SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.20.01.12
Vol.20 (33), No.1 (2024), 137–148

THE HALF-INVERSE TRANSMISSION PROBLEM FOR A STURM-LIOUVILLE-TYPE DIFFERENTIAL EQUATION WITH THE FIXED DELAY AND NON ZERO INITIAL FUNCTION

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Dedicated to the 75th Birthday of Professor Academician Mirjana Vukovic with deep esteem

ABSTRACT. In this paper, we consider the boundary value problem for the Sturm-Liouville type equation with the fixed delay $\frac{\pi}{2}$ and a non zero initial function under transmission conditions at the delay point. We study the case when all parameters within the transmission conditions are known and the potential function is known on the interval $\left(0, \frac{\pi}{2}\right)$. We will prove the uniqueness theorem from two spectra, first with Neumann boundary conditions and second with Cauchy boundary condition. Additionally, we will present an algorithm for the construction of the potential function over the interval $\left(\frac{\pi}{2}, \pi\right)$.

1. INTRODUCTION

Inverse spectral problems are generally defined as the determination of operators using some known spectral characteristics. Such problems have been successfully solved for various classes of operators or boundary value problems (BVPs). One of the most important classes of the BVPs are BVPs generated with the Sturm–Liouville differential equation $-y''(x) + q(x)y(x) = \lambda y(x)$ under different kinds of boundary conditions, so-called classical Sturm-Liouville problem(inverse or direct). A very important paper in inverse spectral theory devoted to this class BVPs is the paper of Borg [1], who proved that an L^1-potential is uniquely recovered from two spectra (sequences of eigenvalues) of boundary value problems with a common differential equation and one common boundary condition at $x = 0$. In addition to this result, many mathematicians contributed to the determination of the operator generated by the Sturm-Liouville differential equation based on other spectral characteristics and we can say that this inverse problem is fully solved. One of the directions for further research was BVPs generated with a Sturm-Liouville type differential equation with a constant delay $-y''(x) + q(x)y(x-a) = \lambda y(x)$. Most of the papers ([2, 4–9, 13–15]) in this direction are devoted to BVPs defined with

2020 *Mathematics Subject Classification.* 34K29,34B24.
Key words and phrases. Differential operators with delay, half-inverse problem, transmission conditions, Fourier trigonometric coefficients.

this differential equation together with different kinds of boundary conditions and initial functions equal to zero. In the paper [3] authors defined and solved Borg-type inverse problem for BVPs generated with Sturm-Liouville type differential equation with a constant delay and non zero initial function. In the paper [3], the authors make the assumption that initial function is a constant function, denoted as $y(0)$, where $y(0) \neq 0$. This assumption is made without a loss of generality.

The focus of our research in this paper is Borg-type transmission inverse problem for boundary value problems (BVPs) generated by the differential equation with a constant delay, where the delay is known and equal to $\frac{\pi}{2}$ and the initial function is not equal to zero, which is defined similar like in the paper [3] but with an essential difference in that BVPs in this paper defined with discontinuity conditions at the delay point or we can say that in this paper we study transmission inverse problem.

Boundary value problems with discontinuity conditions, as well as inverse problems dedicated to this BVPs or transmission inverse problems, have attracted the attention of many researchers ([10–12]). In the paper [10] the authors successfully solve a transmission inverse problem for a BVP generated by a Sturm-Liouville type differential equation with a constant delay, under discontinuity conditions at the delay point, where the delay is greater than $\frac{\pi}{2}$. Additionally, in the paper [16], the authors solve the inverse problem for the same BVPs where the delay is equal to $\frac{\pi}{2}$. It is important to mention, both of these papers are devoted to BVPs with the initial function equal to zero. Boundary value problems containing discontinuities within the interval are prevalent in various disciplines such as mathematics, mechanics, physics, geophysics, and other branches of the natural sciences. Frequently, these problems arise in association with materials exhibiting discontinuous properties.

This paper is devoted to studying two boundary value problems, denoted as L_0 and L_1, characterized as follows. The boundary value problem L_0 is defined with (1.1, 1.2, 1.3, 1.5) and the boundary value problem L_1 is defined with (1.1, 1.2, 1.4, 1.5), where:

$$-y''(x)+q(x)y\left(x-\frac{\pi}{2}\right)=\lambda y(x),\, x\in\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right), \tag{1.1}$$

$$y'(0)=0, \tag{1.2}$$

$$y(\pi)=0, \tag{1.3}$$

$$y'(\pi)=0. \tag{1.4}$$

The jump conditions are:

$$y\left(\frac{\pi}{2}+0\right)=by\left(\frac{\pi}{2}-0\right), y'\left(\frac{\pi}{2}+0\right)=b^{-1}y'\left(\frac{\pi}{2}-0\right)+cy\left(\frac{\pi}{2}-0\right). \tag{1.5}$$

The following notation is used: λ is the spectral parameter; $q(x)$ is a complex-valued function which we call potential, such that $q\in L^2(0,\pi)$.

We assume that $q(x)$ for $x \in (0, \frac{\pi}{2})$ and real parameters b, c are known. Also we assume that the initial function $y\left(x - \frac{\pi}{2}\right) \equiv y(0)$ on $[0, \frac{\pi}{2}]$.

It is known that both spectra L_0 and L_1 are countable. Let $(\lambda_n)_{n=1}^{\infty}$ be the eigenvalues of L_0 and $(\mu_n)_{n=1}^{\infty}$ be the eigenvalues of L_1. We will prove the Theorem of uniqueness for BVPs L_0 and L_1.

Considering the assumption that the potential q is known at $\left(0, \frac{\pi}{2}\right)$, the inverse problem is to prove that q is uniquely determined on the interval $\left(\frac{\pi}{2}, \pi\right)$ from $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$, and find q from $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$.

2. Preliminaries

Let the function $Y_1(x)$ be the solution of the differential equation (1.1) on the interval $\left(0, \frac{\pi}{2}\right)$ satisfying initial conditions $Y_1'(0) = 0, Y_1(0) = 1$. From these conditions we conclude $y\left(x - \frac{\pi}{2}\right) \equiv 1$ on $[0, \frac{\pi}{2}]$ and since the differential equation (1.1) has the form

$$-y''(x) + q(x)y\left(x - \frac{\pi}{2}\right) = \lambda y(x)$$

on the interval $\left(0, \frac{\pi}{2}\right)$, we solve this differential equation and we have

$$Y_1(x) = \cos zx + \frac{1}{z}\int_0^x q(t)\sin z(x-t)\,dt\,,$$

where $\lambda = z^2$.

The differential equation (1.1) on the interval $\left(\frac{\pi}{2}, \pi\right)$ has the form

$$-y''(x) + q(x)y\left(x - \frac{\pi}{2}\right) = \lambda y(x)\,,$$

i.e.

$$-y''(x) + q(x)\left[\cos z\left(x - \frac{\pi}{2}\right) + \frac{1}{z}\int_0^{x-\frac{\pi}{2}} q(t)\sin z\left(x - \frac{\pi}{2} - t\right)dt\right] = z^2 y(x).$$

Using the method of variation of parameters we have the general solution of the differential equation (1.1) on the interval $\left(\frac{\pi}{2}, \pi\right)$

$$\begin{aligned} Y_2(x) = {} & C_1\cos zx + C_2\sin zx - \frac{\cos zx}{z}\int_{\frac{\pi}{2}}^x q(t)\sin zt\cos z\left(t - \frac{\pi}{2}\right)dt \\ & - \frac{\cos zx}{z^2}\int_{\frac{\pi}{2}}^x\int_0^{t-\frac{\pi}{2}} q(t)q(t_1)\sin zt\sin z\left(t - \frac{\pi}{2} - t_1\right)dt_1\,dt + \frac{\sin zx}{z}\int_{\frac{\pi}{2}}^x q(t)\cos zt\cos z\left(t - \frac{\pi}{2}\right)dt \\ & + \frac{\sin zx}{z^2}\int_{\frac{\pi}{2}}^x\int_0^{t-\frac{\pi}{2}} q(t)q(t_1)\cos zt\sin z\left(t - \frac{\pi}{2} - t_1\right)dt_1\,dt\,. \end{aligned}$$

Since

$$\begin{aligned}
Y_1\left(\frac{\pi}{2}-0\right) &= \cos\frac{z\pi}{2}+\frac{1}{z}\int_0^{\frac{\pi}{2}} q(t)\sin z\left(\frac{\pi}{2}-t\right)dt\,,\\
Y_1'\left(\frac{\pi}{2}-0\right) &= -z\sin\frac{z\pi}{2}+\int_0^{\frac{\pi}{2}} q(t)\cos z\left(\frac{\pi}{2}-t\right)dt\,,\\
Y_2\left(\frac{\pi}{2}+0\right) &= C_1\cos\frac{z\pi}{2}+C_2\sin\frac{z\pi}{2}\,,\\
Y_2'\left(\frac{\pi}{2}+0\right) &= -zC_1\sin\frac{z\pi}{2}+zC_2\cos\frac{z\pi}{2}\,,
\end{aligned}$$

from transmission conditions (1.5) we have

$$C_1\cos\frac{z\pi}{2}+C_2\sin\frac{z\pi}{2}=b\cos\frac{z\pi}{2}+\frac{b}{z}\int_0^{\frac{\pi}{2}} q(t)\sin z\left(\frac{\pi}{2}-t\right)dt\,,$$

$$-zC_1\sin\frac{z\pi}{2}+zC_2\cos\frac{z\pi}{2}=-b^{-1}z\sin\frac{z\pi}{2}+b^{-1}\int_0^{\frac{\pi}{2}} q(t)\cos z\left(\frac{\pi}{2}-t\right)dt$$

$$+c\cos\frac{z\pi}{2}+\frac{c}{z}\int_0^{\frac{\pi}{2}} q(t)\sin z\left(\frac{\pi}{2}-t\right)dt\,.$$

This system of linear equations has a unique solution

$$C_1=b^{-1}\sin^2\frac{z\pi}{2}+b\cos^2\frac{z\pi}{2}-\frac{c}{2z}\sin z\pi$$

$$+\left(\frac{b}{z}\cos\frac{z\pi}{2}-\frac{c}{z^2}\sin\frac{z\pi}{2}\right)\int_0^{\frac{\pi}{2}} q(t)\sin z\left(\frac{\pi}{2}-t\right)dt-\frac{b^{-1}}{z}\sin\frac{z\pi}{2}\int_0^{\frac{\pi}{2}} q(t)\cos z\left(\frac{\pi}{2}-t\right)dt,$$

and

$$C_2=\frac{b-b^{-1}}{2}\sin z\pi+\frac{c}{z}\cos^2\frac{z\pi}{2}+\left(\frac{b}{z}\sin\frac{z\pi}{2}+\frac{c}{z^2}\cos\frac{z\pi}{2}\right)\int_0^{\frac{\pi}{2}} q(t)\sin z\left(\frac{\pi}{2}-t\right)dt$$

$$+\frac{b^{-1}}{z}\cos\frac{z\pi}{2}\int_0^{\frac{\pi}{2}} q(t)\cos z\left(\frac{\pi}{2}-t\right)dt.$$

From (1.3) we have the characteristic function of the boundary value problem L_0

$$F_0(z)=\Delta_0(\lambda)=Y_2(\pi)=\frac{1}{z}\int_{\frac{\pi}{2}}^{\pi} q(t)\sin z(\pi-t)\cos z\left(t-\frac{\pi}{2}\right)dt$$
$$+\frac{1}{z^2}\int_{\frac{\pi}{2}}^{\pi}\int_{0}^{t-\frac{\pi}{2}} q(t)q(t_1)\sin z(\pi-t)\sin z\left(t-\frac{\pi}{2}-t_1\right)dt_1dt$$
$$+\left(\frac{b}{z}\cos\frac{z\pi}{2}+\frac{c}{z^2}\sin\frac{z\pi}{2}\right)\int_{0}^{\frac{\pi}{2}} q(t)\sin z\left(\frac{\pi}{2}-t\right)dt$$
$$+\frac{b^{-1}}{z}\sin\frac{z\pi}{2}\int_{0}^{\frac{\pi}{2}} q(t)\cos z\left(\frac{\pi}{2}-t\right)dt+\frac{b-b^{-1}}{2}+\frac{b+b^{-1}}{2}\cos z\pi+\frac{c}{2z}\sin z\pi, \quad (2.1)$$

and from (1.4) we have the characteristic function of the boundary value problem L_1

$$F_1(z)=\Delta_1(\lambda)=Y_2'(\pi)=\int_{\frac{\pi}{2}}^{\pi} q(t)\cos z(\pi-t)\cos z\left(t-\frac{\pi}{2}\right)dt$$
$$+\frac{1}{z}\int_{\frac{\pi}{2}}^{\pi}\int_{0}^{t-\frac{\pi}{2}} q(t)q(t_1)\cos z(\pi-t)\sin z\left(t-\frac{\pi}{2}-t_1\right)dt_1dt$$
$$+\left(-b\sin\frac{z\pi}{2}+\frac{c}{z}\cos\frac{z\pi}{2}\right)\int_{0}^{\frac{\pi}{2}} q(t)\sin z\left(\frac{\pi}{2}-t\right)dt$$
$$+b^{-1}\cos\frac{z\pi}{2}\int_{0}^{\frac{\pi}{2}} q(t)\cos z\left(\frac{\pi}{2}-t\right)dt-z\frac{b+b^{-1}}{2}\sin z\pi+\frac{c}{2}\cos z\pi+\frac{c}{2}. \quad (2.2)$$

The functions $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ are entire in λ of order $1/2$. It is clear that the set of zeros of functions $\Delta_0(\lambda),\Delta_1(\lambda)$ is corresponds to the spectra of BVP L_0,L_1, respectively (see [2]). By using Hadamard's factorization theorem we conclude that the spectra of the BVPs L_0 and L_1 uniquely determine the functions $\Delta_0(\lambda),\Delta_1(\lambda)$.

3. Main results

After elementary transformations of the products of the trigonometric functions into sums/differences, the characteristic functions transform into

$$F_0(z)=\frac{1}{2z}\int_{\frac{\pi}{2}}^{\pi} q(t)\sin z\left(\frac{3\pi}{2}-2t\right)dt+\frac{1}{2z}\sin\frac{z\pi}{2}\int_{\frac{\pi}{2}}^{\pi} q(t)dt$$
$$+\frac{1}{2z^2}\int_{\frac{\pi}{2}}^{\pi}\int_{0}^{t-\frac{\pi}{2}} q(t)q(t_1)\cos z\left(\frac{3\pi}{2}-2t+t_1\right)dt_1dt$$

$$
\begin{aligned}
&-\frac{1}{2z^2}\int_{\frac{\pi}{2}}^{\pi}\int_{0}^{t-\frac{\pi}{2}} q(t)q(t_1)\cos z\left(\frac{\pi}{2}-t_1\right)dt_1dt \\
&+\left(\frac{b}{z}\cos\frac{z\pi}{2}+\frac{c}{z^2}\sin\frac{z\pi}{2}\right)\int_{0}^{\frac{\pi}{2}} q(t)\sin z\left(\frac{\pi}{2}-t\right)dt \\
&+\frac{b^{-1}}{z}\sin\frac{z\pi}{2}\int_{0}^{\frac{\pi}{2}} q(t)\cos z\left(\frac{\pi}{2}-t\right)dt \\
&+\frac{b-b^{-1}}{2}+\frac{b+b^{-1}}{2}\cos z\pi+\frac{c}{2z}\sin z\pi\,, \qquad (3.1)
\end{aligned}
$$

and

$$
\begin{aligned}
F_1(z)=&\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} q(t)\cos z\left(\frac{3\pi}{2}-2t\right)dt+\frac{1}{2}\cos\frac{z\pi}{2}\int_{\frac{\pi}{2}}^{\pi} q(t)dt \\
&-\frac{1}{2z}\int_{\frac{\pi}{2}}^{\pi}\int_{0}^{t-\frac{\pi}{2}} q(t)q(t_1)\sin z\left(\frac{3\pi}{2}-2t+t_1\right)dt_1dt \\
&+\frac{1}{2z}\int_{\frac{\pi}{2}}^{\pi}\int_{0}^{t-\frac{\pi}{2}} q(t)q(t_1)\sin z\left(\frac{\pi}{2}-t_1\right)dt_1dt \\
&+\left(-b\sin\frac{z\pi}{2}+\frac{c}{z}\cos\frac{z\pi}{2}\right)\int_{0}^{\frac{\pi}{2}} q(t)\sin z\left(\frac{\pi}{2}-t\right)dt \\
&+b^{-1}\cos\frac{z\pi}{2}\int_{0}^{\frac{\pi}{2}} q(t)\cos z\left(\frac{\pi}{2}-t\right)dt \\
&-z\frac{b+b^{-1}}{2}\sin z\pi+\frac{c}{2}\cos z\pi+\frac{c}{2}\,. \qquad (3.2)
\end{aligned}
$$

Lemma 3.1. *If $(\lambda_n)_{n=1}^{\infty}$ are the eigenvalues of L_0 and $(\mu_n)_{n=1}^{\infty}$ are the eigenvalues of L_1, then the integral $\int_{\frac{\pi}{2}}^{\pi} q(t)dt$ is uniquely determined by $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$.*

Proof. In (3.2) we put $z=4n$, using $q\in L^2(0,\pi)$, and then obviously

$$
\int_{\frac{\pi}{2}}^{\pi} q(t)dt = 2\lim_{n\to\infty}\left(F_1(4n)-c\right). \qquad \square
$$

Now, we define functions $q^*, Q_1, Q_2 : [0,\pi] \to \mathbb{C}$

$$q^*(t) = \begin{cases} q(t+\frac{\pi}{4}), & t \in [\frac{\pi}{4}, \frac{3\pi}{4}] \\ 0, & otherwise \end{cases}$$

$$Q_1(s) = \begin{cases} \int\limits_{2s}^{\pi} \int\limits_{0}^{2s-\frac{\pi}{2}} q(t)q(t_1)dt_1dt + \\ \int\limits_{s+\frac{\pi}{4}}^{2s} \int\limits_{0}^{2t-2s-\frac{\pi}{2}} q(t)q(t_1)dt_1dt, & s \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \\ 0, & otherwise \end{cases}$$

$$Q_2(s) = \begin{cases} \int\limits_{s+\frac{\pi}{4}}^{\pi} \int\limits_{0}^{2t-2s-\frac{\pi}{2}} q(t)q(t_1)dt_1dt, & s \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right] \\ 0, & otherwise \end{cases}$$

Since $q \in L^2(0,\pi)$, it is obvious that $q^*, Q_1, Q_2 \in L^2(0,\pi)$.
We introduce notation

$$a_c(z) = \int_{\frac{\pi}{2}}^{\pi} q(t)\cos z\left(\frac{3\pi}{2} - 2t\right)dt = \int_0^{\pi} q^*(t)\cos z(\pi - 2t)dt\,,$$

$$a_s(z) = \int_{\frac{\pi}{2}}^{\pi} q(t)\sin z\left(\frac{3\pi}{2} - 2t\right)dt = \int_0^{\pi} q^*(t)\sin z(\pi - 2t)dt\,.$$

$$B_{1c} = \int\limits_0^{\pi} Q_1(s)\cos z(\pi - 2s)ds,\, B_{1s} = \int\limits_0^{\pi} Q_1(s)\sin z(\pi - 2s)ds,$$

$$B_{2c} = \int\limits_0^{\pi} Q_2(s)\cos z(\pi - 2s)ds,\, B_{2s} = \int\limits_0^{\pi} Q_2(s)\sin z(\pi - 2s)ds.$$

Let' s notice that

$$\frac{1}{2z^2}\int\limits_{\frac{\pi}{2}}^{\pi}\int\limits_{0}^{t-\frac{\pi}{2}} q(t)q(t_1)\cos z\left(\frac{3\pi}{2} - 2t + t_1\right)dt_1dt$$

$$-\frac{1}{2z^2}\int\limits_{\frac{\pi}{2}}^{\pi}\int\limits_{0}^{t-\frac{\pi}{2}} q(t)q(t_1)\cos z\left(\frac{\pi}{2} - t_1\right)dt_1dt =$$

$$= \frac{1}{z}\int\limits_{\frac{\pi}{2}}^{\pi}\int\limits_{0}^{t-\frac{\pi}{2}}\int\limits_{\frac{t_1}{2}+\frac{\pi}{4}}^{t-\frac{t_1}{2}-\frac{\pi}{4}} q(t)q(t_1)\sin z\,(\pi - 2s)\,ds dt_1 dt,$$

and

$$-\frac{1}{2z}\int_{\frac{\pi}{2}}^{\pi}\int_{0}^{t-\frac{\pi}{2}} q(t)q(t_1)\sin z\left(\frac{3\pi}{2}-2t+t_1\right)dt_1dt$$

$$+\frac{1}{2z}\int_{\frac{\pi}{2}}^{\pi}\int_{0}^{t-\frac{\pi}{2}} q(t)q(t_1)\sin z\left(\frac{\pi}{2}-t_1\right)dt_1dt =$$

$$=\int_{\frac{\pi}{2}}^{\pi}\int_{0}^{t-\frac{\pi}{2}}\int_{\frac{t_1}{2}+\frac{\pi}{4}}^{t-\frac{t_1}{2}-\frac{\pi}{4}} q(t)q(t_1)\cos z(\pi-2s)\,ds dt_1 dt\,.$$

By substituting the order of integration into these triple integrals and by applying the introduced notations we obtain characteristic functions $F_0(z), F_1(z)$ in the form

$$F_0(z)=\frac{1}{2z}a_s+\frac{1}{2z}\sin\frac{z\pi}{2}\int_{\frac{\pi}{2}}^{\pi}q(t)dt+\frac{1}{z}(B_{1s}+B_{2s})+\frac{b-b^{-1}}{2}+\frac{b+b^{-1}}{2}\cos z\pi$$

$$+\frac{c}{2z}\sin z\pi+\left(\frac{b}{z}\cos\frac{z\pi}{2}+\frac{c}{z^2}\sin\frac{z\pi}{2}\right)\int_0^{\frac{\pi}{2}}q(t)\sin z\left(\frac{\pi}{2}-t\right)dt$$

$$+\frac{b^{-1}}{z}\sin\frac{z\pi}{2}\int_0^{\frac{\pi}{2}}q(t)\cos z\left(\frac{\pi}{2}-t\right)dt\,,$$

$$F_1(z)=\frac{1}{2}a_c+\frac{1}{2}\cos\frac{z\pi}{2}\int_{\frac{\pi}{2}}^{\pi}q(t)dt+B_{1c}+B_{2c}-z\frac{b+b^{-1}}{2}\sin z\pi$$

$$+\frac{c}{2}\cos z\pi+\frac{c}{2}+\left(-b\sin\frac{z\pi}{2}+\frac{c}{z}\cos\frac{z\pi}{2}\right)\int_0^{\frac{\pi}{2}}q(t)\sin z\left(\frac{\pi}{2}-t\right)dt$$

$$+b^{-1}\cos\frac{z\pi}{2}\int_0^{\frac{\pi}{2}}q(t)\cos z\left(\frac{\pi}{2}-t\right)dt\,.$$

We introduce functions

$$M(z)=zF_0(z)-\frac{1}{2}\sin\frac{z\pi}{2}\int_{\frac{\pi}{2}}^{\pi}q(t)dt-z\left[\frac{b-b^{-1}}{2}+\frac{b+b^{-1}}{2}\cos z\pi+\frac{c}{2z}\sin z\pi\right]$$

$$-\left(b\cos\frac{z\pi}{2}+\frac{c}{z}\sin\frac{z\pi}{2}\right)\int_0^{\frac{\pi}{2}}q(t)\sin z\left(\frac{\pi}{2}-t\right)dt-b^{-1}\sin\frac{z\pi}{2}\int_0^{\frac{\pi}{2}}q(t)\cos z\left(\frac{\pi}{2}-t\right)dt,$$

$$N(z) = F_1(z) - \frac{1}{2}\cos\frac{z\pi}{2}\int\limits_{\frac{\pi}{2}}^{\pi} q(t)dt - \left[-z\frac{b+b^{-1}}{2}\sin z\pi\right.$$
$$\left.+\frac{c}{2}\cos z\pi + \frac{c}{2}\right] - \left(-b\sin\frac{z\pi}{2} + \frac{c}{z}\cos\frac{z\pi}{2}\right)\int\limits_0^{\frac{\pi}{2}} q(t)\sin z\left(\frac{\pi}{2}-t\right)dt$$
$$-b^{-1}\cos\frac{z\pi}{2}\int\limits_0^{\frac{\pi}{2}} q(t)\cos z\left(\frac{\pi}{2}-t\right)dt\,.$$

Considering the assumption that the potential is known at $\left(0,\frac{\pi}{2}\right)$, the integrals $\int\limits_0^{\frac{\pi}{2}} q(t)dt\,;\int\limits_0^{\frac{\pi}{2}} q(t)\sin z\left(\frac{\pi}{2}-t\right)dt\,;\int\limits_0^{\frac{\pi}{2}} q(t)\cos z\left(\frac{\pi}{2}-t\right)dt$ are also known. Using Lemma 3.1 and since the parameters b,c, functions $F_0(z),F_1(z)$ are known, the functions $M(z)$ and $N(z)$, defined as above, are also known or ordered by spectra $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$.

On the other hand using the form of the functions $F_0(z),F_1(z)$ we have that the expressions $M(z)$ and $N(z)$ are equal to

$$M(z) = \frac{1}{2}a_s + B_{1s} + B_{2s}\,,$$

$$N(z) = \frac{1}{2}a_c + B_{1c} + B_{2c}\,.$$

If we put $z=m, m\in Z$ in last two equations and multiply them by $i\frac{(-1)^m}{\pi}e^{2imt}$ and $\frac{(-1)^m}{\pi}e^{2imt}$ respectively, after summing we obtain the integral equation

$$f(t) = \frac{1}{2}q^*(t) + Q_1(t) + Q_2(t), \tag{3.3}$$

where $f(t)$ is

$$f(t) = \sum_{m\in\mathbb{Z}}\frac{(-1)^m}{\pi}\left(iM(m)+N(m)\right)e^{2imt}\,.$$

Theorem 3.1. *Let $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$ are the spectra of boundary spectral problems L_0 and L_1 respectively, then the potential q is uniquely determined by $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$ on the interval $(0,\pi)$.*

Proof. The potential q satisfies the integral equation (3.3) and we will show the uniqueness of the solution of this equation. In other words, given that the potential is known on the set $\left(0,\frac{\pi}{2}\right)$, we will show that it is uniquely determined on the set $\left[\frac{\pi}{2},\pi\right]$.

- For $t \in [0, \frac{\pi}{4}]$ and $t \in (\frac{3\pi}{4}, \pi]$, the integral equation has the form

$$f(t) = 0$$

and it does not give us information about the potential.

- For $t \in (\frac{\pi}{2}, \frac{3\pi}{4}]$, the integral equation has the form

$$f(t) = \frac{1}{2} q^*(t) + Q_2(t)$$

which can be transformed into

$$f(t) = \frac{1}{2} q\left(t + \frac{\pi}{4}\right) + \int_{t+\frac{\pi}{4}}^{\pi} \int_0^{2t_1 - 2t - \frac{\pi}{2}} q(t_1) q(t_2) dt_2 dt_1.$$

Introducing that $t + \frac{\pi}{4} = s$, we obtain

$$f\left(s - \frac{\pi}{4}\right) = \frac{1}{2} q(s) + \int_s^{\pi} q(t_1) \left[\int_0^{2t_1 - 2s} q(t_2) dt_2\right] dt_1.$$

The expression $\left[\int_0^{2t_1-2s} q(t_2)dt_2\right]$ represents the kernel of Voltaire's integral equation by s and it is known. Then we conclude that the potential $q(x)$ is determined on $\left(\frac{3\pi}{4}, \pi\right]$ because $s \in \left(\frac{3\pi}{4}, \pi\right]$.

- For $t \in (\frac{\pi}{4}, \frac{\pi}{2}]$, the integral equation has the form

$$f(t) = \frac{1}{2} q^*(t) + Q_1(t)$$

which can be transformed into

$$\begin{aligned} f(t) - \int_{\frac{3\pi}{4}}^{\pi} q(t_1) \left[I_{[2t,\pi]}(t_1) \int_0^{2t-\frac{\pi}{2}} q(t_2) dt_2 + I_{[t+\frac{\pi}{4},2t]}(t_1) \int_0^{2t_1-2t-\frac{\pi}{2}} q(t_2) dt_2 \right] dt_1 \\ = \frac{1}{2} q\left(t + \frac{\pi}{4}\right) + \int_{t+\frac{\pi}{4}}^{\frac{3\pi}{4}} q(t_1) \Bigg[I_{[2t,\pi]}(t_1) \int_0^{2t-\frac{\pi}{2}} q(t_2) dt_2 \\ + I_{[t+\frac{\pi}{4},2t]}(t_1) \int_0^{2t_1-2t-\frac{\pi}{2}} q(t_2) dt_2 \Bigg] dt_1 \, . \end{aligned}$$

The left side of this expression is known and we denote it by $f_1(t)$. We conclude that the starting integral equation transforms into Voltaire's integral equation i.e.

$$\begin{aligned} f_1(t) = \frac{1}{2} q\left(t + \frac{\pi}{4}\right) + \int_{t+\frac{\pi}{4}}^{\frac{3\pi}{4}} q(t_1) \Bigg[I_{[2t,\pi]}(t_1) \int_0^{2t-\frac{\pi}{2}} q(t_2) dt_2 \\ + I_{[t+\frac{\pi}{4},2t]}(t_1) \int_0^{2t_1-2t-\frac{\pi}{2}} q(t_2) dt_2 \Bigg] dt_1 \, . \end{aligned}$$

Introducing that $t + \frac{\pi}{4} = u$, we obtain

$$f_1\left(u - \frac{\pi}{4}\right) = \frac{1}{2} q(u) + \int_u^{\frac{3\pi}{4}} q(t_1) K\left(u - \frac{\pi}{4}, t_1\right) dt_1 \, ,$$

which represents Voltaire's integral equation by u. We conclude that the potential $q(x)$ is determined on $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right]$. The theorem is proved. □

Let us state the following algorithm for solving the given inverse problem before concluding the paper.
Algorithm:

1. We determine the characteristic functions $F_0(z), F_1(z)$ of boundary value problems L_0, L_1 respectively, from spectra $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$ using Hadamard factorization theorem.
2. We determine the integral $\int\limits_{\frac{\pi}{2}}^{\pi} q(t)dt$ using Lemma 3.1.
3. We define the functions $M(z)$ and $N(z)$ which are known from the spectra, Lemma 3.1 and the initial assumption that the potential q is known at $\left(0, \frac{\pi}{2}\right)$.
4. We construct an integral equation by potential q and solve it by observing the mentioned segments.
5. In this way we show that the potential q is uniquely determined from two spectra on $\left(\frac{\pi}{2}, \pi\right)$.

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(Received: January 29, 2024)
(Revised: May 3, 2024)

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