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THE RATE OF CONVERGENCE OF A CERTAIN MIXED MONOTONE DIFFERENCE EQUATION

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Dedicated to the 75th birthday of the dear Acc. Professor Mirjana Vuković

ABSTRACT. This paper investigates the rate of convergence of a certain mixed monotone rational second-order difference equation with quadratic terms. More precisely we give the precise rate of convergence for all attractors of the difference equation $x_{n+1}=\frac{Ax_n^2+Ex_{n-1}}{x_n^2+f}$, where all parameters are positive and initial conditions are non-negative. The mentioned methods are illustrated in several characteristic examples.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we investigate the rate of convergence of solutions of the following second-order difference equation with quadratic terms

$$x_{n+1}=\frac{Ax_n^2+Ex_{n-1}}{x_n^2+f}, \quad n=0,1,..., \tag{1.1}$$

where $A,E,f\in(0,\infty)$ and where the initial conditions x_{-1} and x_0 are arbitrary non-negative real numbers such that $x_{-1}+x_0>0$. Equation (1.1) has very interesting global dynamics that was investigated in [14] and is summarized in Table 1. Equation (1.1) is a perturbation of the sigmoid Beverton-Holt equation also known as the Thomson's equation that is obtained when $E=0$:

$$x_{n+1}=\frac{Ax_n^2}{x_n^2+f}, \quad n=0,1,...,. \tag{1.2}$$

Thomson's equation (1.2) is one of the major models in population dynamics [2,9,23]. Thomson's function in equation (1.2) which is $T(x)=\frac{Ax^2}{x^2+f}$ represents an increasing growth rate, while the *per capita* growth rate, given as $g(x)=T(x)/x$ changes its monotonicity. See [2], Section 3.2, p. 90–92 for discussion of different types of growth rate. So the Thomson's model can be called compensatory, see [2], Section 3.2. and Caswell [6] and Kulenović and Yakubu [16]. This means that in both the Beverton-Holt model and the sigmoid Beverton-Holt model, the decrease

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in the per capita growth rate with population size x_n, is exactly compensated for by the increase in population size x_{n+1}. See also Kulenović and Yakubu [16] for a more detailed explanation. The introduction of the perturbation term Ex_{n-1} in equation (1.1) will not only effect the global dynamics of Thomson's model, but also the rate of convergence of the solutions toward all the attractors, as we will explain in the conclusion section.

Equation (1.1) is called mixed monotone since the right hand side of this equation is always increasing in x_{n-1} and it is either increasing or decreasing in x_n. The local stability analysis of a more general version of Equation (1.1) was done in [11].

The global behavior of difference equation

$$x_{n+1} = F(x_n, x_{n-1}), \quad n = 0, 1, ..., \tag{1.3}$$

where F is continuous function increasing in second variable and increasing or decreasing in first is well developed, and under some mild additional conditions, all solutions of such equation converge to either an equilibrium solution or the period-two solutions, [5, 7]. Based on our results in [14], which are similar in nature to the convergence to an equilibrium solution or period-two solution, we will find the rate of convergence to those attractors by using either the classical Poincaré's theorem or an extension of Perron's theorem, see [1, 8, 9, 13, 17–21] for different versions of Perron's theorem. In all cases, we exhibit explicit non-autonomous linear difference equation satisfied by either the error term $x_n - \bar{x}$, where $\bar{x}$ is an equilibrium solution of Equation (1.1), or by $x_{2n} - \phi$ and $x_{2n+1} - \psi$, where (ϕ, ψ) is a period-two solution of Equation (1.1).

In [1] R. P. Agarwal and M. Pituk studied the scalar equation

$$x_{n+k} + p_1(n)x_{n+k-1} + \cdots + p_k(n)x_n = 0, \tag{1.4}$$

where k is a positive integer and $p_i(n)$ are real or complex sequences for $i = 1, 2, ..., k$, where all coefficients are asymptotically constants, that is

$$q_i = \lim_{n\to\infty} p_i(n), \quad i = 1, 2, ..., k, \tag{1.5}$$

exist in $\mathbb{C}$ when the convergence in (1.5) is at a geometric rate. The cooresponding *limiting equation* of (1.4) is

$$x_{n+k} + q_1 x_{n+k-1} + \cdots + q_k x_n = 0. \tag{1.6}$$

They extended and improved some asymptotic results from [12] for second order linear fractional difference equation.

In [3, 4], S. Bodine and D. A. Lutz greatly improved the estimates of Agarwal and Pituk, considering the (matrix) Poincaré system

$$\mathbf{x}_{n+1} = (A + B_n)\mathbf{x}_n, \tag{1.7}$$

where $\mathbf{x}_n$ is an m-vector, and A, B_n are $m \times m$ matrices for $n = 1, 2, \ldots$ such that $|B_n| \to 0, n \to \infty$ geometrically. Their results were further extended in [10].

The following two theorems give us precise information about the asymptotics of solutions of (1.4) (see [8, 9, 12, 15, 20]).

Theorem 1.1. *(Poincaré's theorem) Consider (1.4) subject to condition (1.5). Let* $\lambda_1,\dots,\lambda_k$ *be the roots of the characteristic equation*

$$\lambda^k + q_1\lambda^{k-1} + \cdots + q_k = 0 \tag{1.8}$$

of the limiting equation (1.6), and suppose that

$$|\lambda_i| \neq |\lambda_j| \quad for \quad i \neq j. \tag{1.9}$$

If x_n *is a solution of (1.4), then either* $x_n = 0$ *or there exists an index* $j \in \{1,\dots,k\}$ *such that*

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda_j.$$

The related results were obtained by Perron [9], and one of Perron's result was improved by Pituk [20, 21].

Theorem 1.2. *Suppose that (1.5) holds. If* x_n *is a solution of (1.4), then either* $x_n = 0$ *or eventually*

$$\limsup_{n\to\infty} (|x_n|)^{1/n} = |\lambda_j|,$$

where $\lambda_1,\dots,\lambda_k$ *are the (not necessarily distinct) roots of the characteristic equation (1.8).*

2. Rate of convergence

In this section, we first discuss the rate of convergence of Thomson's equation, which is a special case of Equation (1.1).

2.1. Rate of convergence of the Thomson equation

The special case of Equation (1.1), when $E = 0$, is the well-known sigmoid Beverton-Holt or Thomson equation

$$x_{n+1} = \frac{Ax_n^2}{x_n^2 + f}, \quad n = 0, 1, \dots, \tag{2.1}$$

which is used in modeling of fish population [2, 23].

The equilibrium points of Equation (2.1) are the positive solutions of the following equation

$$\bar{x} = \frac{A\bar{x}^2}{\bar{x}^2 + f},$$

or equivalently

$$\bar{x}\left(\bar{x}^2 - A\bar{x} + f\right) = 0, \tag{2.2}$$

from which $\bar{x}_1 = 0$ and $\bar{x}_\pm = \frac{A\pm\sqrt{A^2-4f}}{2}$ for $A^2 - 4f \geq 0$. Thus $\bar{x}_\pm > 0$ if $f \leq \frac{1}{4}A^2$.

If we set $h(u) = \frac{Au^2}{u^2+f}$, we obtain $h'(u) = \frac{2Afu}{(u^2+f)^2}$ from which, using (2.2),

$$h'(\bar{x}_1) = 0, \quad h'(\bar{x}_\pm) = \frac{2f}{A\bar{x}_\pm}.$$

It means that $\bar{x}_1$ is locally asymptotically stable for all values of positive parameters A and f. As is well known, the equilibrium $\bar{x}_1 = 0$ is globally asymptotically stable for $A^2 - 4f \leq 0$, for all initial values and for $A^2 - 4f > 0$ within its basin of attraction $(0, \bar{x}_-)$. The larger positive equilibrium $\bar{x}_+$ is globally asymptotically stable for $A^2 - 4f > 0$ within its basin of attraction $(\bar{x}_-, \infty)$.

Consider the rate of convergence to $\bar{x}_1 = 0$ when it is globally asymptotically stable. We have

$$\lim_{n\to\infty} \frac{x_{n+1} - \bar{x}_1}{(x_n - \bar{x}_1)^2} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n^2} = \lim_{n\to\infty} \frac{A}{x_n^2 + f} = \frac{A}{f}.$$

When $\bar{x} = \bar{x}_+$ is globally asymptotically stable, all solutions of Equation (2.1), which are different from all equilibrium points satisfy the following

$$\lim_{n\to\infty} \frac{x_{n+1} - \bar{x}_+}{x_n - \bar{x}_+} = \frac{2f}{A\bar{x}_+} = \frac{4f}{A\left(A + \sqrt{A^2 - 4f}\right)}.$$

Thus, we conclude that the rate of convergence for Thomson's equation is quadratic for the zero equilibrium and linear for the positive equilibrium solution within the basins of attraction of these solutions.

2.2. **Rate of convergence of Equation (1.1)**

	The conditions	The equilibrium points	P2-solutions	
1)	$0 < E < f - \frac{1}{4}A^2$	$\bar{x}_1 = 0$ (LAS)	-	
2)	$0 < E = f - \frac{1}{4}A^2$	$\begin{cases} \bar{x}_1 = 0 \text{ (LAS)} \\ \bar{x}_2 = \frac{A}{2} \\ \text{(NH, } \lambda_1 = 1, \lambda_2 \in (-1,0)\text{)} \end{cases}$	-	$\bar{x}_- = \bar{x}_+ = \frac{A}{2} < A < \frac{Af}{E}$
3)	$f - \frac{1}{4}A^2 < E < f$	$\begin{cases} \bar{x}_1 = 0 \text{ (LAS)} \\ \bar{x}_2 = \bar{x}_- \text{ (SP)} \\ \bar{x}_3 = \bar{x}_+ \text{ (LAS)} \end{cases}$	-	$\frac{A}{2} < \bar{x}_- < \bar{x}_+ < A < \frac{Af}{E}$
4)	$E = f$	$\begin{cases} \bar{x}_1 = 0 \text{ (NH, } \lambda_{1,2} = \pm 1\text{)} \\ \bar{x}_2 = \bar{x}_+ = A \text{ (LAS)} \end{cases}$	-	$\bar{x}_+ = A = \frac{Af}{E}$
5)	$f < E < f + \frac{3}{4}A^2$	$\begin{cases} \bar{x}_1 = 0 \text{ (R)} \\ \bar{x}_2 = \bar{x}_+ \text{ (LAS)} \end{cases}$	-	$\bar{x}_+ > A > \frac{Af}{E}$
6)	$E = f + \frac{3}{4}A^2$	$\begin{cases} \bar{x}_1 = 0 \text{ (R)} \\ \bar{x}_2 = \bar{x}_+ = \frac{3}{2}A \\ \text{(NH, } \lambda_1 = -1, \lambda_2 \in (0,1)\text{)} \end{cases}$	-	$\bar{x}_+ > A > \frac{Af}{E}$
7)	$f + \frac{3}{4}A^2 < E < f + A^2$	$\begin{cases} \bar{x}_1 = 0 \text{ (R)} \\ \bar{x}_2 = \bar{x}_+ \text{ (SP)} \end{cases}$	$\begin{cases} (\phi, \psi) \text{ (LAS)} \\ (\psi, \phi) \text{ (LAS)} \end{cases}$	$\psi > \bar{x}_+ > \phi > A > \frac{Af}{E}$
8)	$E \geq f + A^2$	$\begin{cases} \bar{x}_1 = 0 \text{ (R)} \\ \bar{x}_2 = \bar{x}_+ \text{ (SP)} \end{cases}$	-	$\bar{x}_+ > A > \frac{Af}{E}$

Table 1: *Existence and local stability of equilibrium and period-two solutions. Here LAS stands for locally asymptotically stable, SP stands for saddle point, R stands for repeller and NH stands for non-hyperbolic.*

Table 1 gives an overview of the existence and local stability of equilibrium and period-two solutions of Equation (1.1). Based on that table, we will present the rate of convergence to each equilibrium or period-two solution in each of these cases.

The following result, which gives global dynamics of Equation (1.1) is based on Theorem 4.3 in [14].

Theorem 2.1. *If* $0 < E < f - \frac{A^2}{4}$*, then all solutions of Equation (1.1) which are eventually different from the equilibrium* $\bar{x} = 0$ *satisfy the following*

$$\limsup_{n\to\infty} \left(|x_n|\right)^{1/n} = \sqrt{\frac{E}{f}}.$$

Proof. The error on n-th step term $e_n = x_n - \bar{x}$ satisfies

$$x_{n+1} - \bar{x} = \frac{Ax_n}{x_n^2 + f}\left(x_n - \bar{x}\right) + \frac{E}{x_n^2 + f}\left(x_{n-1} - \bar{x}\right).$$

which implies

$$e_{n+1} + c_n e_n + d_n e_{n-1} = 0,$$

where

$$c_n = -\frac{Ax_n}{x_n^2 + f}, \quad d_n = -\frac{E}{x_n^2 + f}.$$

Since the equilibrium point $\bar{x} = 0$ is a global attractor, we have

$$\lim_{n\to\infty} c_n = 0, \quad \lim_{n\to\infty} d_n = -\frac{E}{f}.$$

The limiting equation of Equation (1.1) is the linearized equation $z_{n+1} - \frac{E}{f} z_{n-1} = 0$ whose characteristic equation is

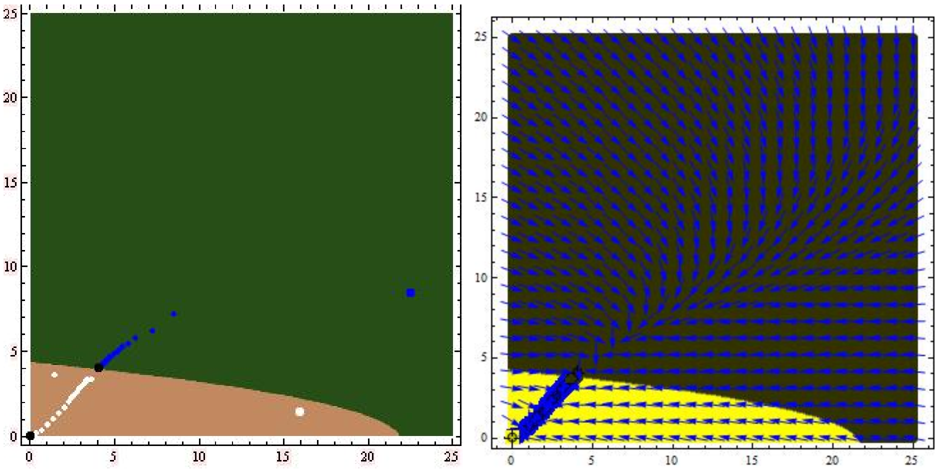
$$\lambda^2 - \frac{E}{f} = 0.$$

Since $|\lambda_1| = |\lambda_2| = \sqrt{\frac{E}{f}}$, the conclusion follows from Theorem 1.2. $\square$

Remark 2.1. In the case $0 < E = f - \frac{A^2}{4}$, Equation (1.1) has two equilibrium points (see Theorem 4.4. in [14]): $\bar{x}_1 = 0$ (which is locally asymptotically stable) and $\bar{x}_2 = \frac{A}{2}$ (which is a non-hyperbolic and semi-stable). Then there exists an invariant curve $\mathcal{C}$ which is a graph of a strictly decreasing continuous function on interval and separates $[0,\infty)^2$ into two connected and invariant components $\mathcal{W}_-\left((\bar{x}_2,\bar{x}_2)\right)$, and $\mathcal{W}_+\left((\bar{x}_2,\bar{x}_2)\right)$, such that $\mathcal{W}_-$ is a basin of attraction of $(0,0)$ and $\mathcal{W}_+$ is a basin of attraction of $(\frac{A}{2}, \frac{A}{2})$. It means that $(0,0)$ is globally asymptotically stable with respect to the set $\mathcal{W}_-$ and the same result about the rate of convergence to $\bar{x}_1 = 0$ as in Theorem 2.1 holds, that is

$$\limsup_{n\to\infty} \left(|x_n|\right)^{1/n} = \sqrt{\frac{E}{f}} = \sqrt{1 - \frac{A^2}{4f}}.$$

(See Figure 1).



(a) (b)

FIGURE 1. *Basins of attraction (for $A=8$, $E=4$, $f=20$, $\bar{x}_1=0$, $\bar{x}_2=4$) of:*
(a) $(\bar{x}_1,\bar{x}_1)$ *(brown),* $(\bar{x}_2,\bar{x}_2)$ *(green) generated by Dynamica 4,*
(b) $(\bar{x}_1,\bar{x}_1)$ *(yelow),* $(\bar{x}_2,\bar{x}_2))$ *(blue) generated by Code Bif2D [22].*

The following result is based on Theorem 4.1 in [14].

Theorem 2.2. *If $E=f$, then all solutions of Equation (1.1), which are eventually different from the equilibrium points $\bar{x}_1=0$ and $\bar{x}_2=A$, satisfy the following*

$$\limsup_{n\to\infty}\left(|x_n|\right)^{1/n}=\sqrt{\frac{E}{A^2+E}}.$$

Proof. The linearized equation of Equation (1.1) about $\bar{x}=\bar{x}_2=A$ is

$$z_{n+1}-\frac{E}{A^2+E}z_{n-1}=0, \tag{2.3}$$

whose characteristic equation is $\lambda^2-\frac{E}{A^2+E}=0$ with the roots $\lambda_{1,2}=\pm\sqrt{\frac{E}{A^2+E}}$.

It can be shown that

$$x_{n+1}-\bar{x}=\frac{Ax_n^2+Ex_{n-1}}{x_n^2+E}-\bar{x}=\frac{(A-\bar{x})\,x_n}{x_n^2+E}\left(x_n-\bar{x}\right)+\frac{E}{x_n^2+E}\left(x_{n-1}-\bar{x}\right),$$

i.e.,

$$e_{n+1}+c_ne_n+d_ne_{n-1}=0,$$

where

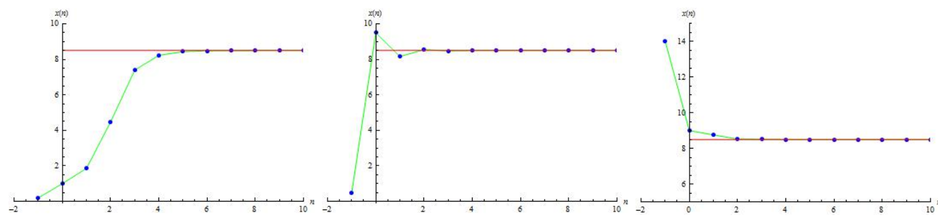
$$e_n=x_n-\bar{x},\quad c_n=-\frac{(A-\bar{x})\,x_n}{x_n^2+E},\quad d_n=-\frac{E}{x_n^2+E}.$$

As the equilibrium point $\bar{x}_2=A$ is a global attractor, we obtain

$$\lim_{n\to\infty}c_n=0,\quad \lim_{n\to\infty}d_n=-\frac{E}{A^2+E}.$$

Thus, the limiting equation of (1.1) is the linearized equation (2.3). Now, the conclusion follows as an immediate consequence of Theorem 1.2 and the fact that $|\lambda_1| = |\lambda_2|$. □

Remark 2.2. In Figure 2, the time series plots show the rate of convergence in the case of Theorem 2.2.



(a) (b) (c)

FIGURE 2. *Time series plot when* $\overline{x} = A = 8.5$, $E = f = 4$ *and* (a) $x_0 = 1.0$, $x_{-1} = 0.2$, (b) $x_0 = 9.5$, $x_{-1} = 0.5$ *and* (c) $x_0 = 9.0$, $x_{-1} = 14.0$

The following result is based on Theorem 4.5 in [14].

Theorem 2.3. *Assume that* $\max\left\{0, f - \frac{A^2}{4}\right\} < E < f$. *Then, all solutions of Equation (1.1) with initial point* $(x_{-1}, x_0) \in \mathcal{W}_+ ((\overline{x}_2, \overline{x}_2))$, *and which are eventually different from the equilibrium* $\overline{x}_3 = \overline{x}_+$, *satisfy the following*

$$\lim_{n\to\infty} \frac{x_{n+1} - \overline{x}_+}{x_n - \overline{x}_+} = \lambda_+ \quad or \quad \lim_{n\to\infty} \frac{x_{n+1} - \overline{x}_+}{x_n - \overline{x}_+} = \lambda_-,$$

where $\lambda_\pm$ *are the real roots of the following equation*

$$\lambda^2 - \frac{2(f-E)}{f + \overline{x}_+^2}\lambda - \frac{E}{f + \overline{x}_+^2} = 0 \tag{2.4}$$

which is the characteristic equation of the linearized equation of Equation (1.1) about $\overline{x}_3 = \overline{x}_+$.

Here, $\mathcal{W}_+ ((\overline{x}_2, \overline{x}_2))$ *is one of two connected and invariant components to which the set* $\mathcal{C}$, *which is an invariant subset of the basin of attraction of* $\overline{x}_2$, *separates* $\mathcal{R} = [0, +\infty)^2$ *(see Figures 3 and 4).*

Proof. Under the assumption of Theorem 4.5 in [14], we proved the folllowing: if $(x_{-1}, x_0) \in \mathcal{W}_+ ((\overline{x}_2, \overline{x}_2))$, then $\lim_{n\to\infty} x_n = \overline{x}_+$.

It can be shown that

$$\begin{aligned} x_{n+1} - \overline{x}_+ &= \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + f} - \overline{x}_+ \\ &= \frac{(A - \overline{x}_+)x_n + f - E}{x_n^2 + f}(x_n - \overline{x}_+) + \frac{E}{x_n^2 + f}(x_{n-1} - \overline{x}_+). \end{aligned}$$

Then we have that

$$e_{n+1}+c_ne_n+d_ne_{n-1}=0, \tag{2.5}$$

where

$$e_n=x_n-\bar{x},\quad c_n=-\frac{(A-\bar{x}_+)x_n+f-E}{x_n^2+f},\quad d_n=-\frac{E}{x_n^2+f}.$$

Since the equilibrium point $\bar{x}_3=\bar{x}_+$ is a global attractor in $\mathcal{W}_+(\bar{x}_2,\bar{x}_2)$, we obtain

$$\lim_{n\to\infty}c_n=\frac{2(f-E)}{\bar{x}_+^2+f},\quad \lim_{n\to\infty}d_n=-\frac{E}{\bar{x}_+^2+f}.$$

Thus, the limiting equation of Equation (2.5) is the linearized equation of Equation (1.1) about $\bar{x}_3=\bar{x}_+$ whose characteristic equation is (2.4). Now, the conclusion follows as an immediate consequence of Theorem 1.1 and the fact that the discriminant of Equation (2.4) is positive. □

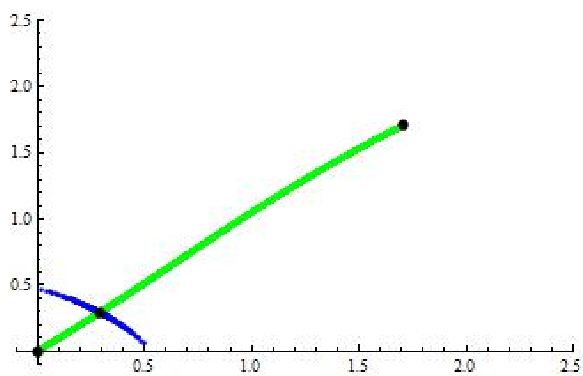


FIGURE 3. *Stable (blue) and unstable (green) manifolds of saddle point $(\bar{x}_2,\bar{x}_2)$, $\bar{x}_2\approx 0.293$ for $A=2$, $E=4.5$ and $f=5$.*

The following result is based on Theorem 4.8 in [14].

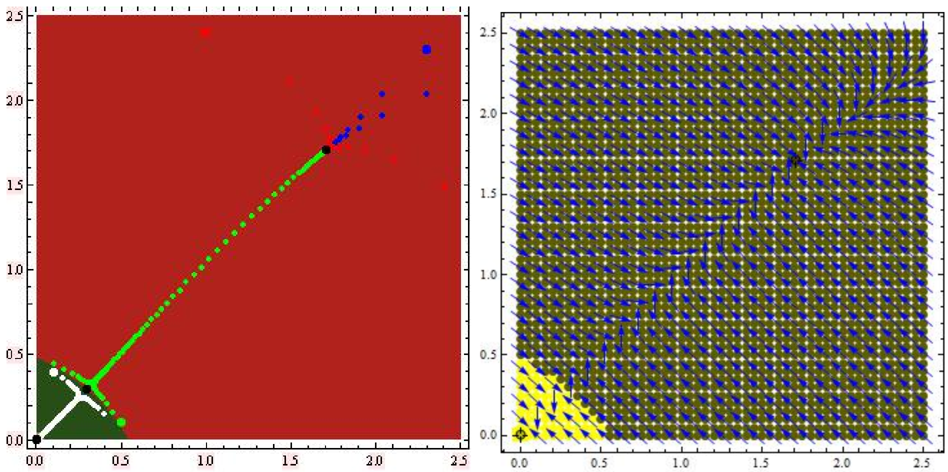
Theorem 2.4. *Assume that $E=f+\frac{3}{4}A^2$. Then, all solutions of Equation (1.1), which are eventually different from the equilibrium points $\bar{x}_1=0$ and $\bar{x}_2=\bar{x}_+=\frac{3}{2}A$, satisfy the following*

$$\lim_{n\to\infty}\frac{x_{n+1}-\bar{x}_+}{x_n-\bar{x}_+}=\lambda_+\quad or\quad \lim_{n\to\infty}\frac{x_{n+1}-\bar{x}_+}{x_n-\bar{x}_+}=\lambda_-,$$

where $\lambda_\pm$ are the real roots of the following equation

$$\lambda^2+\frac{3A^2}{3A^2+2E}\lambda-\frac{2E}{3A^2+2E}=0 \tag{2.6}$$

which is the characteristic equation of the linearized equation of Equation (1.1) about $\bar{x}_2=\bar{x}_+=\frac{3}{2}A$.



(a) (b)

FIGURE 4. *Basins of attraction* ($A=2$, $E=4.5$, $f=5$, $\bar{x}_1=0$, $\bar{x}_3\approx 1.71$) *of:*
(a) $(\bar{x}_1,\bar{x}_1)$ (green), $(\bar{x}_3,\bar{x}_3)$ (red) generated by Dynamica 4,
(b) $(\bar{x}_1,\bar{x}_1)$ (yellow), $(\bar{x}_3,\bar{x}_3)$ (black) generated by Code Bif2D [22].

Proof. Under the assumption of Theorem 4.8 in [14], we proved the following: if $(x_{-1},x_0)\in(0,+\infty)$, then $\lim\limits_{n\to\infty} x_n=\bar{x}_+=\frac{3}{2}A$. It can be shown that

$$\begin{aligned} x_{n+1}-\bar{x}_+ &= \frac{Ax_n^2+Ex_{n-1}}{x_n^2+f}-\bar{x}_+ \\ &= -\frac{\frac{1}{2}A\left(x_n+\bar{x}_+\right)}{x_n^2+E-\frac{3}{4}A^2}\left(x_n-\bar{x}_+\right)+\frac{E}{x_n^2+E-\frac{3}{4}A^2}\left(x_{n-1}-\bar{x}_+\right). \end{aligned}$$

Then we have that

$$e_{n+1}+c_ne_n+d_ne_{n-1}=0, \tag{2.7}$$

where

$$e_n=x_n-\bar{x},\quad c_n=\frac{\frac{1}{2}A\left(x_n+\bar{x}_+\right)}{x_n^2+E-\frac{3}{4}A^2},\quad d_n=-\frac{E}{x_n^2+E-\frac{3}{4}A^2}.$$

Since the equilibrium point $\bar{x}_+=\frac{3}{2}A$ is globally attractor in $(0,+\infty)$, we obtain

$$\lim_{n\to\infty} c_n=\frac{3A^2}{3A^2+2E},\quad \lim_{n\to\infty} d_n=-\frac{2E}{3A^2+2E}.$$

Thus, the limiting equation of Equation (2.7) is the linearized equation of Equation (1.1) about $\bar{x}_2=\bar{x}_+=\frac{3}{2}A$ whose characteristic equation is (2.6). Now, the conclusion follows as an immediate consequence of Theorem 1.1 and the fact that the discriminant of Equation (2.6) is positive. $\square$

The following result is based on Theorem 4.9 in [14].

Theorem 2.5. *Assume that $f+\frac{3}{4}A^2<E<f+A^2$. Then, Equation (1.1) has two equilibrium points: $\bar{x}_1=0$ which is a repeller and $\bar{x}_2=\bar{x}_+$ which is a saddle point, and has the unique minimal period-two solution $\{\ldots\phi,\psi,\phi,\psi,\ldots\}$, which is locally asymptotically stable. There exists a set $\mathcal{C}\subset Q_1(\bar{x}_2,\bar{x}_2)\cup Q_3(\bar{x}_2,\bar{x}_2)$ and $\mathcal{W}^s((\bar{x}_2,\bar{x}_2))=\mathcal{C}$ is the basin of attraction of $(\bar{x}_2,\bar{x}_2)$. The set $\mathcal{C}$ is a graph of a strictly increasing continuous function of the first variable on an interval and separates $\mathcal{R}_1=[0,+\infty)^2\setminus\{(0,0)\}$ into two connected and invariant parts, $\mathcal{W}_-^{(1)}((\bar{x}_2,\bar{x}_2))$ and $\mathcal{W}_+^{(1)}((\bar{x}_2,\bar{x}_2))$, where*

$$\mathcal{W}_-^{(1)}((\bar{x}_2,\bar{x}_2)):=\left\{(x,y)\in\mathcal{R}_1\setminus\mathcal{C}:\exists\left(x',y'\right)\in\mathcal{C}\text{ with }(x,y)\preceq_{se}\left(x',y'\right)\right\},$$
$$\mathcal{W}_+^{(1)}((\bar{x}_2,\bar{x}_2)):=\left\{(x,y)\in\mathcal{R}_1\setminus\mathcal{C}:\exists\left(x',y'\right)\in\mathcal{C}\text{ with }\left(x',y'\right)\preceq_{se}(x,y)\right\}.$$

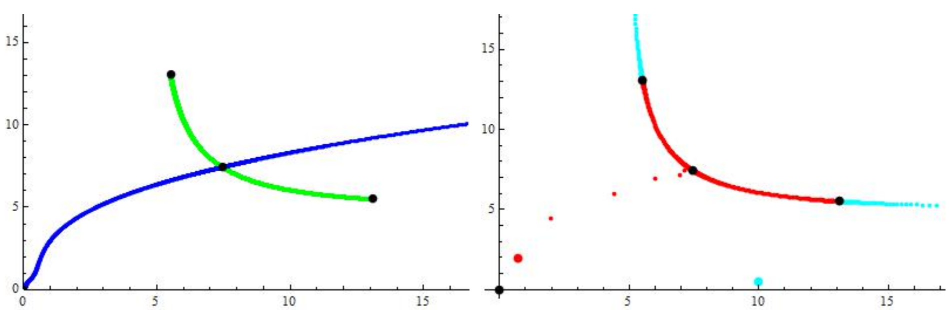
Also, every solution with initial point $(x_{-1},x_0)\notin\left(\mathcal{W}_+^{(1)}(\bar{x}_2,\bar{x}_2)\cup\mathcal{W}_-^{(1)}(\bar{x}_2,\bar{x}_2)\right)$, and which is eventually different from a period-two solution, that converges to the period-two solution, satisfies one of two the following asymptotic relations:

$$\lim_{n\to\infty}\frac{x_{2n+1}-\psi}{x_{2n-1}-\psi}=\lambda_+\quad or\quad\lim_{n\to\infty}\frac{x_{2n+1}-\psi}{x_{2n-1}-\psi}=\lambda_-,$$
$$\lim_{n\to\infty}\frac{x_{2n+2}-\phi}{x_{2n}-\phi}=\lambda_+\quad or\quad\lim_{n\to\infty}\frac{x_{2n+2}-\phi}{x_{2n}-\phi}=\lambda_-,$$

where $\lambda_\pm$ are the roots of the following equation

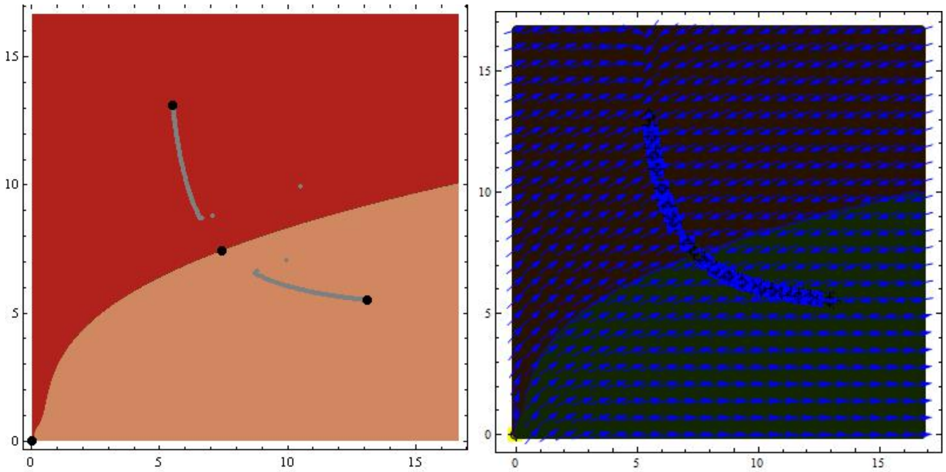
$$\lambda^2-\left(\frac{4\phi\psi(A-\phi)(A-\psi)}{(f+\psi^2)(f+\phi^2)}+\frac{E}{f+\phi^2}+\frac{E}{f+\psi^2}\right)\lambda+\frac{E^2}{(f+\psi^2)(f+\phi^2)}=0, \qquad (2.8)$$

which is the characteristic equation of the linearized equation of Equation (1.1) about (ϕ,ψ). (See Figures 5 and 6).



(a) (b)

FIGURE 5. (a) *Stable (blue) and unstable (green) manifolds of saddle point $(\bar{x}_2,\bar{x}_2)$, $\bar{x}_2\approx 7.44951$;* (b) *orbits with initial conditions $(\bar{x}_{-1},\bar{x}_0)=(10.0,0.5)$ (cyan) and $(\bar{x}_{-1},\bar{x}_0)=(0.6907,1.973)$ (red) for $A=4.899$, $E=23$ and $f=4$.*



(a) (b)

FIGURE 6. *Basins of attraction of P2 solution* (ψ,φ) *and* (φ,ψ), $\psi \approx 13.1073$, $\varphi \approx 5.50817$ *for* $A = 4.899$, $E = 23$ *and* $f = 4$*:* (a) *generated by Dynamica 4,* (b) *generated by Code Bif2D [22].*

Proof. Under the assumption of Theorem 4.9 in [14], we proved the part of this theorem without the rate of convergence of the solutions of Equation (1.1). Also, in [14] the following statements were proved:

(i) if $(x_{-1},x_0) \in \mathcal{W}_{+}^{(1)}(\bar{x}_2,\bar{x}_2)$, then $\lim_{n\to\infty} x_{2n} = \phi$ and $\lim_{n\to\infty} x_{2n+1} = \psi$;

(ii) if $(x_{-1},x_0) \in \mathcal{W}_{-}^{(1)}(\bar{x}_2,\bar{x}_2)$, then $\lim_{n\to\infty} x_{2n} = \psi$ and $\lim_{n\to\infty} x_{2n+1} = \phi$.

Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Equation (1.1). Then

$$\phi = \frac{A\psi^2 + E\phi}{\psi^2 + f}, \quad \psi = \frac{A\phi^2 + E\psi}{\phi^2 + f},$$

and

$$x_{n+1} - \psi = \frac{(Af - E\psi)\,x_n}{(\phi^2+f)\,(x_n^2+f)} x_n + \frac{E\left(\phi^2+f\right)}{(\phi^2+f)\,(x_n^2+f)} x_{n-1} - \frac{f\psi}{x_n^2+f},$$

$$x_{n+1} - \phi = \frac{(Af - E\phi)\,x_n}{(\psi^2+f)\,(x_n^2+f)} x_n + \frac{E\left(\psi^2+f\right)}{(\psi^2+f)\,(x_n^2+f)} x_{n-1} - \frac{f\phi}{x_n^2+f}.$$

and

$$\begin{aligned} x_{2k+1} - \psi &= \frac{(Af - E\psi)\,x_{2k}}{(\phi^2+f)\left(x_{2k}^2+f\right)}(x_{2k}-\phi) + \frac{E\left(\phi^2+f\right)}{(\phi^2+f)\left(x_{2k}^2+f\right)}(x_{2k-1}-\psi) \\ &\quad + \frac{(Af-E\psi)\,\phi}{(\phi^2+f)\left(x_{2k}^2+f\right)}(x_{2k}-\phi) + \frac{(Af-E\psi)\,\phi^2}{(\phi^2+f)\left(x_{2k}^2+f\right)} + \frac{\psi\,(E-f)\left(\phi^2+f\right)}{(\phi^2+f)\left(x_{2k}^2+f\right)} \\ &= \frac{(Af - E\psi)\,(x_{2k}+\phi)}{(\phi^2+f)\left(x_{2k}^2+f\right)}(x_{2k}-\phi) + \frac{E\left(\phi^2+f\right)}{(\phi^2+f)\left(x_{2k}^2+f\right)}(x_{2k-1}-\psi), \end{aligned}$$

i.e.,

$$x_{2k+1}-\psi=\frac{C_k}{A_k}\left(x_{2k}-\phi\right)+\frac{E\left(\phi^2+f\right)}{A_k}\left(x_{2k-1}-\psi\right), \tag{2.9}$$

where

$$A_k=\left(\phi^2+f\right)\left(x_{2k}+f\right),\quad C_k=\left(Af-E\psi\right)\left(x_{2k}+\phi\right).$$

Also, in a similar way we obtain

$$\begin{aligned}
x_{2k}-\phi=&\frac{\left(Af-E\phi\right)x_{2k-1}}{\left(\psi^2+f\right)\left(x_{2k-1}^2+f\right)}\left(x_{2k-1}-\psi\right)+\frac{E\left(\psi^2+f\right)}{\left(\psi^2+f\right)\left(x_{2k-1}^2+f\right)}\left(x_{2k-2}-\phi\right)\\
&+\frac{\left(Af-E\phi\right)\psi}{\left(\psi^2+f\right)\left(x_{2k-1}^2+f\right)}\left(x_{2k-1}-\psi\right)+\frac{\left(Af-E\phi\right)\psi^2}{\left(\psi^2+f\right)\left(x_{2k-1}^2+f\right)}+\frac{\phi\left(E-f\right)\left(\psi^2+f\right)}{\left(\psi^2+f\right)\left(x_{2k-1}^2+f\right)}\\
=&\frac{\left(Af-E\phi\right)\left(x_{2k-1}+\psi\right)}{\left(\psi^2+f\right)\left(x_{2k-1}^2+f\right)}\left(x_{2k-1}-\psi\right)+\frac{E\left(\psi^2+f\right)}{\left(\psi^2+f\right)\left(x_{2k-1}^2+f\right)}\left(x_{2k-2}-\phi\right),
\end{aligned}$$

i.e.,

$$x_{2k}-\phi=\frac{D_k}{B_k}\left(x_{2k-1}-\psi\right)+\frac{E\left(\psi^2+f\right)}{B_k}\left(x_{2k-2}-\phi\right), \tag{2.10}$$

where

$$B_k=\left(\psi^2+f\right)\left(x_{2k-1}^2+f\right),\quad D_k=\left(Af-E\phi\right)\left(x_{2k-1}+\psi\right).$$

By using (2.10) and (2.9) we get

$$x_{2k+2}-\phi=\left(\frac{C_kD_{k+1}}{A_kB_{k+1}}+\frac{E\left(\psi^2+f\right)}{B_{k+1}}\right)\left(x_{2k}-\phi\right)+\frac{E\left(\phi^2+f\right)D_{k+1}}{A_kB_{k+1}}\left(x_{2k-1}-\psi\right). \tag{2.11}$$

It is obvious that (2.10) implies the following relation

$$x_{2k-1}-\psi=\frac{B_k}{D_k}\left(x_{2k}-\phi\right)-\frac{E\left(\psi^2+f\right)}{D_k}\left(x_{2k-2}-\phi\right). \tag{2.12}$$

By substituting (2.12) into (2.11) we obtain

$$\begin{aligned}
x_{2k+2}-\phi=&\left(\frac{C_kD_k}{A_kB_{k+1}}+\frac{E\left(\psi^2+f\right)}{B_{k+1}}+\frac{E\left(\phi^2+f\right)B_kD_{k+1}}{A_kB_{k+1}D_k}\right)\left(x_{2k}-\phi\right)\\
&-\frac{E^2\left(\phi^2+f\right)\left(\psi^2+f\right)D_{k+1}}{A_kB_{k+1}D_k}\left(x_{2k-2}-\phi\right).
\end{aligned} \tag{2.13}$$

If we set $e_k=x_{2k}-\phi$, Equation (2.13) becomes

$$e_{k+1}=c_ke_k+d_ke_{k-1},\quad k=0,1,..., \tag{2.14}$$

where

$$c_k=\frac{C_kD_k}{A_kB_{k+1}}+\frac{E\left(\psi^2+f\right)}{B_{k+1}}+\frac{E\left(\phi^2+f\right)B_kD_{k+1}}{A_kB_{k+1}D_k},\quad d_k=-\frac{E^2\left(\phi^2+f\right)\left(\psi^2+f\right)D_{k+1}}{A_kB_{k+1}D_k},$$

with

$$\lim_{k\to\infty} c_k = \frac{4\phi\psi(A-\phi)(A-\psi)}{(f+\psi^2)(f+\phi^2)} + \frac{E}{f+\psi^2} + \frac{E}{f+\phi^2},$$

$$\lim_{k\to\infty} d_k = -\frac{E^2}{(f+\psi^2)(f+\phi^2)}.$$

In view of

$$\lim_{k\to\infty} A_k = \left(f+\phi^2\right)^2, \quad \lim_{k\to\infty} B_k = \left(f+\psi^2\right)^2,$$

$$\lim_{k\to\infty} C_k = 2\phi(Af-E\psi) = 2\phi(A-\psi)\left(f+\phi^2\right),$$

$$\lim_{k\to\infty} D_k = 2\psi(Af-E\phi) = 2\psi(A-\phi)\left(f+\psi^2\right),$$

(see [14], page 139), the limiting equation of (2.14) is

$$e_{k+1} - c_1 e_k - d_1 e_{k-1} = 0, \tag{2.15}$$

where

$$c_1 = \frac{4\phi\psi(A-\phi)(A-\psi)}{(f+\psi^2)(f+\phi^2)} + \frac{E}{f+\psi^2} + \frac{E}{f+\phi^2},$$

$$d_1 = -\frac{E^2}{(f+\psi^2)(f+\phi^2)}.$$

The characteristic equation of (2.15) is

$$\lambda^2 - \left(\frac{4\phi\psi(A-\phi)(A-\psi)}{(f+\psi^2)(f+\phi^2)} + \frac{E}{f+\psi^2} + \frac{E}{f+\phi^2}\right)\lambda + \frac{E^2}{(f+\psi^2)(f+\phi^2)} = 0. \tag{2.16}$$

Note that (2.16) is the characteristic equation of the map T^2, evaluated at the period-two solution. Also, we see that the discriminant of (2.16) is a positive number.

In an analogous way it is obtained that

$$e_{k+1} - a_k e_k - b_k e_{k-1} = 0, \quad k = 0,1,..., \tag{2.17}$$

where

$$e_k = x_{2k-1} - \psi,$$

$$a_k = \frac{C_k D_k}{A_k B_k} + \frac{E\left(\phi^2+f\right)}{A_k} + \frac{E\left(\psi^2+f\right)A_{k-1}C_k}{A_k B_k C_{k-1}},$$

$$b_k = -\frac{E^2\left(\phi^2+f\right)\left(\psi^2+f\right)C_k}{A_k B_k C_{k-1}},$$

with

$$c_1 = \lim_{k\to\infty} a_k = \lim_{k\to\infty} c_k, \quad d_1 = \lim_{k\to\infty} b_k = \lim_{k\to\infty} d_k.$$

Thus, the limiting equation of (2.17) is (2.15). Using Theorem 1.1 the conclusion of the theorem follows. □

3. Conclusion

As we mentioned in the the introduction section, the perturbation term Ex_{n-1} in equation (1.1) effects both the global dynamics of Thomson's model by creating an attracting period-two solution (Case 7 in Table 1) and also the rate of convergence of the solutions toward all the attractors. The effect can be seen in the simplest case of convergence to the globally asymptotically stable zero equilibrium $\bar{x}_0 = 0$ in both equations. Namely, in the original Thomson's model (1.2) the rate of convergence toward $\bar{x}_0 = 0$ is A/f when $A^2 - 4f \geq 0$, while in the perturbed equation (1.1), the rate of convergence is $\sqrt{\frac{E}{f}}$ when $4E < 4f - A^2$.

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