_{2^{\frac{1-s}{s}}r}(y), \quad x,y \in \mathbb{R}^d.$$

or equivalently

$$m_{-r}^s(x-y) m_{-r}^s(y) \leq m^s_{-2^{\frac{s-1}{s}}r}(x), \quad x,y \in \mathbb{R}^d.$$

Then we obtain

$$\begin{aligned} m_{-r}^s * m_{-r}^s(x) &= \int_{\mathbb{R}^d} m_{-r}^s(x-y) m_{-r}^s(y) dy \\ &\leq \int_{\mathbb{R}^d} m_{-r/2}^s(x-y) m_{-r/2}^s(y) m_{-r/2}^s(y) dy \lesssim m_{-2^{-1/s}r}^s(x), \quad x \in \mathbb{R}^d. \end{aligned} \tag{2.4}$$

2.3. **STFT and modulation spaces**

By (f, φ) we denote the dual pairing between $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, and the dual pairing in the context of Gelfand-Shilov type spaces in Section 4.

Let

$$\pi(Z)g(t) = M_\xi T_x g(t) = e^{2\pi i t \cdot \xi} g(t-x), \;\; g \in \mathcal{S}\left(\mathbb{R}^d\right) \setminus \{0\}, \;\; Z = (x,\xi) \in \mathbb{R}^{2d}.$$

The short-time Fourier transform (STFT) of $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$ with respect to a given window $g \in \mathcal{S}\left(\mathbb{R}^d\right) \setminus \{0\}$ is defined as

$$V_g f(x,\xi) = \langle f, \pi(Z) g \rangle = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i t \xi} dt.$$

The same formula over $\mathbb{R}^{2d}$ is given by

$$\mathcal{V}_g f(X,\Xi) = \int_{\mathbb{R}^{2d}} f(t_1,t_2)\overline{g((t_1,t_2)-(x_1,x_2))}e^{-2\pi i(t_1,t_2)\cdot(\xi_1,\xi_2)}dt_1dt_2, \qquad (2.5)$$

where $X=(x_1,x_2)$ and $\Xi=(\xi_1,\xi_2)$.

The (cross-)Wigner distribution is given by

$$W(f,g)(x,\xi) = \int_{\mathbb{R}^d} f(x-\frac{t}{2})\overline{g(x+\frac{t}{2})}e^{-2\pi i\xi t}dt, \qquad f,g \in L^2(\mathbb{R}^d), \qquad (2.6)$$

and when $g \in \mathcal{S}\left(\mathbb{R}^d\right)$ it extends to $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$ by duality.

If $X=(x_1,x_2)$ and $\Xi=(\xi_1,\xi_2)$ belong to $\mathbb{R}^{2d}$, we write

$$\begin{aligned}&\mathcal{W}(f,g)(X,\Xi)\\ &\quad= \int_{\mathbb{R}^{2d}} f((x_1,x_2)-\frac{(t_1,t_2)}{2})\overline{g((x_1,x_2)+\frac{(t_1,t_2)}{2})}e^{-2\pi i(t_1,t_2)\cdot(\xi_1,\xi_2)}dt_1dt_2.\end{aligned}$$

We recall that for $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$ and $g \in \mathcal{S}\left(\mathbb{R}^d\right)$,

$$W(f,g)(x,\xi) = 2^d e^{4\pi i x\xi} V_{g^*} f(2x,2\xi), \quad x,\xi \in \mathbb{R}^d,$$

holds, where $g^*(x)=g(-x)$ (see [12]).

Modulation spaces are introduced by imposing mixed Lebesgue spaces norm to the STFT as follows.

Let $1 \le p,q \le \infty$ and $m \in \mathcal{M}_v$, and let $g \in \mathcal{S}\left(\mathbb{R}^d\right)\setminus\{0\}$. Then the weighted modulation space $M^{p,q}_m\left(\mathbb{R}^d\right)$ consists of $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$ with the property

$$\|f\|_{M^{p,q}_m} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,\xi)|^p\,|m(x,\xi)|^p\,dx\right)^{q/p} d\xi\right)^{1/q} < \infty, \qquad (2.7)$$

with obvious changes if $p,q=\infty$. It is a Banach space with the norm $\|\cdot\|_{M^{p,q}_m}$, and it is well known that the definition does not depend on the choice of the window $g \in \mathcal{S}\left(\mathbb{R}^d\right)\setminus\{0\}$ in the sense that different Schwartz functions yield equivalent norms.

The original source for modulation spaces is [9], see also [12], and the recent monograph [5].

2.4. **Pseudo-differential operators**

We end this section with some remarks on the Weyl-Hörmander pseudo-differential operators given by (1.3). By a straightforward calculation one can show that for $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$ formula (1.3) is (in the weak sense) equivalent to

$$\langle a^w f, g\rangle = \langle a, W(g,f)\rangle \qquad g \in \mathcal{S}\left(\mathbb{R}^d\right),$$

where W is the Wigner distribution given by (2.6). It is well known that the mapping

$$a^w : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$$

is continuous.

The relation between a^w and the STFT is given by the use of the symplectic structure on $\mathbb{R}^d$. This is an important observation when considering metrics different than the Euclidean ones, see Section 5.

For $(\xi,\eta) \in \mathbb{R}^{2d}$ we denote $j(\xi,\eta) = (\eta,-\xi)$, Let $g \in \mathcal{S}\left(\mathbb{R}^d\right) \setminus \{0\}$. Recall that Lemma [12, Lemma 14.5.1] implies the following formula:

$$|(a^w\pi(X)g,\pi(Y)g)| = \left|\mathcal{V}_\Phi a\left(\frac{X+Y}{2}, j(Y-X)\right)\right|, \quad X,Y \in \mathbb{R}^{2d}, \tag{2.8}$$

where $\Phi = W(g,g) \in \mathcal{S}\left(\mathbb{R}^{2d}\right)$. By the change of variables we obtain

$$\left|\mathcal{V}_\Phi \sigma(U,V)\right| = \left|\left\langle \sigma^w \pi\left(U - \frac{j^{-1}(V)}{2}\right) g, \pi\left(U + \frac{j^{-1}(V)}{2}\right) g\right\rangle\right|, \tag{2.9}$$

$U,V \in \mathbb{R}^{2d}$. Note that the standard symplectic form on $\mathbb{R}^{2d}$ is related to j by

$$[(x,\xi),(y,\eta)] = \langle j(x,\xi),(y,\eta)\rangle = \langle \xi, y\rangle - \langle x,\eta\rangle, \quad x,y,\xi,\eta \in \mathbb{R}^d.$$

Also,

$$[(x,\xi),(y,\eta)] = \begin{bmatrix} y & \eta \end{bmatrix} \cdot \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad x,y,\xi,\eta \in \mathbb{R}^d,$$

where the standart symplectic matrix $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ is $2d \times 2d$ block matrix and I is $d \times d$ identity matrix.

3. Novel spaces for symbols of pseudo-differential operators

To extend results from [23] into the framework of Gelfand–Shilov spaces given in [5] (see Section 4) we need a careful preparation. This section contains original material, namely the construction of particular Wiener-amalgam spaces $W\left(L^\infty, \mathcal{A}_r^s\right)$ which are used in the definition of new modulation spaces $\widetilde{M}^\infty_{proj,s}$ and $\widetilde{M}^\infty_{ind,s}$. Then the symbols of pseudo-differential operators are distributions from $\widetilde{M}^\infty_{proj,s}$ or $\widetilde{M}^\infty_{ind,s}$.

3.1. Spaces of sequences

We introduce spaces of sequences which are convenient for our investigations. We note that these sequences are considered in [26] in the context of mapping properties of the Bargmann transforms (cf. [26, Definition 3.1]).

Let Λ be a discrete subgroup of $\mathbb{R}^d$, and

$$m_r^s(\cdot) = e^{r|\cdot|^{1/s}}, \qquad r,s > 0. \tag{3.1}$$

Then the sequence $\mathbf{a} = (a_\lambda)_{\lambda\in\Lambda}$ belongs to the Banach space $l^\infty_{m_r^s}(\Lambda)$ if

$$||\mathbf{a}||_{r,s} = \|\mathbf{a}\, m_r^s\|_\infty = \sup_{\lambda\in\Lambda} |a_\lambda| e^{r|\lambda|^{1/s}} < \infty.$$

We will use the notation $\mathcal{A}_r^s = l^\infty_{m_r^s}(\Lambda)$.

It is well known that $l_v^\infty(\Lambda)$ is Banach algebra with respect to convolution if and only if v is subconvolutive (see [7]). Since m_r^s is not subconvolutive when $0 < s \leq 1$ (see (2.3)), it follows that $\mathcal{A}_r^s$ is not a convolution algebra.

We collect some of the basic properties of $\mathcal{A}_r^s$ in the following Lemma.

Lemma 3.1. *Let $s, r > 0$.*

i) *If $0 < r_1 < r_2$ then $\mathcal{A}_{r_2}^s \hookrightarrow \mathcal{A}_{r_1}^s \hookrightarrow l^1$ where $\hookrightarrow$ denotes continuous embedding. Moreover, this embedding is compact.*
ii) *$\mathcal{A}_r^s$ is involutive, i.e., if $\mathbf{a} \in \mathcal{A}_r^s$ then $\mathbf{a}^* = (a_{-\lambda})_{\lambda\in\Lambda} \in \mathcal{A}_r^s$ and $||\mathbf{a}^*||_{r,s} = ||\mathbf{a}||_{r,s}$.*
iii) *$\mathcal{A}_r^s$ is solid, i.e., if $\mathbf{b} \in \mathcal{A}_r^s$ and $a_\lambda \leq b_\lambda$ for all $\lambda \in \Lambda$ then $\mathbf{a} \in \mathcal{A}_r^s$ and $||\mathbf{a}||_{r,s} \leq ||\mathbf{b}||_{r,s}$.*
iv) *Let $\mathbf{a}, \mathbf{b} \in \mathcal{A}_r^s$ and $\mathbf{c} = \mathbf{a} * \mathbf{b}$. Then there exists $c \in (0,1]$ (depending on s) such that*

$$\|\mathbf{c}\|_{cr,s} \lesssim \|\mathbf{a}\|_{r,s}\|\mathbf{b}\|_{r,s}. \tag{3.2}$$

Proof. We only prove $i)$ and $iv)$ since $ii)$ and $iii)$ are straightforward.
i) The compactness of the embedding is the consequence of the Köthe theory of sequence spaces since the weights satisfy $e^{(r_1-r_2)|\lambda|^{1/s}} \to 0$ as $|\lambda| \to \infty$.
iv) Note that Lemma 2.1 and (2.3) implies that there exists $c \in (0,1]$ ($c = 1$ for $s > 1$ or $c = 2^{-1/s}$ for $0 < s \leq 1$) such that

$$m_{-r}^s * m_{-r}^s(x) \lesssim m_{-cr}^s(x) \quad x \in \mathbb{R}^d. \tag{3.3}$$

Therefore, if $\mathbf{c} = (c_\lambda)_{\lambda\in\Lambda}$ we obtain

$$\begin{aligned} |c_\lambda| &\leq \sum_{\mu\in\Lambda} |a_{\lambda-\mu}||b_\mu| \leq \|\mathbf{a}\|_{r,s}\|\mathbf{b}\|_{r,s}(m_{-r}^s * m_{-r}^s(\lambda)) \\ &\lesssim \|\mathbf{a}\|_{r,s}\|\mathbf{b}\|_{r,s} m_{-cr}^s(\lambda), \quad \lambda \in \Lambda, \end{aligned} \tag{3.4}$$

which implies $|c_\lambda| m_{cr}^s(\lambda) \lesssim \|\mathbf{a}\|_{r,s}\|\mathbf{b}\|_{r,s}$, $\lambda \in \Lambda$. The claim follows by taking the supremum over λ. □

We introduce the Frechét space ($FS-$ space) and the dual Frechet space ($DFS-$ space) of sequences, respectively, by taking the projective and inductive limit topologies:

$$\mathcal{A}_{proj}^s = \varprojlim_{r\to\infty} \mathcal{A}_r^s, \quad \mathcal{A}_{ind}^s = \varinjlim_{r\to 0} \mathcal{A}_r^s, \quad s > 0.$$

These spaces are nuclear, and by Lemma 3.1 it follows that the spaces $\mathcal{A}_{proj}^s$ and $\mathcal{A}_{ind}^s$ are closed under convolution for any $s > 0$. Moreover, for a given $r_0 > 0$ we have

$$\mathcal{A}_{proj}^s \hookrightarrow \mathcal{A}_{r_0}^s \hookrightarrow \mathcal{A}_{ind}^s \hookrightarrow l^1, \quad s > 0.$$

3.2. Wiener amalgam spaces related to $\mathcal{A}_r^s$

Next we introduce the Wiener-Amalgam space related to $\mathcal{A}_r^s$ when $r, s > 0$.

Let $C \in \mathbb{R}^{2d}$ be an open relatively compact set containing the origin, and let Λ be a lattice in $\mathbb{R}^{2d}$ such that $\mathbb{R}^{2d} = \bigcup_{\lambda\in\Lambda}(\lambda + C)$. For a locally bounded function F on $\mathbb{R}^{2d}$ we set

$$F_\lambda = \sup_{Y\in\lambda+C} |F(Y)|, \quad \lambda \in \Lambda. \tag{3.5}$$

Then F belongs to the Banach space $W(L^\infty, \mathcal{A}_r^s) := W_r^s$ if $\mathbf{F} = (F_\lambda)_\lambda \in \mathcal{A}_r^s(\Lambda)$, and the norm in W_r^s is given by $\|F\|_{W_r^s} = \|F_\lambda\|_{r,s}$. In particular, W_r^s contains locally bounded functions whose decay rate at infinity is bounded by $m_{-r}^s(\cdot) = e^{-r|\cdot|^{1/s}}$.

Remark 3.1. Note that $W_r^s = W(L^\infty, \mathcal{A}_r^s) = W(L^\infty, l^\infty_{m_r^s}(\Lambda)) = L^\infty_{m_r^s}(\mathbb{R}^{2d})$. Moreover, by Lemma 3.1 it follows that $\mathcal{A}_r^s \hookrightarrow l^1$, and therefore

$$W_r^s \hookrightarrow W(L^\infty, l^1) := W(l^1),$$

where $W(l^1)$ is the Wiener space defined by $(F_\lambda)_{\lambda\in\Lambda} \in l^1$ (see (3.5)).

If $g \in W(l^1)$ and the lattice Λ is sufficiently dense in $\mathbb{R}^{2d}$, then $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ (see [12] for details).

By taking the projective and inductive limits we obtain

$$W^s_{proj} = \varprojlim_{r\to\infty} W_r^s = W\left(L^\infty, \mathcal{A}^s_{proj}\right) \ \text{ and } \ W^s_{ind} = \varinjlim_{r\to 0} W_r^s = W\left(L^\infty, \mathcal{A}^s_{ind}\right).$$

We can prove the following Lemma.

Lemma 3.2. *Let $s > 0$. W^s_{proj} and W^s_{ind} are involutive algebras with respect to convolution. In particular,*

$$\|F * G\|_{W^s_{cr}} \lesssim \|F\|_{W_r^s} \|G\|_{W_r^s}, \quad r > 0, \tag{3.6}$$

where $c \in (0,1]$ as in part iv) of the Lemma 3.1.

Proof. Directly from the definition it follows that W_r^s is an involutive algebra.

Note that part $iv)$ of Lemma 3.1 implies that $\mathcal{A}_r^s * \mathcal{A}_r^s \hookrightarrow \mathcal{A}_{cr}^s$. Then [8, Theorem 3] implies that $W_r^s * W_r^s \hookrightarrow W_{cr}^s$ and the statement follows. □

3.3. Modulation spaces related to $\mathcal{A}_r^s$

We end the section by introducing modulation spaces with respect to $\mathcal{A}_r^s$. For a symbol a on $\mathbb{R}^{2d}$ we define

$$\mathcal{G}(a)(Y) = \sup_{X\in\mathbb{R}^{2d}} |\mathcal{V}_\Phi(a)(X,Y)|, \quad Y \in \mathbb{R}^{2d},$$

where $\Phi = W(g,g)$, for a given window $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$.

We say that $a \in \widetilde{M}^{\infty,\mathcal{A}_r^s}(\mathbb{R}^{2d})$ if $G(a) \circ j \in W_r^s$, where $j(\xi,\eta) = (\eta, -\xi)$, $(\xi,\eta) \in \mathbb{R}^{2d}$.

The space $\widetilde{M}^{\infty,\mathcal{A}_r^s}(\mathbb{R}^{2d})$ is a Banach space with the norm given by

$$\|\sigma\|_{\widetilde{M}^{\infty,\mathcal{A}_r^s}} = \|G(\sigma) \circ j\|_{W_r^s}.$$

To shorten the notation we will write $\widetilde{M}^{\infty,\mathcal{A}_r^s}(\mathbb{R}^{2d}) := \widetilde{M}^\infty_{r,s}$.

Again, we consider the corresponding projective and inductive limit spaces

$$\widetilde{M}^{\infty}_{proj,s} = \varprojlim_{r\to\infty} \widetilde{M}^{\infty}_{r,s} \quad \text{and} \quad \widetilde{M}^{\infty}_{ind,s} = \varinjlim_{r\to 0} \widetilde{M}^{\infty}_{r,s}.$$

It can be proved that these spaces are related to the standard modulation spaces as given in (2.7) in the following way.

Proposition 3.1. *Let $g \in \mathcal{S}(\mathbb{R}^d)\backslash\{0\}$, and $\Phi = W(g,g)$. Then*

$$\widetilde{M}^{\infty}_{proj,s} = \bigcap_{r>0} M^{\infty,\infty}_{1\otimes m^s_r} \quad \textit{and} \quad \widetilde{M}^{\infty}_{ind,s} = \bigcup_{r>0} M^{\infty,\infty}_{1\otimes m^s_r},$$

where

$$M^{\infty,\infty}_{1\otimes m^s_r} = \{f \in \mathcal{S}'(\mathbb{R}^{2d}) \,|\, \sup_{X,\Xi\in\mathbb{R}^{2d}} |\mathcal{V}_\Phi f(X,\Xi)| e^{r|\Xi|^{1/s}} < \infty\}.$$

The class of symbols $\widetilde{M}^{\infty,\mathcal{A}^s_r}(\mathbb{R}^{2d})$ is an appropriate choice when dealing with pseudo-differential operators on Gelfand-Shilov type spaces with Hörmander metrics.

4. Approximate diagonalization in Gelfand-Shilov spaces

In this section we first recall the definition of standard Gelfand-Shilov spaces, and then recall the results from [4] and [5] which are necessary when extending results from Section 5 to weights that decay at infinity faster then the inverse of any polynomial. Basic facts of Gelfand-Shilov spaces can be found in the original source [11].

Let $s,t,A,B > 0$. We start with the Banach space $S^{s,A}_{t,B}\left(\mathbb{R}^d\right)$ consisting of functions $f \in C^\infty(\mathbb{R}^d)$ with the finite norm

$$||f||_{S^{s,A}_{t,B}} = \sup_{\alpha,\beta\in\mathbb{N}^d_0, x\in\mathbb{R}^d} \frac{\left|x^\alpha \partial^\beta f(x)\right|}{A^{|\alpha|}B^{|\beta|}\alpha!^t\beta!^s}.$$

Then, by taking projective and inductive limits we obtain

$$\Sigma^s_t = \varprojlim_{A>0,B>0} S^{s,A}_{t,B}; \quad S^s_t = \varinjlim_{A>0,B>0} S^{S,A}_{t,B}.$$

Let us denote $S^s_{t,*}$ for Σ^s_t or S^s_t. Also, the usual notation for isotropic spaces is $\mathcal{S}^s(\mathbb{R}^d)$ for $\mathcal{S}^s_s(\mathbb{R}^d), s \geq 1/2$, and $\Sigma^s(\mathbb{R}^d)$ for $\Sigma^s_s(\mathbb{R}^d), s > 1/2$.

The following results from [5] give rise to approximate diagonalization when considering the Euclidean metrics.

Theorem 4.1. *[5, Theorem 5.2.7] Let $s > 0, m \in \mathcal{M}_v\left(\mathbb{R}^d\right), g \in M^1_{v\otimes 1}\left(\mathbb{R}^d\right)\backslash\{0\}$ such that*

$$\|\partial^\alpha g\|_{L^1_v(\mathbb{R}^d)} \lesssim C^{|\alpha|}(\alpha!)^s, \quad \alpha \in \mathbb{N}^d,$$

for some $C > 0$. If $f \in C^\infty\left(\mathbb{R}^d\right)$ the following conditions are equivalent:

(i) *There exists a constant $C > 0$ such that*

$$|\partial^\alpha f(x)| \lesssim m(x) C^{|\alpha|}(\alpha!)^s, \quad x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d.$$

(ii) *There exists a constant* $\varepsilon > 0$ *such that*

$$|V_g f(x,\xi)| \lesssim m(x) e^{-\varepsilon|\xi|^{\frac{1}{s}}}, \quad x,\xi \in \mathbb{R}^{2d}, \alpha \in \mathbb{N}^d.$$

Theorem 4.2. *[5, Theorem 5.2.10] Let* $s \geq 1/2$ *and* $m \in \mathcal{M}_v(\mathbb{R}^{2d})$.

a) *If* $1/2 \leq s$ *let* $g \in S^s_s(\mathbb{R}^d)$.

b) *If* $1/2 < s$ *let* $g \in \Sigma^s_s(\mathbb{R}^d)$.

Assume the following growth condition on the weight v:

$$v(z) \lesssim e^{\varepsilon|z|^{1/s}}, \quad z \in \mathbb{R}^{2d},$$

for every $\varepsilon > 0$.

Let $a \in C^\infty(\mathbb{R}^{2d})$. *Then the following are equivalent:*

(i) *The symbol a satisfies*

$$|\partial^\alpha a(z)| \lesssim m(z) C^{|\alpha|} (\alpha!)^s, \quad z \in \mathbb{R}^{2d}, \alpha \in \mathbb{N}^{2d}.$$

(ii) *There exists* $\varepsilon > 0$ *such that*

$$|\langle a^w \pi(z) g, \pi(w) g\rangle| \lesssim m\left(\frac{w+z}{2}\right) e^{-\varepsilon|w-z|^{\frac{1}{s}}}, \quad z,w \in \mathbb{R}^{2d}. \tag{4.1}$$

Theorem 4.3. *[5, Theorem 5.2.12] Let* $s \geq 1/2$, $g \in S^s_s(\mathbb{R}^d)$, $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, *and let* $\mathcal{G}(g,\Lambda)$ *be a Gabor superframe for* $L^2(\mathbb{R}^d)$. *If* $a \in C^\infty(\mathbb{R}^{2d})$, *then the following properties are equivalent:*

(i) *There exists* $\varepsilon > 0$ *such that the estimate* (4.1) *holds.*

(ii) *There exists* $\varepsilon > 0$ *such that*

$$|\langle a^w \pi(\mu) g, \pi(\lambda) g\rangle| \lesssim m\left(\frac{\lambda+\mu}{2}\right) e^{-\varepsilon|\lambda-\mu|^{\frac{1}{s}}}, \quad \lambda,\mu \in \Lambda.$$

Now we state a result on approximate diagonalization in the spirit of [14]. The proof follows from a careful inspection of the proofs of Theorems 4.1, 4.2, and 4.3, and will be given elsewhere.

Here below $\mathcal{A}^s_*$ denotes $\mathcal{A}^s_{proj}$ or $\mathcal{A}^s_{ind}$, W^s_* stands for W^s_{proj} or W^s_{ind}, and by $\widetilde{M}^\infty_{*,s}$ we denote $\widetilde{M}^\infty_{proj,s}$ or $\widetilde{M}^\infty_{ind,s}$.

Theorem 4.4. *[5, Theorem 5.2.12] Let* $s \geq 1/2$ *and* $g \in \mathcal{S}^s_s(\mathbb{R}^d)$. *Then the following are equivalent:*

a) $a \in \widetilde{M}^\infty_{*,s}$,

b) *There exists* $H \in W^s_*$ *such that*

$$|\langle a^w \pi(X) g, \pi(Y) g\rangle| \leq H(Y-X), \quad X,Y \in \mathbb{R}^{2d}, \tag{4.2}$$

c) *There exists a sequence* $h \in \mathcal{A}^s_*(\Lambda)$ *such that*

$$|\langle a^w \pi(\mu) g, \pi(\nu) g\rangle| \leq h(\nu-\mu), \quad \mu,\nu \in \Lambda.$$

5. Extensions on the spaces with the Hörmander metric

In this section we present some general results from [23] in the context of tempered distributions, and symbol classes $S(M,g)$, see (5.1). Extension of these results to Gelfand-Shilov spaces and their dual spaces of tempered ultradistributions is a highly nontrivial task and will be the subject of our future investigations.

Before explaining the main results, we need to recall necessary notions related to the geometry in the observed spaces. We refer to [20] for more details on the subject.

We assume that a Riemannian metric g on $\mathbb{R}^{2d}$ is a Borel measurable section of the 2-covariant tensor bundle $T^2T^*\mathbb{R}^{2d}$ that is symmetric and positive-definite at every point. The corresponding quadratic forms are denoted by the same symbol: $g_X(T) := g_X(T,T)$, $T \in T_X\mathbb{R}^{2d}$. For each $X \in \mathbb{R}^{2d}$, we identify $\mathbb{R}^{2d}$ with $T_X\mathbb{R}^{2d}$ that sends every $Y \in \mathbb{R}^{2d}$ to the directional derivative in direction Y at X. Let $T \in \mathbb{R}^{2d}\backslash\{0\}$. We denote by ∂_T the vector field on $\mathbb{R}^{2d}$ given by the directional derivative in direction T at every point $X \in \mathbb{R}^{2d}$. We denote by $\sigma : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ the isomorphism induced by the symplectic form; notice that ${}^t\sigma = -\sigma$, $\sigma_X \in L(\mathbb{R}^{2d},\mathbb{R})$, $\sigma_X(T) = [X,T]$, $T \in \mathbb{R}^{2d}$. Let g be a Riemannian metric on $\mathbb{R}^{2d}$ and, for $X \in \mathbb{R}^{2d}$, denote by $Q_X : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ the isomorphism induced by g_X. In particular, Q_X is $2d \times 2d$ diagonal matrix with elements $q_X(E_j,E_j)$ where $\{E_j\}_{j=1}^{2d}$ is the basis of $\mathbb{R}^{2d}$.

For $X \in \mathbb{R}^{2d}$, set $Q_X^\sigma := {}^t\sigma Q_X^{-1}\sigma : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ and let $g_X^\sigma(T,S) := \langle Q_X^\sigma T, S\rangle$, $T,S \in \mathbb{R}^{2d}$. Then g^σ is again a Riemannian metric on $\mathbb{R}^{2d}$ called the symplectic dual of g; it is also given by

$$g_X^\sigma(T) = \sup_{S\in\mathbb{R}^{2d}\backslash\{0\}} [T,S]^2/g_X(S).$$

The Riemannian metric g is said to be a Hörmander metric [18] (i.e., an admissible metric in the terminology of [2, 20]) if the following three conditions are satisfied:

(i) (slow variation) There exist $C_0 \geq 1$ and $r_0 > 0$ such that

$$g_X(X-Y) \leq r_0^2 \Rightarrow C_0^{-1} g_Y(T) \leq g_X(T) \leq C_0 g_Y(T),$$

for all $X,Y,T \in \mathbb{R}^{2d}$;

(ii) (temperance) There exist $C_0 \geq 1$ and $N_0 \geq 0$ such that

$$(g_X(T)/g_Y(T))^{\pm 1} \leq C_0(1+g_X^\sigma(X-Y))^N, \quad \text{for all } X,Y,T \in \mathbb{R}^{2d};$$

(iii) (the uncertainty principle) $g_X(T) \leq g_X^\sigma(T)$, for all $X,T \in \mathbb{R}^{2d}$.

Next we need an admissibility condition. For a given metric g, a positive Borel measurable function M on $\mathbb{R}^{2d}$ is said to be g-admissible if there are $C \geq 1$, $r > 0$ and $N \geq 0$ such that

$$g_X(X-Y) \leq r^2 \Rightarrow C^{-1}M(Y) \leq M(X) \leq CM(Y), \quad \text{and}$$
$$(M(X)/M(Y))^{\pm 1} \leq C(1+g_X^{\sigma}(X-Y))^N, \quad \text{for all } X,Y \in \mathbb{R}^{2d}.$$

All the constants mentioned above are called *admissibility constants*.

Given a g-admissible weight M, the space of symbols $S(M,g)$ is defined as the space of all $a \in C^\infty(\mathbb{R}^{2d})$ for which

$$\|a\|_{S(M,g)}^{(k)} = \sup_{l \leq k} \sup_{\substack{X \in \mathbb{R}^{2d} \\ T_1,\ldots,T_l \in \mathbb{R}^{2d}\setminus\{0\}}} \frac{|a^{(l)}(X;T_1,\ldots,T_l)|}{M(X)\prod_{j=1}^{l} g_X(T_j)^{1/2}} < \infty, \quad \forall k \in \mathbb{N}. \tag{5.1}$$

With this system of seminorms, $S(M,g)$ becomes a Fréchet space. One can always regularize the metric making it smooth without changing the notion of g-admissibility of a weight and the space $S(M,g)$; the same can be done for any g-admissible weight (see [18]).

For any symbol $a \in \mathcal{S}(\mathbb{R}^{2d})$, we consider the Weyl quantization a^w, see (1.3). This correspondence extends to symbols in $\mathcal{S}'(\mathbb{R}^{2d})$ in a usual manner, and in this case $a^w : \mathcal{S}(\mathbb{R}^{2d}) \to \mathcal{S}'(\mathbb{R}^{2d})$ is a continuous mapping. For any $a \in S(M,g)$ with a g-admissible weight M, the operator a^w is continuous on $\mathcal{S}(\mathbb{R}^{2d})$ and uniquely extends to an operator on $\mathcal{S}'(\mathbb{R}^{2d})$ (cf. [18]).

5.1. **The symplectic short-time Fourier transform**

We extended in [22] the short time Fourier transform to the spaces of functions over the ground space equipped with the Hörmader metrics with the aim to analyze the almost diagonalization of a class of pseudo differential operators that is, the estimate of the action of a pseudo-differential operator on wave packages accommodated to the involved metrics.

We denote by $\mathcal{F}_\sigma$ the symplectic Fourier transform on $\mathbb{R}^{2d}$:

$$\mathcal{F}_\sigma f(X) = \int_{\mathbb{R}^{2d}} e^{-2\pi i[X,Y]} f(Y)dY, \quad f \in L^1(\mathbb{R}^{2d});$$

recall that $\mathcal{F}_\sigma \mathcal{F}_\sigma = \mathrm{Id}$. ($[X,Y]$ is the symplectic product)

Let $\varphi \in \mathcal{S}(\mathbb{R}^{2d})$ and set $\varphi_X := \varphi(X)$, $X \in \mathbb{R}^{2d}$. We define the *symplectic short-time Fourier transform* $\mathcal{V}_\varphi f$ of $f \in \mathcal{S}'(\mathbb{R}^{2d})$ with respect to φ as

$$\mathcal{V}_\varphi f(X,\Xi) := \mathcal{F}_\sigma(f\overline{\varphi_X})(\Xi) = \langle f, e^{-2\pi i[\Xi,\cdot]}\overline{\varphi_X}\rangle, \quad X,\Xi \in \mathbb{R}^{2d}.$$

When $f \in L^1_{(1+|\cdot|)^{-s}}(\mathbb{R}^{2d})$ for some $s \geq 0$ ($|\cdot|$ is (any) norm on $\mathbb{R}^{2d}$), we have

$$\mathcal{V}_\varphi f(X,\Xi) = \int_{\mathbb{R}^{2d}} e^{-2\pi i[\Xi,Y]} f(Y)\overline{\varphi_X(Y)}dY.$$

The mapping $\Xi \mapsto e^{-2\pi i[\Xi,\cdot]}$, from $\mathbb{R}^{2d}$ to the space of ϕ such that $\|\phi^{(\alpha)}(\cdot)(1+|\cdot|)^{-1}\|_{L^\infty} < \infty$, for every $\alpha \in \mathbb{N}_0^d$, is well-defined and smooth. Therefore the mapping

$$(X,\Xi) \mapsto e^{-2\pi i[\Xi,\cdot]}\overline{\varphi}_X, \quad \text{from} \quad \mathbb{R}^{2d} \times \mathbb{R}^{2d} \to \mathcal{S}(\mathbb{R}^{2d}),$$

is strongly Borel measurable. Consequently, the function

$$(X,\Xi)\mapsto \mathcal{V}_\varphi f(X,\Xi) \quad \text{from} \quad \mathbb{R}^{2d}\times\mathbb{R}^{2d}\to\mathbb{C},$$

is always Borel measurable, and if φ is of class $\mathcal{C}^k$, $0\le k\le\infty$, then

$$\mathcal{V}_\varphi f\in C^k(\mathbb{R}^{2d}\times\mathbb{R}^{2d}), \quad 0\le k\le\infty.$$

For the study of $\mathcal{V}_\varphi$, we consider the Fréchet space

$$\varprojlim_{s\to\infty} L^\infty_{(1+|\cdot|)^s}(\mathbb{R}^{2d}\times\mathbb{R}^{2d}),$$

and the inductive limit (LB)-spaces,

$$\varinjlim_{s\to\infty} L^\infty_{(1+|\cdot|)^{-s}}(\mathbb{R}^{2d}\times\mathbb{R}^{2d}), \quad \text{and} \quad \varinjlim_{s\to\infty} L^1_{(1+|\cdot|)^{-s}}(\mathbb{R}^{2d}\times\mathbb{R}^{2d}).$$

We note that $|\cdot|$ is any norm on $\mathbb{R}^{2d}\times\mathbb{R}^{2d}$ and none of these spaces depend on the particular choice of $|\cdot|$. We have the following embeddings:

$$\begin{aligned}\mathcal{S}(\mathbb{R}^{2d}\times\mathbb{R}^{2d}) &\hookrightarrow \varprojlim_{s\to\infty} L^\infty_{(1+|\cdot|)^s}(\mathbb{R}^{2d}\times\mathbb{R}^{2d}) \hookrightarrow \varinjlim_{s\to\infty} L^\infty_{(1+|\cdot|)^{-s}}(\mathbb{R}^{2d}\times\mathbb{R}^{2d})\\ &\hookrightarrow \varinjlim_{s\to\infty} L^1_{(1+|\cdot|)^{-s}}(\mathbb{R}^{2d}\times\mathbb{R}^{2d}) \hookrightarrow \mathcal{S}'(\mathbb{R}^{2d}\times\mathbb{R}^{2d}). \qquad (5.2)\end{aligned}$$

The following assertions are proved in [22]:

(i) The sesquilinear mapping

$$\mathcal{S}'(\mathbb{R}^{2d})\times\mathcal{S}(\mathbb{R}^{2d})\to \varinjlim_{s\to\infty} L^\infty_{(1+|\cdot|)^{-s}}(\mathbb{R}^{2d}\times\mathbb{R}^{2d}), \quad (f,\varphi)\mapsto \mathcal{V}_\varphi f, \qquad (5.3)$$

is well-defined and hypocontinuous. Furthermore, for any bounded subset B of $\mathcal{S}'(\mathbb{R}^{2d})$ there is $s>0$ such that $\mathcal{V}_\varphi f\in L^\infty_{(1+|\cdot|)^{-s}}(\mathbb{R}^{2d})$, for all $f\in B$, $\varphi\in\mathcal{S}(\mathbb{R}^{2d})$, and the set of linear mappings

$$\mathcal{S}(\mathbb{R}^d)\to L^\infty_{(1+|\cdot|)^{-s}}(\mathbb{R}^{2d}\times\mathbb{R}^{2d}), \quad \varphi\mapsto\mathcal{V}_\varphi f, \qquad f\in B,$$

is an equicontinuous subset of $\mathcal{L}(\mathcal{S}(\mathbb{R}^{2d}), L^\infty_{(1+|\cdot|)^{-s}}(\mathbb{R}^{2d}\times\mathbb{R}^{2d}))$.

(ii) The sesquilinear mapping

$$\mathcal{S}(\mathbb{R}^{2d}\times\mathbb{R}^{2d})\to \varprojlim_{s\to\infty} L^\infty_{(1+|\cdot|)^s}(\mathbb{R}^{2d}\times\mathbb{R}^{2d}), \quad (\psi,\varphi)\mapsto\mathcal{V}_\varphi\psi, \qquad (5.4)$$

is well-defined and continuous.

(iii) Under a suitable (geometric) condition, and if $\varphi\in\mathcal{S}(\mathbb{R}^{2d})$, then the conjugate-linear mapping $\mathcal{S}(\mathbb{R}^{2d})\to\mathcal{S}(\mathbb{R}^{2d}\times\mathbb{R}^{2d})$, $\psi\mapsto\mathcal{V}_\varphi\psi$, is well-defined and continuous.

Our main result in [22] is devoted to the almost diagonalization of $a\in S(M,g)$. We present a result which is a consequence of the main theorem in [22]. Recall, $\mathbb{R}^{2d}$ is occupied with the symplectic structure $[(x,\xi),(y,\eta)]=\langle\xi,y\rangle-\langle\eta,x\rangle$.

We denote by $\Psi^{g,L}_X$ the topological isomorphism over the Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$,

$$\Psi^{g,L}_X:\mathcal{S}(\mathbb{R}^{2d})\to\mathcal{S}(\mathbb{R}^{2d}),\ (\Psi^{g,L}_X\varphi)(Y)=\varphi({Q_X}^{-1/2}Y),\, Y\in\mathbb{R}^{2d},$$

and we extend it, by duality, to the topological isomorphism

$$\Psi_X^{g,L} : \mathcal{S}'(\mathbb{R}^{2d}) \to \mathcal{S}'(\mathbb{R}^{2d}), \quad \langle \Psi_X^{g,L} f, \varphi \rangle = |\det Q_X|^{1/2} \langle f, \varphi \circ Q_X^{1/2} \rangle. \tag{5.5}$$

In the simplest case when the metric g is symplectic ($g = g^\sigma$, see below), then our main Theorem in [22] has the following form: for each $N \in \mathbb{N}$,

$$\mathbb{R}^{2d} \times \mathbb{R}^{2d} \ni (X, \Xi) \mapsto M((X+\Xi)/2)^{-1}(1 + g_{\frac{X+\Xi}{2}}(X - \Xi))^N \tag{5.6}$$
$$\times \left| \left\langle \left(\Psi^{g,L}_{\frac{X+\Xi}{2}} a \right)^w \pi\left((Q_{\frac{X+\Xi}{2}})^{1/2} X \right) \chi, \overline{\pi\left((Q_{\frac{X+\Xi}{2}})^{1/2} \Xi \right) \chi} \right\rangle \right| \in L^\infty(\mathbb{R}^{2d} \times \mathbb{R}^{2d}).$$

This is a generalization of the result of Gröchenig and Rzeszotnik [14, Theorem 4.2 (i)-(ii)], since when g is the Euclidean metric on $\mathbb{R}^{2d}$ and $M(X) = 1$ (which corresponds to the Hörmander class $S^0_{0,0}$), then $\Psi^{g,L}_X = \mathrm{Id}$, $Q_X = \mathrm{Id}$, $\forall X \in \mathbb{R}^{2d}$ and

$$(X, \Xi) \mapsto \langle X - \Xi \rangle^N |\langle a^w \pi(X)\chi, \overline{\pi(\Xi)\chi} \rangle| \in L^\infty(\mathbb{R}^{4n}).$$

The right hand side of (5.6) can be explained by the use of simplectic and metaplectic transformations. Denote by $Mp(\mathbb{R}^{2d})$ and $Sp(\mathbb{R}^{2d})$ the spaces of metaplectic operators and of symplectic transformations over $\mathbb{R}^{2d}$ (cf. [10], [20]). Note that the interesting approach to the analysis of metaplectic transforms is given in [6] as well as in [10]. The so called lifting theorems using the Hörmander metric are considered in [1] and [17].

Let Π be the surjective homomorphism $\Pi : \mathrm{Mp}(\mathbb{R}^{2d}) \to \mathrm{Sp}(\mathbb{R}^{2d})$. Since $Q_X^{1/2} \in \mathrm{Sp}(\mathbb{R}^{2d})$ for each $X \in \mathbb{R}^{2d}$, there exists $\Phi^g_X \in \mathrm{Mp}(\mathbb{R}^{2d})$ such that

$$\Pi(\Phi^g_X) = Q_X^{-1/2} \quad \text{and} \quad (\Psi^g_X a)^w = (\Phi^g_X)^* a^w \Phi^g_X, \quad a \in \mathcal{S}'(\mathbb{R}^{2d}),\ X \in \mathbb{R}^{2d}.$$

Let τ_X, $X \in \mathbb{R}^{2d}$, be a metaplectic operator

$$\tau_{(x,\xi)} \kappa = e^{2\pi i \langle y - x/2, \xi \rangle} \chi(\cdot - x), \quad \kappa(X, \Xi) \in \mathcal{S}(\mathbb{R}^{2d}).$$

Clearly, $\tau_{(x,\xi)} = e^{-\pi i \langle x, \xi \rangle} \pi(x, \xi)$. Then, since

$$\Omega \tau_X \Omega^* = \tau_{\Pi(\Omega)X} \text{ for all } \Omega \in \mathrm{Mp}(\mathbb{R}^{2d}), X \in \mathbb{R}^{2d},$$

(see [10, Theorem 7.13, p. 205]) we infer that for each $X, Y \in \mathbb{R}^{2d}$, $\pi(Q_Y^{1/2} X)$ is equal to $(\Phi^g_Y)^* \pi(X) \Phi^g_Y$ up to a constant of modulus 1. Thus, (5.6) is equivalent to the following: for each $N \in \mathbb{N}$,

$$(X, \Xi) \mapsto M((X+\Xi)/2)^{-1}(1 + g_{\frac{X+\Xi}{2}}(X - \Xi))^N$$
$$\times \left| \left\langle a^w \pi(X) \Phi^g_{\frac{X+\Xi}{2}} \chi, \overline{\pi(\Xi) \Phi^g_{\frac{X+\Xi}{2}} \chi} \right\rangle \right| \in L^\infty(W \times \mathbb{R}^{2d}).$$

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