SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.20.01.13
Vol.20 (33), No.1 (2024), 149–171

CHARACTERIZATION OF WEYL FUNCTIONS IN THE CLASS OF OPERATOR-VALUED GENERALIZED NEVANLINNA FUNCTIONS

MUHAMED BOROGOVAC

Dedicated to Prof. Mirjana Vuković for her jubilee.

ABSTRACT. We provide the necessary and sufficient conditions for a generalized Nevanlinna function Q ($Q \in N_\kappa(\mathcal{H})$) to be a Weyl function (also known as a Weyl-Titchmarch function).

We also investigate an important subclass of $N_\kappa(\mathcal{H})$, the functions that have a boundedly invertible derivative at infinity $Q'(\infty) := \lim_{z\to\infty} zQ(z)$. These functions are regular and have the operator representation $Q(z) = \tilde{\Gamma}^+(A-z)^{-1}\tilde{\Gamma}, z \in \rho(A)$, where A is a bounded self-adjoint operator in a Pontryagin space $\mathcal{K}$. We prove that every such strict function Q is a Weyl function associated with the symmetric operator $S := A_{|(I-P)\mathcal{K}}$, where P is the orthogonal projection, $P := \tilde{\Gamma}\left(\tilde{\Gamma}^+\tilde{\Gamma}\right)^{-1}\tilde{\Gamma}^+$.

Additionally, we provide the relation matrices of the adjoint relation S^+ of S, and of $\hat{A}$, where $\hat{A}$ is the representing relation of $\hat{Q} := -Q^{-1}$. We illustrate our results through examples, wherein we begin with a given function $Q \in N_\kappa(\mathcal{H})$ and proceed to determine the closed symmetric linear relation S and the boundary triple Π so that Q becomes the Weyl function associated with Π.

1. INTRODUCTION

1.1. We denote the sets of positive integers, real numbers, and complex numbers by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. Let $(\mathcal{K}, [.,.])$ represent a Krein space. That is a complex vector space equipped with a scalar product $[.,.]$, which is a Hermitian sesquilinear form. It admits the following decomposition of $\mathcal{K}$:

$$\mathcal{K} = \mathcal{K}_+[+]\mathcal{K}_-,$$

where $(\mathcal{K}_+, [.,.])$ and $(\mathcal{K}_-, -[.,.])$ are Hilbert spaces that are mutually orthogonal with respect to the form $[.,.]$. Elements $x, y \in \mathcal{K}$ are *orthogonal* if $[x,y] = 0$, denoted by $x[\perp]y$. Every Krein space $(\mathcal{K}, [.,.])$ is *associated* with a Hilbert space $(\mathcal{K}, (.,.))$, defined as a direct and orthogonal sum of the Hilbert spaces $(\mathcal{K}_+, [.,.])$ and $(\mathcal{K}_-, -[.,.])$. The topology in the Krein space $\mathcal{K}$ is induced by the associated

---

2020 *Mathematics Subject Classification.* 34B20, 47B50, 47A06, 47A56.
Key words and phrases. Weyl function; ordinary boundary triple; generalized Nevanlinna function; Pontryagin space.

Hilbert space $(\mathcal{K},(.,.))$. The *orthogonal companion* $A^{[\perp]}$ of the set A is defined by $A^{[\perp]} := \{y \in \mathcal{K} : x[\perp]y, \forall x \in A\}$, and the *isotropic* part M of A is defined by $M := A \cap A^{[\perp]}$. For properties of Krein spaces, one can refer to e.g., [6, Chapter V].

If the scalar product $[.,.]$ has $\kappa \in \mathbb{N}$ negative squares, then we call it a *Pontryagin space of negative index* κ. If $\kappa = 0$, then it is a Hilbert space. More information about Pontryagin space can be found, for example, in [18].

The following definitions of a linear relation and basic concepts related to it can be found in [1,14,24]. In the following, X, Y, and W represent Krein spaces which include Pontryagin and Hilbert spaces.

A *linear relation* $T : X \to Y$ is a linear manifold $T \subseteq X \times Y$.

If $X = Y$, then T is said to be a *linear relation in* X. A linear relation T is closed if it is a (closed) subspace with respect to the product topology of $X \times Y$. As usual, for a linear relation or operator $T : X \to Y$, or $T \subseteq X \times Y$, the symbols $\operatorname{dom} T$, $\operatorname{ran} T$, and $\ker T$ represent the domain, range and kernel, respectively. Additionally, we will use the following concepts and notation for two linear relations, T and S from X into Y, and a linear relation U from Y into W:

$$\begin{aligned}
\operatorname{mul} T &:= \{g \in Y : \{0,g\} \in T\}, \\
T(f) &:= \{g \in Y, : \{f,g\} \in T\}, (f \in D(T)), \\
T^{-1} &:= \{\{g,f\} \in Y \times X : \{f,g\} \in T\}, \\
zT &:= \{\{f,zg\} \in X \times Y : \{f,g\} \in T\}, (z \in \mathbb{C}), \\
S + T &:= \{\{f,g+k\} : \{f,g\} \in S, \{f,k\} \in T\}, \\
S \hat{+} T &:= \{\{f+h,g+k\} : \{f,g\} \in S, \{h,k\} \in T\}, \\
S \dot{+} T &:= \{\{f+h,g+k\} : \{f,g\} \in S, \{h,k\} \in T, S \cap T = \{0\}\}, \\
UT &:= \{\{f,k\} \in X \times W : \{f,g\} \in T, \{g,k\} \in U \, for some \, g \in Y\}, \\
T^* &:= \{\{k,h\} \in Y \times X : [f,h] = [g,k] \, for all \, \{f,g\} \in T\}, \\
T_\infty &:= \{\{0,g\} \in T\}.
\end{aligned}$$

If $T(0) = \{0\}$, we say that T is *single-valued* linear relation, i.e. *operator*. The sets of closed linear relations, closed operators, and bounded operators in X are denoted by $\tilde{C}(X)$, $C(X)$, $B(X)$, respectively.

Let A be a linear relation in a Krein space $\mathcal{K}$. When $X = Y = \mathcal{K}$ we use the notation A^+ rather than A^*. We say that A is *symmetric* (*selfadjoint*) if it satisfies $A \subseteq A^+$ $(A = A^+)$.

Every point $\alpha \in \mathbb{C}$ for which $\{f, \alpha f\} \in A$, with some $f \neq 0$, is called a *finite eigenvalue*, denoted by $\alpha \in \sigma_p(A)$. The corresponding vectors are *eigenvectors belonging to the eigenvalue* α. If for some $z \in \mathbb{C}$ the operator $(A - z)^{-1}$ is bounded, not necessarily densely defined in $\mathcal{K}$, then z is a *point of regular type of* A, symbolically, $z \in \hat{\rho}(A)$. If for $z \in \mathbb{C}$ the relation $(A - z)^{-1}$ is a bounded operator and $\overline{\operatorname{ran}(A - z)} = \mathcal{K}$, then z is a *regular point of* A, symbolically $z \in \rho(A)$.

In a Pontryagin space $\mathcal{K}$, an isometric operator U is called *unitary* if $\operatorname{dom} U = \operatorname{ran} U = \mathcal{K}$, see [18, Definition 5.4].

According to the definition [5, Definition 1.3.7], linear relations $T : \mathcal{K} \to \mathcal{K}$ and $T' : \mathcal{K}' \to \mathcal{K}'$ are unitarily equivalent if there exists a unitary operator $U : \mathcal{K} \to \mathcal{K}'$ such that $T' = \{\{U(x), U(x')\} : \{x, x'\} \in T\}$.

Let $\mathcal{L}(\mathcal{H})$ denote the Banach space of bounded operators in a Hilbert space $\mathcal{H}$. Recall that an operator valued function $Q : \mathcal{D}(Q) \subset \mathbb{C} \to \mathcal{L}(\mathcal{H})$ belongs to the *generalized Nevanlinna class* $N_\kappa(\mathcal{H})$ if it is meromorphic on $\mathbb{C}\backslash\mathbb{R}$, such that $Q(z)^* = Q(\bar{z})$, for all points z of holomorphy of Q, and the kernel $N_Q(z,w) := \frac{Q(z)-Q(w)^*}{z-\bar{w}}$ has κ negative squares. A generalized Nevanlinna function $Q \in N_\kappa(\mathcal{H})$ is called *regular* if the operator $Q(w)$ is boundedly invertible at least for one point $w \in \mathcal{D}(Q)$, see [22].

We will need the following, Krein-Langer representation of generalized Nevanlinna functions.

Theorem 1.1. *A function $Q : \mathcal{D}(Q) \subset \mathbb{C} \to \mathcal{L}(\mathcal{H})$ is a generalized Nevanlinna function of some index κ if and only if it has a representation of the form*

$$Q(z) = Q(w)^* + (z - \bar{w})\Gamma_w^+ \left(I + (z - w)(A - z)^{-1}\right)\Gamma_w, z \in \mathcal{D}(Q), \tag{1.1}$$

where, A is a self-adjoint linear relation in some Pontryagin space $(\mathcal{K}, [.,.])$ of index $\tilde{\kappa} \geq \kappa; \Gamma_w : \mathcal{H} \to \mathcal{K}$ is a bounded operator; $w \in \rho(A) \cap \mathbb{C}^+$ is a fixed point of reference. This representation can be chosen to be minimal, that is

$$\mathcal{K} = c.l.s. \left\{\Gamma_z h : z \in \rho(A), h \in \mathcal{H}\right\} \tag{1.2}$$

where

$$\Gamma_z := \left(I + (z - w)(A - z)^{-1}\right)\Gamma_w. \tag{1.3}$$

If realization (1.1) is minimal, then $\tilde{\kappa} = \kappa$. In that case $\mathcal{D}(Q) = \rho(A)$ and the triple $(\mathcal{K}, A, \Gamma_w)$ is uniquely determined (up to unitary equivalence).

The linear relation A in (1.1) is called a *representing relation (operator)* of Q. Such operator representations were developed by M. G. Krein and H. Langer, see e.g. [19, 20] and later converted to representations in terms of linear relations, see e.g. [15, 17].

Functions $Q \in N_\kappa(\mathcal{H})$ which fulfill the condition

$$\bigcap_{z \in D(Q)} \ker \frac{Q(z) - Q(\bar{w})}{z - \bar{w}} = \{0\} \tag{1.4}$$

for one, and hence for all, $w \in \mathcal{D}(Q)$, are called *strict*, see e.g. [3, p. 619].

In what follows, S denotes a closed symmetric relation or operator, not necessarily densely defined in a separable Pontryagin space $(\mathcal{K}[.,.])$, and S^+ denotes an adjoint linear relation of S in $(\mathcal{K}[.,.])$. For definitions and notation of concepts related to an ordinary boundary triple Π for the linear relation S^+, see e.g. [5,9,10].

We copy some of those definitions here with adjusted notation. For example, the operator denoted by Γ_2 in [9] is denoted by Γ_0 in [5, 10] and here, while Γ_1 denotes the same operator in all papers. Elements of S^+ are denoted by $\hat{f}$, $\hat{g}$, ..., where e.g. $\hat{f} := \begin{pmatrix} f \\ f' \end{pmatrix} = \{f, f'\}$. Let

$$\mathcal{R}_z := \mathcal{R}_z(S^+) = \ker\left(S^+ - z\right), z \in \hat{\rho}(S),$$

be the *defect subspace* of S. Then

$$\hat{\mathcal{R}}_z := \left\{ \begin{pmatrix} f_z \\ zf_z \end{pmatrix} : f_z \in \mathcal{R}_z \right\}, \ \mathcal{R} := (\operatorname{dom} S)^{[\perp]}, \ \hat{\mathcal{R}} := \left\{ \begin{pmatrix} 0 \\ f \end{pmatrix} : f \in \mathcal{R} \right\}. \quad (1.5)$$

Definition 1.1. [9, Definition 2.1] *A triple* $\Pi = (\mathcal{H}, \Gamma_0, \Gamma_1)$*, where* $\mathcal{H}$ *is a Hilbert space and* Γ_0, Γ_1 *are bounded operators from* S^+ *to* $\mathcal{H}$*, is called an ordinary boundary triple for the relation* S^+ *if the abstract Green's identity*

$$\left[f', g\right] - \left[f, g'\right] = \left(\Gamma_1 \hat{f}, \Gamma_0 \hat{g}\right)_{\mathcal{H}} - \left(\Gamma_0 \hat{f}, \Gamma_1 \hat{g}\right)_{\mathcal{H}}, \forall \hat{f}, \hat{g} \in S^+, \quad (1.6)$$

holds, and the mapping $\Gamma : \hat{f} \to \begin{pmatrix} \Gamma_0 \hat{f} \\ \Gamma_1 \hat{f} \end{pmatrix}$ *from* S^+ *to* $\mathcal{H} \times \mathcal{H}$ *is surjective.*
The operator Γ *is called the boundary or reduction operator.*

An extension $\tilde{S}$ of S is called proper, if $S \subsetneq \tilde{S} \subseteq S^+$. The set of proper extensions of S is denoted by $Ext\, S$. Two proper extensions S_0, $S_1 \in Ext\, S$ are called *disjoint* if $S_0 \cap S_1 = S$, and *transversal* if, additionally, $S_0 \hat{+} S_1 = S^+$.

Each ordinary boundary triple is naturally associated with two self-adjoint extensions of S, defined by $S_i := \ker \Gamma_i, i = 0, 1$, i.e., we have $S_i = S_i^+, i = 0, 1$, see [9, p. 4425].

Under above notation, the function

$$\emptyset \neq \rho(S_0) \ni z \mapsto \gamma_z = \left\{ \left\{ \Gamma_0 \hat{f}_z, f_z \right\} : \hat{f}_z \in \hat{\mathcal{R}}_z(S^+) \right\}$$

is called the γ-*field* associated with the boundary triple $\Pi = (\mathcal{H}, \Gamma_0, \Gamma_1)$, and the function

$$\emptyset \neq \rho(S_0) \ni z \mapsto M(z) = \left\{ \left\{ \Gamma_0 \hat{f}_z, \Gamma_1 \hat{f}_z \right\} : \hat{f}_z \in \hat{\mathcal{R}}_z(S^+) \right\} \quad (1.7)$$

is called the *Weyl function* associated with the boundary triple $\Pi = (\mathcal{H}, \Gamma_0, \Gamma_1)$, see e.g. [5, 9, 13]. Let us mention that functions $\gamma_z : \mathcal{H} \to \mathcal{R}_z$ are bijections and satisfy the formula (1.3).

1.2. **The following is a summary of the results presented in this paper.** Basic concepts of the Weyl function and γ-field of the symmetric operator S in the Hilbert space setting were introduced in the classical papers, see [13, 14]. For later developments in the field of boundary relations and Weyl functions, we refer the reader to [2, 4, 9, 12].

In this paper, we prove a characterization of the Weyl functions in the class of operator valued regular generalized Nevanlinna functions. Therefore, we use operator (relation) representations in the Pontryagin space $(\mathcal{K}, [.,.])$ setting of the regular generalized Nevanlinna function $Q \in N_\kappa(\mathcal{H})$. We denote by A the representing self-adjoint relation of Q and by $\hat{A}$ the representing self-adjoint relation of $\hat{Q} = -Q^{-1}$.

In Section 2, in Proposition 2.1 and Example 2.1, we show how to derive the strict part of a generalized Nevanlinna function. It is well known that a strict function need not to be invertible, see e.g. [11]. In Example 2.1, we see that a regular function Q need not to be strict.

In Theorem 2.1, one of the main results of the paper, we give a characterization of the Weyl functions in terms of regular and strict generalized Nevanlinna functions. In Theorem 2.1 (b), we prove the more difficult converse part. It is a generalization of the converse part of [13, Theorem 1] in several levels. Namely, in the converse part of [13, Theorem 1], authors start with a Krein Q-function of a given symmetric operator S in a Hilbert space. This means they assume the existence of the symmetric operator S, and then they prove the existence of the corresponding boundary triple that has the Weyl function equal to the given Q-function.

We solve a more general problem. We only assume that a regular and strict generalized Nevanlinna function is given, i.e. we do not assume the existence of a symmetric operator or relation S. We first have to prove the existence of the symmetric linear relation S in a Pontryagin space to be in a position to find the corresponding triple. In order to prove the existence of the symmetric relation S, we use much later results from [22].

Similar issues were studied for the definitizable matrix function, see [2, Theorem 3.5].

Section 3 can be viewed as an application of [7] in the area of boundary triples and Weyl functions. In this section, we deal with an important subclass of regular functions $Q \in N_\kappa(\mathcal{H})$, the functions that have a boundedly invertible derivative $Q'(\infty) := \lim_{z\to\infty} zQ(z)$. We are again focused on finding a symmetric operator S and a boundary triple Π for a given function Q. We start with such a function Q with the representing bounded operator A, and in Theorem 3.1 we prove that there exists a symmetric operator S such that Q is the Weyl function corresponding to S and A. Hence, here we also give a solution of the converse problem. Moreover, we give matrix representations of A, $\hat{A}$, S, and S^+. Theorem 3.1 also gives us useful new relationships between linear relations A, $\hat{A}$, S, S^+ and $\hat{\mathcal{R}}$ associated with a given function $Q \in N_\kappa(\mathcal{H})$.

In Corollary 3.1, we prove that $\hat{A}$, A and S^+ are $\mathcal{R}$-regular extensions of S if the corresponding function Q is strict and $Q'(\infty)$ is boundedly invertible.

In Section 4, we make use of the abstract results of sections 2 and 3. In examples 4.1 and 4.3, the functions have a boundedly invertible derivative $Q'(\infty)$, i.e. they satisfy the assumptions of Theorem 3.1. Therefore, we apply Theorem 3.1 to find the closed symmetric relation S and the corresponding ordinary boundary triple

Π in each of the examples so that Q is the Weyl function associated with Π. In Example 4.3, we use Theorem 3.1 also to find relation matrices $\hat{\mathcal{R}}$, $\hat{A}$, S and S^+ for the given function $Q \in N_\kappa(\mathcal{H})$ represented by A.

In Example 4.2 we prove that the strict part $\tilde{Q}$ of the function Q used in Example 2.1 is indeed a Weyl function corresponding to some symmetric relation S and the corresponding boundary triple Π.

2. Characterization of Weyl functions in the set of regular generalized Nevanlinna functions $N_\kappa(\mathcal{H})$

2.1 We will need the following lemma and proposition.

Lemma 2.1. *[8, Lemma 4.2] Let $Q \in N_\kappa(\mathcal{H})$ be a minimally represented function by a triplet $(\mathcal{K}, A, \Gamma_w)$ in representation (1.1).*

(i) *If $z \in \mathcal{D}(Q)$, then*

$$\ker\Gamma_z = \ker\Gamma_w =: \ker\Gamma; \forall w \in \mathcal{D}(Q),$$

(ii)
$$\ker\Gamma = \left\{ h \in \mathcal{H} : \frac{Q(z) - Q(\bar{w})}{z - \bar{w}} h = 0, \forall z, \forall w \in \mathcal{D}(Q) \right\}.$$

According to Lemma 2.1 we can introduce the Hilbert space $\tilde{\mathcal{H}} := (\ker\Gamma)^\perp$ and operators $\tilde{\gamma}_w := (\Gamma_w)_{|\tilde{\mathcal{H}}}$.

Proposition 2.1. *Let $Q \in N_\kappa(\mathcal{H})$ be a function minimally represented by (1.1) with operators $\Gamma_z : \mathcal{H} \to \mathcal{K}$ defined by (1.3) that satisfy (1.2). Then the following hold:*

(i) *Operators $\Gamma_z, z \in \mathcal{D}(Q)$ are one-to-one if and only if the function $Q(z) : \mathcal{H} \to \mathcal{H}$ is strict.*

(ii) *For every function $Q \in N_\kappa(\mathcal{H})$ minimaly represented by (1.1) with the triple $(\mathcal{K}, A, \Gamma_w)$, there exists a unique, up to multiplicative constant, strict function $\tilde{Q} \in N_\kappa\left(\tilde{\mathcal{H}}\right)$ defined by (1.1) with the triple $(\mathcal{K}, A, \tilde{\gamma}_w)$. Functions Q and $\tilde{Q}$ have the same number of positive squares as well.*

Proof. (i) This is an obvious consequence of the previous lemma.

(ii) Since, for every $w \in \mathcal{D}(Q) = \mathcal{D}\left(\tilde{Q}\right)$, the operator $\tilde{\gamma}_w : \tilde{\mathcal{H}} \to \operatorname{ran}\Gamma_w$ coincides with Γ_w everywhere except on $\ker\Gamma_w$, the Pontryagin space defined by (1.2) with $\tilde{\gamma}_w$ instead Γ_w coincides with $\mathcal{K}$. Because $\tilde{\gamma}_w, \forall w \in \mathcal{D}(\tilde{Q})$, are injections

$$\bigcap_{z,w \in \mathcal{D}\left(\tilde{Q}\right)} \ker \frac{\tilde{Q}(z) - \tilde{Q}(\bar{w})}{z - \bar{w}} = \emptyset.$$

holds, i.e., $\tilde{Q}$ is a strict function. The representing relation A remains the same because functions $\tilde{\gamma}_w, \forall w \in \mathcal{D}(Q)$ do not change anything in $\mathcal{K}$.

For elements $h,k \in \mathcal{H} = \tilde{\mathcal{H}}(+)\ker\Gamma$ we have the corresponding unique orthogonal decomposition $h = \tilde{h}(+)h_0 \wedge k = \tilde{k}(+)k_0$. Therefore,

$$\left[\frac{\tilde{Q}(z)-\tilde{Q}(\bar{w})}{z-\bar{w}}\tilde{h},\tilde{k}\right] = \left[\tilde{\gamma}_z\tilde{h},\tilde{\gamma}_w\tilde{k}\right] = \left[\Gamma_z h,\Gamma_w k\right].$$

This means that the numbers of both negative and positive squares of Q and of $\tilde{Q}$ are the same. □

The function $\tilde{Q} \in N_\kappa(\tilde{\mathcal{H}})$, introduced in Proposition 2.1, will be referred to as the *strict part* of Q. Additionally, we will call the Hilbert space $\tilde{\mathcal{H}}$ the *domain* of the strict part $\tilde{Q}$.

Example 2.1. *Consider the following regular matrix function*

$$Q(z) = \begin{pmatrix} \frac{z}{2}-1 & \frac{z}{2} \\ \frac{z}{2} & \frac{z}{2}+1 \end{pmatrix}.$$

Then for vector $h = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

$$N(z,w)h = \frac{Q(z)-Q(\bar{w})}{z-\bar{w}}h = 0, \forall w,z \in \mathcal{D}(Q).$$

Therefore, this is an example of a regular function that is not strict.

Our task is to find the strict part $\tilde{Q}$ *of* Q.

Let us switch from the basis $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to the new ortho-normal basis $f_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $f_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. With respect to the new basis, we have

$$Q(z) = \begin{pmatrix} 0 & -1 \\ -1 & z \end{pmatrix} \wedge f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \wedge f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \wedge h = \sqrt{2}f_1.$$

According to Proposition 2.1, we can introduce the domain of $\tilde{Q}$ by $\tilde{\mathcal{H}} = l.s.\{f_2\}$. Then, if we denote by $P_{|\tilde{\mathcal{H}}}$ the orthogonal projection onto $\tilde{\mathcal{H}}$ we get the strict part of Q

$$\tilde{Q}(z) = P_{|\tilde{\mathcal{H}}}Q(z)_{|\tilde{\mathcal{H}}} = z, z \in \mathcal{D}(Q).$$

Recall that the strict part preserves the numbers of positive and negative squares. □

Later, in Example 4.2, we will find the corresponding triple of $\tilde{Q}$, and we will show that $\tilde{Q}$ is the corresponding Weyl function.

2.2 Most of the statements in the first part of the following theorem about the Weyl function Q are already known, as cited. We added a proof of regularity of Q in order to obtain a characterization. Part (b) is more interesting. In part (b) we start from a generalized Nevanlinna function Q and under the condition of regularity of

Q we prove the existence of a simple closed operator S so that Q becomes a Weyl function of S. Part (b) is a generalization of the converse part of [13, Theorem 1].

Theorem 2.1. (a) *Let S, $\{0\} \subseteq S \subsetneq A$, be a simple closed symmetric operator in a Pontryagin space $\mathcal{K}$ of index κ. Let $A^+ = A$, $\rho(A) \neq \emptyset$, let $\Pi = (\mathcal{H}, \Gamma_0, \Gamma_1)$ be an ordinary boundary triple for S^+ ($A = \ker \Gamma_0$), and let $Q(z)$ be the Weyl function of A corresponding to Π. Assume that $Q(w)$ is invertible for at least one point $w \in \mathcal{D}(Q)$.*

Then $Q \in N_\kappa(\mathcal{H})$, Q is a regular and strict function uniquely determined by the relation A in the minimal representation of the form (1.1).

(b) *Conversely, let $Q \in N_\kappa(\mathcal{H})$ be a regular and strict function given by a minimal representation (1.1) with a representing relation A.*

Then there exists a unique closed simple linear operator S, $\{0\} \subseteq S \subsetneq A \subsetneq S^+$ and there exists an ordinary boundary triple $\Pi = (\mathcal{H}, \Gamma_0, \Gamma_1)$ for S^+ such that $A = \ker \Gamma_0$. The function $Q(z)$ is the Weyl function of A corresponding to Π.

(c) *In this case, the following hold:*
   (i) *The representing relation $\hat{A}$ of $\hat{Q} := -Q^{-1}$ satisfies $\hat{A} = \ker \Gamma_1$.*
   (ii) *A and $\hat{A}$ are transfersal extensions of $S := A \cap \hat{A}$.*

Proof. (a) The assumptions are appropriate. Namely, the existence of the boundary triple $\Pi = (\mathcal{H}, \Gamma_0, \Gamma_1)$, with $A := \ker \Gamma_0$, has been proven in [9, Proposition 2.2 (2)]. The existence of the corresponding (well defined) Weyl function with bounded values $Q(z)$ has been proven in [9, p. 4427].

According to the terminology of [3, p. 619], the assumption that the closed linear relation S is *simple* means

$$\mathcal{K} = c.l.s. \left\{ \mathcal{R}_z(S^+) : z \in \rho(A) \right\}. \tag{2.1}$$

The relationship between one-to-one operators $\gamma_z \in [\mathcal{H}, \mathcal{R}_z], z \in \rho(A)$, of the γ-field γ and the Weyl function has been established by [9, (2.13)]

$$\frac{Q(z) - Q^*(w)}{z - \bar{w}} = \gamma_w^+ \gamma_z, \; \forall w, z \in \rho(A), \tag{2.2}$$

where, according to [9, (2.6)], γ-filed satisfies

$$\gamma_z = \left(I + (z - w)(A - z)^{-1} \right) \gamma_w. \tag{2.3}$$

For all $h, k \in \mathcal{H}$,

$$\left(\frac{Q(z) - Q^*(w)}{z - \bar{w}} h, k \right) = \left(\gamma_w^+ \gamma_z(h), k \right) = [\gamma_z(h), \gamma_w(k)] = [f, g], f \in \mathcal{R}_z, g \in \mathcal{R}_w.$$

Because $(\mathcal{K}, [.,.])$ given by (2.1) is a Pontryagin space with a negative index κ, we conclude that Q has κ negative squares. Because $Q(z)$ are bounded operators, $Q \in N_\kappa(\mathcal{H})$ holds.

Let us note that the corresponding claim for Weyl families and generalized Nevanlinna families has been proven in [4, Theorem 4.8].

From (2.2) and (2.3) it follows that

$$Q(z) = Q(\bar{w}) + (z - \bar{w})\gamma_w^+ \left(I + (z-w)(A-z)^{-1}\right)\gamma_w, z \in \rho(A). \tag{2.4}$$

Because $\gamma_z(\mathcal{H}) = \mathcal{R}_z$, according to (2.1) and (2.3), the minimality condition (1.2) is fulfilled with $A = \ker\Gamma_0$ and with γ-field (2.3). Then, according to Theorem 1.1, the state space $\mathcal{K}$, the representing relation A, the γ-field and the function Q given by (2.4) are uniquely determined (up to unitary equivalence).

By the definition of a γ-field, the operators $\gamma_z : \mathcal{H} \to \mathcal{R}_z$ are one-to-one for all $z \in \mathcal{D}(Q)$. Then, according to Proposition 2.1 (i), the function $Q(z)$ is strict.

Let us prove that the function Q is regular. According to our assumptions, there exists at least one point $\bar{w} \in \mathcal{D}(Q)$ such that $\hat{Q}(\bar{w}) := -Q(\bar{w})^{-1}$ is an operator. Because of the symmetry of the function Q, $Q(w)^{-1}$ is also an operator. According to definition (1.7) of the Weyl function, it is obvious that $\mathcal{D}(\hat{Q}(z)) = \operatorname{ran}\Gamma_1 = \mathcal{H}, \forall z \in \mathcal{D}(Q)$. Therefore $\left(-Q(w)^{-1}\right)^* = \left(-Q(w)^*\right)^{-1} = \left(-Q(\bar{w})\right)^{-1}$ is an operator. This further means that $\hat{Q}(w)$ is a closed operator. It is also defined on entire $\mathcal{H}$, i.e., $\hat{Q}(w)$ is bounded operator. This proves that $Q(w)$ is boundedly invertible operator. By definition Q is a regular function. This completes the proof of (a).

(b) The assumption that $Q \in N_\kappa(\mathcal{H})$ is a regular function with the representing relation A in the minimal representation (1.1) includes that (1.2) and (1.3) hold, and $\rho(A) \neq \emptyset$. According to [22, Proposition 2.1], the inverse $\hat{Q} = -Q^{-1} \in N_\kappa(\mathcal{H})$ admits the representation

$$\hat{Q}(z) = \hat{Q}(\overline{w}) + (z - \overline{w})\hat{\Gamma}_w^+ \left(I + (z-w)(\hat{A}-z)^{-1}\right)\hat{\Gamma}_w, \tag{2.5}$$

where $w \in \rho(A) \cap \rho(\hat{A})$ is an arbitrarily selected point of reference,

$$\hat{\Gamma}_w := -\Gamma_w Q(w)^{-1}, \tag{2.6}$$

and

$$(\hat{A} - z)^{-1} = (A-z)^{-1} - \Gamma_z Q(z)^{-1}\Gamma_{\bar{z}}^+, \ \forall z \in \rho(A) \cap \rho(\hat{A}) \tag{2.7}$$

holds.

According to Proposition 2.1 (i), the assumption that $Q \in N_\kappa(\mathcal{H})$ is a strict function means that operators $\Gamma_z, z \in \mathcal{D}(Q)$, in representation (1.1) are one-to-one.

We need to prove that there exists a closed symmetric relation S, a boundary triple $\Pi = (\mathcal{H}, \Gamma_0, \Gamma_1)$ and a corresponding Weyl function $M(z) = Q(z)$.

We define the closed symmetric relation S by

$$S := A \cap \hat{A}. \tag{2.8}$$

Because representations (1.1) and (2.5) are uniquely determined, the linear relation S is also uniquely determined. This also means that the self-adjoint relation A is an extension of S.

The linear relation S defined by (2.8) has equal (finite or infinite) defect numbers in the separable Pontryagin space $\mathcal{K}$ because it has a self-adjoint extension A within $\mathcal{K}$. Let us denote that defect number by $d(S)$. We already observed that $\Gamma_z : \mathcal{H} \to \Gamma_z(\mathcal{H}), z \in \rho(A)$, are one-to-one operators. Therefore, $\dim \mathcal{H} = d(S)$.

We can here apply [9, Proposition 2.2]. Therefore, there exists a boundary triple $\tilde{\Pi} = (\tilde{\mathcal{H}}, \Gamma_0, \Gamma_1)$ for S^+ such that $A = \ker \Gamma_0$, with a γ-field $\gamma_z, z \in \rho(A)$, that satisfies (2.3).

According to [9, Proposition 2.2 (3)], $\gamma_z : \tilde{\mathcal{H}} \to \mathcal{R}_z = \gamma_z(\tilde{\mathcal{H}}), \forall z \in \rho(A)$, is a one-to-one operator. Recall that γ_z and $\tilde{\mathcal{H}}$ were introduced so that $\dim(\tilde{\mathcal{H}}) = d(S)$ holds. This means $\dim(\tilde{\mathcal{H}}) = \dim \mathcal{H} = d(S)$. Therefore, we can consider $\mathcal{H} = \tilde{\mathcal{H}}$, hence $\tilde{\Pi} = (\mathcal{H}, \Gamma_0, \Gamma_1)$.

Let $M(z)$ be the Weyl function corresponding to $\tilde{\Pi} = (\mathcal{H}, \Gamma_0, \Gamma_1)$. Then $M(z)$ and $\gamma(z)$ satisfy [9, (2.13)]. According to [9, Remark 2.2], the operator valued function $M(z)$ is a Q-function of S represented by $A = \ker \Gamma_0$ in some Pontryagin space $\tilde{\mathcal{K}}$. (For a definition of the Q-function of S see e.g. [21].) The minimal Pontryagin space of the Q-function $M(z)$ is given by means of $\gamma_z(\mathcal{H}) = \mathcal{R}_z(S^+)$, which is

$$\tilde{\mathcal{K}} := c.l.s.\left\{\mathcal{R}_z(S^+) : z \in \rho(A)\right\} \subseteq \mathcal{K}. \tag{2.9}$$

According to [9, (2.13)] and (2.3)

$$M(z) = M(w)^* + (z - \bar{w})\gamma_w^+ \left(I + (z - w)(A - z)^{-1}\right)\gamma_w, z \in \rho(A). \tag{2.10}$$

Let us now use the so called ε_z-model, see [20,23]. According to that model, we can identify the building blocks of $\tilde{\mathcal{K}}$ with $\gamma_z(h)\,(h \in \mathcal{H}, z \in \rho(A))$, and the building blocks of $\mathcal{K}$ with $\Gamma_z(h), (h \in \mathcal{H}, z \in \rho(A))$. Therefore, we can define one-to-one operator $U : \tilde{\mathcal{K}} \to \mathcal{K}$ by

$$U(\gamma_z(h)) = \Gamma_z(h), \forall h \in \mathcal{H}, \forall z \in \rho(A),$$

and we can set

$$[\gamma_z(h), \gamma_w(k)] = [\Gamma_z(h), \Gamma_w(k)], \forall h, k \in \mathcal{H}, \forall z, w \in \rho(A).$$

Obviously, the operator U is a unitary operator. Therefore, the spaces $\tilde{\mathcal{K}}$ and $\mathcal{K}$ are unitarily equivalent. This, together with $\mathcal{H} = \tilde{\mathcal{H}}$, means that the representations (1.1) and (2.10), both represented by the same relation A, are unitarily equivalent. In other words, we can consider $Q = M$.

According to (2.9), by definition S is a simple relation with respect to $\tilde{\mathcal{K}} = \mathcal{K}$. We know that a simple linear relation S is an operator.

(c) (i) According to [9, (2.3)], there exists a bijective correspondence between proper extensions $\tilde{S} \in Ext\, S$ and closed sub-spaces θ in $\mathcal{H} \times \mathcal{H}$ defined by

$$S_\theta \in Ext\, S \Leftrightarrow \theta := \Gamma S_\theta = \left\{ \begin{pmatrix} \Gamma_0 \hat{f} \\ \Gamma_1 \hat{f} \end{pmatrix} : \hat{f} \in S_\theta \right\} \in \tilde{\mathcal{C}}(\mathcal{H}). \tag{2.11}$$

Then the Krein (a.k.a. Krein-Naimark) formula

$$(S_\theta - z)^{-1} = (A - z)^{-1} + \Gamma_z (\theta - Q(z))^{-1} \Gamma_{\bar{z}}^{+} \tag{2.12}$$

holds. Let us set $S_\theta := \hat{A}$, where $\hat{A}$ is the linear relation that represents the inverse function $\hat{Q}$ in representation (2.5). Then according to (2.7), the pair: $S_\theta = \hat{A}, \theta = O_{\mathcal{H}}$ (a zero function on $\mathcal{H}$), satisfies (2.12). Because the correspondence defined by (2.11) is a bijection, it follows

$$\theta = \Gamma \hat{A} = \left\{ \begin{pmatrix} \Gamma_0 \hat{f} \\ 0 \end{pmatrix} : \hat{f} \in \hat{A} \right\}. \tag{2.13}$$

Therefore, $\hat{A} = \ker \Gamma_1 =: S_1$. This proves (ii).

(ii) $S := A \cap \hat{A}$ has been defined in (b). It suffices to prove $S^+ \subseteq \ker \Gamma_0 \hat{+} \ker \Gamma_1$.

Assume $\hat{k} \in S^+$ and $\hat{h} = \Gamma \hat{k}$. Then, because Γ is surjective, we have

$$\begin{pmatrix} h \\ h' \end{pmatrix} = \begin{pmatrix} 0 \\ h' \end{pmatrix} + \begin{pmatrix} h \\ 0 \end{pmatrix} = \Gamma \hat{t} + \Gamma \hat{r}, \hat{t} \in \ker \Gamma_0, \hat{r} \in \ker \Gamma_1.$$

Hence, $\hat{s} := \hat{k} - \hat{t} - \hat{r} \in S \subseteq ker \Gamma_0$, i.e. $\hat{k} := (\hat{s} + \hat{t}) + \hat{r} =: \hat{u} + \hat{r} \in \ker \Gamma_0 \hat{+} \ker \Gamma_1$. This proves $S^+ \subseteq \ker \Gamma_0 \hat{+} \ker \Gamma_1$. □

Corollary 2.1. *Let $\mathcal{K}$ be a Pontryagin space of negative index κ and let $M(z)$ be the Weyl function associated with the ordinary boundary triple $\Pi = (\mathcal{H}, \Gamma_0, \Gamma_1)$. If $\hat{M} := -M^{-1}$ exists then relations $S_i := \ker \Gamma_i, i = 1, 2$, satisfy*

$$(S_1 - z)^{-1} = (S_0 - z)^{-1} + \hat{\gamma}_z \gamma_{\bar{z}}^{+}, z \in \rho(S_0) \cap \rho(S_1), \tag{2.14}$$

where γ_z and $\hat{\gamma}_z$ are γ-fields associated with S_0 and S_1, respectively.

Proof. By definition of the Weyl function, the operator Γ_1 is for $\hat{M}$ what Γ_0 is for M. According to Theorem 2.1 (c), $\hat{A} = S_1$. Therefore, we can substitute S_0 and S_1 for A and $\hat{A}$ into (2.7), respectively. Hence, we can rewrite (2.6) with $w = z$, $\Gamma_w = \gamma_z$, $\hat{\Gamma}_w = \hat{\gamma}_z$ and substitute (2.6) into (2.7) to obtain (2.14). □

2.3. Identity (2.14) gives us a relationship between resolvents of $A = \ker \Gamma_0$ and $\hat{A} := \ker \Gamma_1$ when $S := A \cap \hat{A}$ and A is the representing relation of the Weyl function Q, i.e. of the regular and strict generalized Nevanlinna function Q. In the following proposition, we will establish a direct relationship between any two closed linear relations A and B that satisfy $\rho(A) \cap \rho(B) \neq \emptyset$. Then we will apply it to the representing relations A and $\hat{A}$ of Q and $\hat{Q}$, respectively.

Recall, for the *defect subspace* of a linear relation T we use the notation

$$\hat{\mathcal{R}}_z(T) = \left\{ \begin{pmatrix} t \\ zt \end{pmatrix} \in T \right\}.$$

Proposition 2.2. *Let A and B be linear relations in a Krein space $\mathcal{K}$, let B be a closed relation, and $\rho(A) \cap \rho(B) \neq \emptyset$. Then*

$$A \subseteq B \dotplus \hat{\mathcal{R}}_z(A \hat{+} B), \forall z \in \rho(A) \cap \rho(B). \tag{2.15}$$

Equality holds if and only if $A = B$.

Proof. For $z \in \rho(A) \cap \rho(B)$ and for every $\begin{pmatrix} f \\ f' \end{pmatrix} \in A$ we have $\begin{pmatrix} f \\ f' - zf \end{pmatrix} \in A - z$. Because $z \in \rho(B)$, and B is closed, there exists $\begin{pmatrix} g \\ g' \end{pmatrix} \in B$ such that

$$f' - zf = g' - zg \Rightarrow f' - g' = z(f - g)$$

holds. Therefore

$$\begin{pmatrix} f \\ f' \end{pmatrix} - \begin{pmatrix} g \\ g' \end{pmatrix} = \begin{pmatrix} f - g \\ f' - g' \end{pmatrix} = \begin{pmatrix} f - g \\ z(f - g) \end{pmatrix} \in \hat{\mathcal{R}}_z(A \hat{+} B).$$

Thus

$$\begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} g \\ g' \end{pmatrix} + \begin{pmatrix} f - g \\ z(f - g) \end{pmatrix}. \tag{2.16}$$

The sum (2.16) is direct because $0 \neq \begin{pmatrix} t \\ zt \end{pmatrix} \in B \cap \hat{\mathcal{R}}_z(A \hat{+} B) \Rightarrow z \in \sigma_p(B)$, which contradicts the assumption $z \in \rho(B)$. This proves (2.15).

To prove the second claim, let us assume $A = B \dotplus \hat{\mathcal{R}}_z(A \hat{+} B), z \in \rho(A) \cap \rho(B)$. Then for $S := A \cap B$ we have

$$S = B \subseteq A \Rightarrow A \hat{+} B = A \Rightarrow \hat{\mathcal{R}}_z(A \hat{+} B) = \emptyset \Rightarrow A = B.$$

The converse implication follows from $\hat{\mathcal{R}}_z(B) = \{0\}$. □

Corollary 2.2. *Let $Q \in N_\kappa(\mathcal{H})$ be a regular strict function and let A and $\hat{A}$ be the representing relations of Q, and $\hat{Q} := -Q^{-1}$, respectively. For $S = A \cap \hat{A}$,*

$$A \subseteq \hat{A} \dotplus \hat{\mathcal{R}}_z(S^+), \forall z \in \rho(A) \cap \rho(\hat{A}).$$

holds. Equality holds if and only if $A = \hat{A}$.

Proof. The regularity of Q implies $\rho(A) \cap \rho(\hat{A}) \neq \emptyset$. According to Theorem 2.1 (c)(ii), we can substitute S^+ for $A \hat{+} \hat{A}$. Then both claims follow from Proposition 2.2. □

Obviously, the relations A and $\hat{A}$ can exchange places in the above corollary.

3. Weyl function $Q \in N_\kappa(\mathcal{H})$ with boundedly invertible $Q'(\infty)$

3.1 A significant part of this paper is about the class of functions $Q \in N_\kappa(\mathcal{H})$ that are holomorphic at ∞, i.e. the functions Q for which there exists $Q'(\infty) := \lim_{z \to \infty} zQ(z)$.

Lemma 3.1. [7, Lemma 3] *A function $Q \in N_\kappa(\mathcal{H})$ is holomorphic at ∞ if and only if $Q(z)$ has a representation*

$$Q(z) = \tilde{\Gamma}^{+}(A-z)^{-1}\tilde{\Gamma}, z \in \rho(A), \tag{3.1}$$

with a bounded operator A. In this case

$$Q'(\infty) := \lim_{z\to\infty} zQ(z) = -\tilde{\Gamma}^{+}\tilde{\Gamma}, \tag{3.2}$$

where the limit denotes convergence in the Banach space of bounded operators $\mathcal{L}(\mathcal{H})$.

Recall, see [7, Proposition 1], that the operator $\tilde{\Gamma}$ used in (3.1) can be expressed as

$$\tilde{\Gamma} = (A-z)\Gamma_z, \forall z \in \rho(A). \tag{3.3}$$

Then the representation (3.1) is minimal, if and only if

$$\mathcal{K} = c.l.s.\left\{(A-z)^{-1}\tilde{\Gamma}h : z \in \rho(A), h \in \mathcal{H}\right\}.$$

The decomposition of the function $Q \in N_\kappa(\mathcal{H})$ in [7, Remark 1] shows us the important role representations of the form (3.1) play in research of the function $Q \in N_\kappa(\mathcal{H})$.

The following lemma from [7] will be frequently needed in this paper.

Lemma 3.2. [7, Lemma 4] *Let* $\tilde{\Gamma} : \mathcal{H} \to \mathcal{K}$ *be a bounded operator and let* $\tilde{\Gamma}^{+} : \mathcal{K} \to \mathcal{H}$ *be its adjoint operator. Assume also that* $\tilde{\Gamma}^{+}\tilde{\Gamma}$ *is a boundedly invertible operator in the Hilbert space* $(\mathcal{H}, (.,.))$. *Then for the operator*

$$P := \tilde{\Gamma}\left(\tilde{\Gamma}^{+}\tilde{\Gamma}\right)^{-1}\tilde{\Gamma}^{+} \tag{3.4}$$

the following statements hold:

(i) *P is an orthogonal projection in the Pontryagin space* $(\mathcal{K}, [.,.])$.
(ii) *The scalar product* $[.,.]$ *does not degenerate on* $P\mathcal{K} = \tilde{\Gamma}\mathcal{H}$ *and therefore it does not degenerate on* $\tilde{\Gamma}(\mathcal{H})^{[\perp]} = \ker\tilde{\Gamma}^{+}$.
(iii) $\ker\tilde{\Gamma}^{+} = (I-P)\mathcal{K}$.
(iv) *The Pontryagin space* $\mathcal{K}$ *can be decomposed as a direct orthogonal sum of Pontryagin spaces i.e.*

$$\mathcal{K} = (I-P)\mathcal{K}[+]P\mathcal{K}. \tag{3.5}$$

3.2 Let

$$\mathcal{K} := \mathcal{K}_1 [+] \mathcal{K}_2$$

be a Pontryagin space with nontrivial Pontryagin subspaces $\mathcal{K}_l$, $l = 1, 2$, and let $E_l : \mathcal{K} \to \mathcal{K}_l$, $l = 1, 2$, be orthogonal projections. Let T be a linear relation in $\mathcal{K} = \mathcal{K}_1 [+] \mathcal{K}_2$. If for any projection E_i, $i = 1,2$, $E_i(D(T)) \subseteq D(T)$ holds, then according to [8, Lemma 2.2] the following four linear relations can be defined

$$T_i^j := \left\{ \begin{pmatrix} k_i \\ k_i^j \end{pmatrix} : k_i \in D(T) \cap \mathcal{K}_i, k_i^j \in E_j T(k_i) \right\} \subseteq \mathcal{K}_i \times \mathcal{K}_j, i, j = 1, 2.$$

In this notation the subscript "i" is associated with the domain subspace $\mathcal{K}_i$, the superscript "j" is associated with the range subspace $\mathcal{K}_j$. For example $\begin{pmatrix} k_1 \\ k_1^2 \end{pmatrix} \in T_1^2$. We will use "[+]" to denote adjoint relations of T_i^j. Therefore

$$T_1^2 \subseteq \mathcal{K}_1 \times \mathcal{K}_2 \Rightarrow T_1^{2[+]} \subseteq \mathcal{K}_2 \times \mathcal{K}_1.$$

Hence, for the linear relation T and decomposition $\mathcal{K} := \mathcal{K}_1 [+] \mathcal{K}_2$, we can assign the following *relation matrix*

$$\begin{pmatrix} T_1^1 & T_2^1 \\ T_1^2 & T_2^2 \end{pmatrix}.$$

We obtain

$$T = \left(T_1^1 + T_1^2\right) \hat{+} \left(T_2^1 + T_2^2\right).$$

Lemma 3.3. . *Let $Q \in N_\kappa(\mathcal{H})$ satisfy conditions of Lemma 3.1. Then*

$$B := A_{|(I-P)\mathcal{K}} \dot{+} (\{0\} \times P\mathcal{K}) \subseteq (I-P)\mathcal{K} \times \mathcal{K} \tag{3.6}$$

holds, where projection P is defined by (3.4). Then

$$z \in \rho(A) \cap \rho(\hat{A}) \Rightarrow \mathcal{K} \subseteq (B-z)(I-P)\mathcal{K}, \tag{3.7}$$

and

$$z \in \rho(A) \cap \rho(\hat{A}) \Rightarrow z \in \rho(B).$$

Proof. Assume $z \in \rho(A) \cap \rho(\hat{A})$. Then, according to (2.5) and [7, Theorem 3], $z \in \rho(\hat{A})$ if and only if $z \in \rho(\tilde{A})$, where

$$\tilde{A} := (I-P)A_{|(I-P)\mathcal{K}}.$$

Therefore, for any $f = (I-P)f + Pf \in \mathcal{K}$ there exists $g \in (I-P)\mathcal{K}$, such that

$$(I-P)f = \Big((I-P)A_{|(I-P)\mathcal{K}} - z(I-P)\Big)g.$$

Also, there exists $k \in \mathcal{K}$ such that

$$Pk = Pf - PA_{|(I-P)\mathcal{K}}g \Rightarrow Pf = PA_{|(I-P)\mathcal{K}}g + Pk$$

holds. We will also use the identity: $(I-P)A_{|(I-P)\mathcal{K}} + PA_{|(I-P)\mathcal{K}} = A_{|(I-P)\mathcal{K}}$.

Now we have,

$$\begin{aligned} f &= (I-P)f + Pf \\ &= \Big((I-P)A_{|(I-P)\mathcal{K}} - z(I-P)\Big)g + PA_{|(I-P)\mathcal{K}}g + Pk \\ &= \left(A_{|(I-P)\mathcal{K}} - z(I-P)\right)g + Pk \in (B - z(I-P))g \in (B-z)(I-P)\mathcal{K}. \end{aligned}$$

This proves (3.7).

Let us prove that for $z \in \rho(A) \cap \rho(\hat{A})$ and $f \in (I-P)\mathcal{K}$

$$(B-z)f = 0 \Rightarrow f = 0$$

holds. Indeed, we already mentioned that $z\in\rho(A)\cap\rho(\hat{A})\Rightarrow z\in\rho((I-P)A_{|(I-P)\mathcal{K}})$. Now we have for $f\in(I-P)\mathcal{K}$:

$$0=(B-z)f=\left(A_{|(I-P)\mathcal{K}}\dotplus(\{0\}\times P\mathcal{K})-z\right)f\Rightarrow$$
$$\Rightarrow\left((I-P)A_{|(I-P)\mathcal{K}}-z\right)f=0.$$

Because $z\in\rho\left((I-P)A_{|(I-P)\mathcal{K}}\right)$, it follows that $f=0$. This further means that $(B-z)^{-1}$ is an operator. Relation B is closed as a sum of a bounded and closed relation. Then, because of (3.7) the closed operator $(B-z)^{-1}$ is bounded. This proves $z\in\rho(B)$. □

Now we can prove the following lemma.

Lemma 3.4. *Let $Q\in N_\kappa(\mathcal{H})$ satisfy conditions of Lemma 3.1. Then the representing relation $\hat{A}$ of $\hat{Q}:=-Q^{-1}$ satisfies*

$$\hat{A}=A_{|(I-P)\mathcal{K}}\dotplus\hat{A}_\infty,\tag{3.8}$$

where

$$\hat{A}_\infty=\{0\}\times P\mathcal{K}.$$

Proof. Because $\tilde{\Gamma}^+\tilde{\Gamma}$ is boundedly invertible, according to Lemma 3.2 the scalar product $[.,.]$ does not degenerate on the subspace $P(\mathcal{K})=\tilde{\Gamma}(\mathcal{H})$. According to [7, Theorem 3], there exists $\hat{Q}(z):=-Q(z)^{-1}, z\in\rho(A)\cap(\hat{A})$. Let $\hat{Q}$ be represented by a self-adjoint linear relation $\hat{A}$ in representation (2.5). Then $\hat{A}$ satisfies (2.7).

Let us now observe the linear relation B given by (3.6), and let us find the resolvent $(B-z)^{-1}$, which exists according to Lemma 3.3. Let us select a point $z\in\rho(B)$ and a vector

$$f\in\mathcal{K}=(B-zI)(I-P)\mathcal{K},$$

and let us find $(B-z)^{-1}f$.

According to Lemma 3.3 there exists an element $g:=(B-z)^{-1}f\in(I-P)\mathcal{K}$. According to definition (3.6) of B and $P\mathcal{K}=\tilde{\Gamma}\mathcal{H}$,

$$\{g,f+zg\}\in A_{|(I-P)\mathcal{K}}\dotplus\left(\{0\}\times\tilde{\Gamma}\mathcal{H}\right)$$

holds. This means that for some $h\in\mathcal{H}$

$$f+zg=Ag+\tilde{\Gamma}h$$

holds. Then we have

$$Ag-zg=f-\tilde{\Gamma}h.$$

Hence,

$$g=(A-z)^{-1}f-(A-z)^{-1}\tilde{\Gamma}h.$$

Because $\tilde{\Gamma}^+(I-P)=0$, we have

$$0=\tilde{\Gamma}^+g=\tilde{\Gamma}^+(A-z)^{-1}f-\tilde{\Gamma}^+(A-z)^{-1}\tilde{\Gamma}h=\tilde{\Gamma}^+(A-z)^{-1}f-Q(z)h.$$

According to (3.3),

$$\Gamma_{\bar{z}}^{+} f = \tilde{\Gamma}^{+} (A - z)^{-1} f.$$

Therefore,

$$h = Q(z)^{-1} \Gamma_{\bar{z}}^{+} f.$$

This and (3.3) gives

$$(B - z)^{-1} f = g = (A - z)^{-1} f - \Gamma_z h = (A - z)^{-1} f - \Gamma_z Q(z)^{-1} \Gamma_{\bar{z}}^{+} f,$$

which proves that formula (2.7) holds for the linear relation $B \subseteq (I - P)\mathcal{K} \times \mathcal{K}$ defined by (3.6). Therefore, $(B - z)^{-1} = (\hat{A} - z)^{-1}$, and

$$\hat{A} = B = A_{|(I-P)\mathcal{K}} \dot{+} (\{0\} \times P\mathcal{K}).$$

Because A is a single valued, the sum is direct, and $\hat{A}_\infty = (\{0\} \times P\mathcal{K})$, i.e. representation (3.8) of $\hat{A}$ holds. □

Note, identity (3.8) derived here by means of the operator valued function $Q \in N_\kappa(\mathcal{H})$ corresponds to identity [16, (3.5)] which was derived for a scalar function $q \in N_\kappa$. Also note that $\hat{A}_\infty = \{0\} \times P\mathcal{K}$ holds according to [7, Proposition 5] too.

Theorem 3.1. *Let $Q \in N_\kappa(\mathcal{H})$ be holomorphic at infinity with boundedly invertible $Q'(\infty)$ and let Q be minimally represented by (3.1)*

$$Q(z) = \tilde{\Gamma}^{+} (A - z)^{-1} \tilde{\Gamma}, z \in \rho(A),$$

with a bounded operator A. Then, relative to decomposition (3.5)

$$\mathcal{K}_1 [+] \mathcal{K}_2 := (I - P)\mathcal{K} [+] P\mathcal{K},$$

the following hold:

(i) $A = \begin{pmatrix} \tilde{A} & (I-P)A_{|P\mathcal{K}} \\ PA_{|(I-P)\mathcal{K}} & PA_{|P\mathcal{K}} \end{pmatrix}$, *where* $\tilde{A} = (I-P)A_{|(I-P)\mathcal{K}}$.

(ii) $\hat{A} = A_{|I-P} \dot{+} (\{0\} \times P\mathcal{K}) = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \hat{A}_\infty \end{pmatrix}$,

(iii) $S = A_{|(I-P)\mathcal{K}}$, $\mathcal{R} = P\mathcal{K}$. *S is a symmetric, closed, bounded operator.*

(iv) $S^{+} = \begin{pmatrix} \tilde{A} & (I-P)A_{|P\mathcal{K}} \\ (I-P)\mathcal{K} \times P\mathcal{K} & P\mathcal{K} \times P\mathcal{K} \end{pmatrix}$.

(v) $\mathcal{R}_z = \left\{ \begin{pmatrix} -(\tilde{A} - z)^{-1} A P x_P \\ x_P \end{pmatrix} : x_P \in P\mathcal{K} \right\}$, $\mathcal{K} = c.l.s. \{\mathcal{R}_z : z \in \rho(A)\}$, *i.e. S is simple.*

(vi) *If additionally, $\tilde{\Gamma}$ is a one-to-one operator, then Q is the Weyl function associated with (S, A) and $S^{+} = A \hat{+} \hat{A} = A \dot{+} \hat{\mathcal{R}}$.*

Proof. (i) The relation matrix of the operator A, with respect to decomposition (3.5), is obviously

$$A = \begin{pmatrix} (I-P)A_{|(I-P)\mathcal{K}} & (I-P)A_{|P\mathcal{K}} \\ PA_{|(I-P)\mathcal{K}} & PA_{|P\mathcal{K}} \end{pmatrix} = A_{|(I-P)\mathcal{K}} \dot{+} A_{|P\mathcal{K}}. \tag{3.9}$$

(ii) According to [7, Theorem 3 (ii)], the function Q is regular. Therefore, there exists the inverse function $\hat{Q}$ and the representing relation $\hat{A}$. According to (3.8), the condition $(I-P)D(\hat{A}) \subseteq D(\hat{A})$ is satisfied. Hence, according to [8, Lemma 2.2], there exists a relation matrix of $\hat{A}$ relative to decomposition (3.5). Let that relation matrix be

$$\hat{A} = \begin{pmatrix} \hat{A}_1^1 & \hat{A}_2^1 \\ \hat{A}_1^2 & \hat{A}_2^2 \end{pmatrix},$$

where $\hat{A}_i^j \subseteq \mathcal{K}_i \times \mathcal{K}_j$, $i,j = 1,2$. According to Lemma 3.4

$$\hat{A}\,(0) = P\mathcal{K}. \tag{3.10}$$

Therefore, $\hat{A}(0)$ is an ortho-complemented subspace of $\mathcal{K}$. According to [24, Theorem 2.4],

$$\hat{A} = \hat{A}_s[\dot{+}]\hat{A}_\infty, \tag{3.11}$$

where $\hat{A}_s$ is a self-adjoint densely defined operator in $\hat{A}(0)^{[\perp]} = (I-P)\mathcal{K}$, $\operatorname{ran}\hat{A}_s \subseteq (I-P)\,\mathcal{K}$ and denotes direct orthogonal sum of sub-spaces.

For $g \in (I-P)\,\mathcal{K}$, from (3.8) and (3.11), it follows that

$$\left((I-P)\,A_{|(I-P)\mathcal{K}}[\dot{+}]PA_{|(I-P)\mathcal{K}}\right) g + Pk_0 = A_s g[\dot{+}]Pk$$

for some $k_0, k \in \mathcal{K}$. Obviously:

$$A_s g = (I-P)\,A_{|(I-P)\mathcal{K}}\, g = \tilde{A} g. \tag{3.12}$$

Obviously $\hat{A}_\infty \subseteq P\mathcal{K} \times P\mathcal{K}$ and $\hat{A}_\infty = \hat{A}_\infty^+$. Hence, the relation matrix of $\hat{A}$ is

$$\hat{A} = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \hat{A}_\infty \end{pmatrix}, \tag{3.13}$$

.

(iii) Let us now find $S = A \cap \hat{A}$. According (3.13), we have

$$\hat{A} = \left\{ \begin{pmatrix} x_{I-P} \\ \tilde{A} x_{I-P} + p \end{pmatrix} : x_{I-P} \in (I-P)\mathcal{K},\ p \in P\mathcal{K} \right\}.$$

Since $\operatorname{dom} S = (I-P)\mathcal{K}$, elements of $A \cap S$ satisfy

$$\begin{pmatrix} x_{I-P} \\ A x_{I-P} \end{pmatrix} = \begin{pmatrix} x_{I-P} \\ \tilde{A} x_{I-P} + PA x_{I-P} \end{pmatrix} \in \hat{A},$$

thus $S = A_{|(I-P)\mathcal{K}}$.

By definition $\mathcal{R} = ((I-P)\mathcal{K})^{[\perp]} = P\mathcal{K}$ and $\hat{A}_\infty = \tilde{\mathcal{R}}$.

S is a closed symmetric relation in the Pontryagin space $\mathcal{K}$ because it is the intersection of such relations A and $\hat{A}$. S is a bounded operator as a restriction of bounded operator A. This proves (iii).

(iv) Now when we know S, we can find S^+ by definition. It is as claimed in (iv).

(v) By solving equation $(S^{+}-z)\begin{pmatrix} x_{I-P} \\ x_P \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e., by solving equation

$$\begin{pmatrix} \tilde{A}-z & (I-P)A_{|P\mathcal{K}} \\ (I-P)\mathcal{K}\times P\mathcal{K} & P\mathcal{K}\times P\mathcal{K}-z \end{pmatrix}\begin{pmatrix} x_{I-P} \\ x_P \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we obtain

$$\mathcal{R}_z=\left\{\begin{pmatrix} -(\tilde{A}-z)^{-1}(I-P)APx_P \\ x_P \end{pmatrix}: x_P\in P\mathcal{K}\right\}.$$

According to [7, Theorem 4], the function $\hat{Q}_2(z):=\tilde{\Gamma}_2^{+}(\tilde{A}-z)^{-1}\tilde{\Gamma}_2\in\mathcal{N}_{\kappa_2}(\mathcal{H})$, with $\tilde{\Gamma}_2:=(I-P)A\tilde{\Gamma}\left(\tilde{\Gamma}^{+}\tilde{\Gamma}\right)^{-1}$, has κ_2 negative squares, where κ_2 is the negative index of $(I-P)\mathcal{K}$. Then

$$(I-P)\mathcal{K}=c.l.s.\left\{(\tilde{A}-z)^{-1}\tilde{\Gamma}_2\mathcal{H},\, z\in\rho(\tilde{A})\right\}. \tag{3.14}$$

It is easy to verify

$$(\tilde{A}-z)^{-1}(I-P)AP\mathcal{K}=(\tilde{A}-z)^{-1}\tilde{\Gamma}_2\mathcal{H}=(\tilde{A}-z)^{-1}(I-P)A\tilde{\Gamma}(\tilde{\Gamma}^{+}\tilde{\Gamma})^{-1}\mathcal{H}.$$

According to (3.14) we have

$$\begin{pmatrix} f_{I-P} \\ f_P \end{pmatrix}[\perp]\begin{pmatrix} -(\tilde{A}-z)^{-1}(I-P)APx_P \\ x_P \end{pmatrix},\forall z\in\rho(A)\Rightarrow\begin{pmatrix} f_{I-P} \\ f_P \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This further means

$$\mathcal{K}=c.l.s.\left\{\mathcal{R}_z : z\in\rho(A)\right\}.$$

Hence, $S=A_{|(I-P)\mathcal{K}}$ is a simple operator in $\mathcal{K}$.

(vi) If $\tilde{\Gamma}$ is one-to-one, then according to (3.3), $\ker\Gamma_z=\{0\},\forall z\in\rho(A)$. According to Proposition 2.1 (i), the function Q is strict. According to Theorem 2.1 (b), Q is the Weyl function of A corresponding to the boundary triple $\Pi=(\mathcal{H},\Gamma_0,\Gamma_1)$ that satisfies $A=\ker\Gamma_0$. The second claim of (vi) follows from Theorem 2.1 (c).

The claim $A\dotplus\hat{\mathcal{R}}=S^{+}$ we can see by comparing elements of the two relations. Indeed, for an arbitrary $f=f_{I-P}+f_P\in\mathcal{K}$,

$$\left\{\begin{pmatrix} f_{I-P}+f_P \\ \tilde{A}f_{I-P}+(I-P)Af_P+PAf_{I-P}+PAf_P+P\mathcal{K} \end{pmatrix}\right\}=$$

$$=\left\{\begin{pmatrix} f_{I-P}+f_P \\ \tilde{A}f_{I-P}+(I-P)Af_P+P\mathcal{K}+P\mathcal{K} \end{pmatrix}\right\}$$

obviously holds, where we use claim (iv) for S^{+} on the right hand side of the equation. □

Recall that an extension $\tilde{S}\in Ext\,S$ is *$\mathcal{R}$-regular* if $\tilde{S}\hat{+}\hat{\mathcal{R}}$ is a closed linear relation in $\mathcal{K}\times\mathcal{K}$, see [9, Definition 3.1].

Corollary 3.1. *Let $Q\in N_{\kappa}(\mathcal{H})$ be a strict function that satisfies the conditions of Theorem 3.1. Then A, $\hat{A}$, and S^{+} are $\mathcal{R}$-regular extensions of S.*

Proof. The extension A is $\mathcal{R}$-regular because, according to Theorem 3.1 (vi), $S^+ = A \dot{+} \hat{\mathcal{R}}$ and it is a closed relation in $\mathcal{K} \times \mathcal{K}$.

From $\hat{A} = S \dot{+} \hat{\mathcal{R}}$ and $\hat{\mathcal{R}} \hat{+} \hat{\mathcal{R}} = \hat{\mathcal{R}}$, it follows that $\hat{A} = \hat{A} \hat{+} \hat{\mathcal{R}}$. Since $\hat{A}$ is closed, it is the $\mathcal{R}$-regular extension of S. By the same token, S^+ is $\mathcal{R}$-regular. □

4. EXAMPLES

In the following examples we will show how to use results from sections 2 and 3 to find a closed symmetric operator S and a reduction operator Γ for a given generalized Nevanlinna function Q so that Q becomes the Weyl function related to S and Γ. We will also express S and S^+ in terms of the representing operator A of the function Q.

Example 4.1. *Given function the* $Q(z) := -\frac{1}{z}$, $Q \in N_0(\mathbb{C})$. *Find the corresponding symmetric linear realton S,* S^+ *and the triple* $\Pi = (\mathbb{C}, \Gamma_0, \Gamma_1)$.

This function is holomorphic at ∞ and

$$Q^{'}(\infty) := \lim_{z \to \infty} zQ(z) = -I_{\mathbb{C}}$$

is a boundedly invertible operator, i.e. the conditions of Theorem 3.1 are satisfied. It is also easy to verify that Q is a strict function in $\mathcal{D}(Q)$. According to Lemma 3.1, the minimal representation of Q is of the form

$$Q(z) = \tilde{\Gamma}^+ (A - z)^{-1} \tilde{\Gamma}, z \in \rho(A),$$

where A is a bounded operator, and $Q^{'}(\infty) = -\tilde{\Gamma}^+ \tilde{\Gamma} = -I_{\mathbb{C}} = (-1) \in \mathbb{C}^{1\times 1}$.

We know, and it is easy to verify, that in the representation of the function $Q(z) := -\frac{1}{z}$, the minimal state space is $\mathcal{K} = \mathbb{C}$, the representing operator is

$$A = (0) = \left\{ \begin{pmatrix} f \\ 0 \end{pmatrix} : f \in \mathbb{C} \right\} \subseteq \mathbb{C}^2,$$

the resolvent is $(A - z)^{-1} = -\frac{1}{z} I_{\mathbb{C}}$, and $\tilde{\Gamma}^+ = \tilde{\Gamma} = (1) \in \mathbb{C}^{1\times 1}$ holds. According to (3.4), $P = I_{\mathbb{C}}$. Because $P\mathcal{K} = \mathcal{K}$, according to Theorem 3.1, $S = A_{|(I-P)\mathcal{K}} \cap \hat{A} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$. Then according to Theorem 3.1 (v), $\mathcal{R}_z = P\mathcal{K} = \mathcal{K}$. Because $\tilde{\Gamma}$ is a one-to-one operator, according to Theorem 3.1 (vi), $Q(z) := -\frac{1}{z}$ is the Weyl function associated with S and A.

We also know that in the same state space $\mathcal{K} = \mathbb{C}$, there exists a linear relation $\hat{A}$ that minimally represents $\hat{Q}(z) = -Q^{-1}(z) = zI_{\mathbb{C}}$, and $\hat{\mathcal{R}} = (\{0\} \times \mathbb{C}) \subseteq \mathbb{C}^2$. According to Theorem 3.1 (iii), $\hat{A} = \tilde{A}[\dot{+}]\hat{\mathcal{R}} = \hat{\mathcal{R}}$.

Then, according to Theorem 2.1 (c) (ii), $S^+ = A \hat{+} \hat{A} = \mathbb{C}^2$.

Now we need to define the reduction operator $\Gamma = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : S^+ \to \mathcal{H}^2$ that will satisfy identity (1.6) and

$$A = \ker \Gamma_0 \wedge \hat{A} = \ker \Gamma_1.$$

Because, $M = Q \in N_0(\mathbb{C})$, the space $\mathcal{K} = \mathbb{C}$ is endowed with the usual definite scalar product. We can easily verify that the reduction operator that satisfies the above condition is defined by

$$\Gamma \begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} f' \\ -f \end{pmatrix}. \qquad \square$$

Example 4.2. *In Example 2.1 we derived a strict part $\tilde{Q}(z) = z$ from a non-strict matrix Nevanlinna function. Because the strict part remains a Nevanlinna function and it becomes a strict function, according to Theorem 2.1 (b) there exist a reduction operator Γ and a boundary triple Π that correspond to $\tilde{Q}(z) = z$.*

To accomplish this task, we can use results of Example 4.1, because $-\tilde{Q}(z)^{-1} = -\frac{1}{z}$. This means that Γ_0 and Γ_1 exchange roles, i.e., in this example

$$\Gamma \begin{pmatrix} f \\ f' \end{pmatrix} := \begin{pmatrix} f \\ f' \end{pmatrix}.$$

Therefore, now we have

$$A = \ker \Gamma_0 = \left\{ \begin{pmatrix} 0 \\ f' \end{pmatrix} : f' \in \tilde{\mathcal{H}} \right\} \wedge \hat{A} = \ker \Gamma_1 = \left\{ \begin{pmatrix} f \\ 0 \end{pmatrix} : f \in \tilde{\mathcal{H}} \right\},$$

where $\tilde{\mathcal{H}} = \mathbb{C}$. Then $S = A \cap \hat{A} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}, S^+ = \tilde{\mathcal{H}}^2$. Obviously $\ker (S^+ - zI) = \tilde{\mathcal{H}}$. This implies $\hat{\mathfrak{R}}_z (S^+) = \left\{ \begin{pmatrix} f \\ zf \end{pmatrix} : f \in \tilde{\mathcal{H}} \right\}$. Thus

$$\Gamma_0 \begin{pmatrix} f \\ zf \end{pmatrix} = f \wedge \Gamma_1 \begin{pmatrix} f \\ zf \end{pmatrix} = zf.$$

By the definition of the Weyl function, see (1.7), it follows that $\tilde{Q}(z) = z$, i.e. $\tilde{Q}(z)$ is indeed the Weyl function corresponding to the reduction operator Γ. $\square$

Note that in [5, Example 2.4.2], the authors start from the symmetric relation S and the reduction operator Γ to find the corresponding Weyl function M, while in this example we do the converse work, we start from the strict part $\tilde{Q}$ to find Γ and S. At the end we verified that $\tilde{Q}$ is indeed the Weyl function corresponding to those Γ and S.

In the following example, we will show how to use Theorem 3.1 to find linear relations S, $\hat{A}$ and S^+ for a given function Q.

Example 4.3. *Given the function*

$$Q(z) = \begin{pmatrix} \frac{-(1+z)}{z^2} & \frac{1}{z} \\ \frac{1}{z} & \frac{1}{1+z} \end{pmatrix} \in N_2(\mathbb{C}^2)$$

and its operator representation

$$Q(z) = \tilde{\Gamma}^+ (A - z)^{-1} \tilde{\Gamma},$$

where the fundamental symmetry J, and operators A, Γ and Γ^{+} are, respectively:

$$J=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, A=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tilde{\Gamma}=\begin{pmatrix} 0.5 & -1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}, \tilde{\Gamma}^{+}=\begin{pmatrix} 1 & 0.5 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

our task is to find linear relations S, $\hat{A}$ and S^{+}.

It is easy to verify that the function Q satisfies the conditions of Theorem 3.1. Indeed, the limit (3.2) gives

$$\tilde{\Gamma}^{+}\tilde{\Gamma}=\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \left(\tilde{\Gamma}^{+}\tilde{\Gamma}\right)^{-1}=\begin{pmatrix} 0.5 & -0.5 \\ -0.5 & -0.5 \end{pmatrix}.$$

Then, by means of formula (3.4), we get

$$P=\begin{pmatrix} 0.75 & 0.125 & 0.25 \\ 0.5 & 0.75 & -0.5 \\ 0.5 & -0.25 & 0.5 \end{pmatrix}, I-P=\begin{pmatrix} 0.25 & -0.125 & -0.25 \\ -0.5 & 0.25 & 0.5 \\ -0.5 & 0.25 & 0.5 \end{pmatrix}.$$

According to Theorem 3.1 (iii), we can find S:

$$S=A\left(I-P\right)=\begin{pmatrix} -0.5 & 0.25 & 0.5 \\ 0 & 0 & 0 \\ 0.5 & -0.25 & -0.5 \end{pmatrix}.$$

$$\tilde{A}:=(I-P)A\left(I-P\right)=\begin{pmatrix} -0.25 & 0.125 & 0.25 \\ 0.5 & -0.25 & -0.5 \\ 0.5 & -0.25 & -0.5 \end{pmatrix}=-(I-P).$$

By solving equation $Px=x$ and then using the fact $(I-P)\mathcal{K}[\perp]P\mathcal{K}$, we obtain

$$(I-P)\mathcal{K}=l.s.\left\{\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}\right\};\ P\mathcal{K}=l.s.\left\{\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix},\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

According to Theorem 3.1 (ii) we have

$$\hat{A}=\tilde{A}[\dot{+}]\hat{\mathcal{R}}=-I_{I-P}[\dot{+}]\left(\{0\}\times P\mathcal{K}\right).$$

The equivalent, developed form of the linear relation $\hat{A}$ is:

$$\hat{A}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}=\left(\frac{f_1}{4}-\frac{f_2}{8}-\frac{f_3}{4}\right)\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}+c_1\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}+c_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

where $f=\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}\in\mathcal{K}=\mathbb{C}^3$, and $c_i\in\mathbb{C}, i=1,2$, are arbitrary constants.

The easiest way to obtain the developed form of S^{+} is to use Theorem 3.1 (vi) representation $S^{+}=A\dot{+}\hat{\mathcal{R}}$. We get

$$S^+f = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} + P\mathcal{K} = \begin{pmatrix} f_2 \\ 0 \\ -f_3 \end{pmatrix} + c_1 \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

where f and $c_i \in \mathbb{C}$, $i = 1,2$, are as before. □

REFERENCES

[1] R. Arens, *Operational calculus of linear relations*, Pacific J. Math., 11 (1961), 9-23.

[2] J. Behrndt, *Boundary value problems with eigenvalue depending boundary conditions*, Mathematische Nachrichten 282 (2009), 659-689.

[3] J. Behrndt, A. Luger, *An analytic characterization of the eigenvalues of self-adjoint extensions*, J. Funct. Anal. 242 (2007) 607–640.

[4] J. Behrndt, V. A. Derkach, S. Hassi, and H. de Snoo, *A realization theorem for generalized Nevanlinna families*, Operators and Matrices 5 (2011), no. 4, 679–706.

[5] J. Behrndt, S. Hassi, H. de Snoo, *Boundary Value Problems, Weyl Functions, and Differential Operators*, Open access eBook; https://doi.org/10.1007/978-3-030-36714-5

[6] J. Bognar, *Indefinite Inner Product Spaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[7] M. Borogovac, *Inverse of generalized Nevanlinna function that is holomorphic at Infinity*, North-Western European Journal of Mathematics, 6 (2020), 19-43.

[8] M. Borogovac, *Reducibility of Self-Adjoint Linear Relations and Application to Generalized Nevanlinna Functions*, Ukr. Math. J. (2022); https://doi.org/10.1007/s11253-022-02118-x

[9] V. A. Derkach, *On generalized resolvents of Hermitian relations in Krein spaces*, Journal of Math. Sci, Vol. 97, No. 5, 1999.

[10] V. A. Derkach, *Boundary Triplets, Weyl Functions, and the Krein Formula*, In book: Operator Theory, Chapter 10, pp.183-218 (2014); DOI: 10.1007/978-3-0348-0667-1_32

[11] V.A. Derkach, S. Hassi, H.S.V. de Snoo, *Operator models associated with singular perturbations*, Methods Functional Analysis Topology 7 (2001), 1-21,

[12] V. A. Derkach, S. Hassi, M.M. Malamud, and H.S.V. de Snoo, *Boundary relations and their Weyl families*, Trans. Amer. Math. Soc., 358 (2006), 5351–5400.

[13] V. A. Derkach, M. M. Malamud, *Generalized Resolvents and the Boundary Value Problems for Hermitian Operators with Gaps*, J. Funct. Anal. 95 (1991) 1-95.

[14] V. A. Derkach, M. M. Malamud, *The extension theory of hermitian operators and the moment problem*, J. Math. Sciences, 73 (1995), 141–242.

[15] A. Dijksma, H. Langer and H. S. V. de Snoo, *Eigenvalues and pole functions of Hamiltonian systems with eigenvalue depending boundary conditions*, Math. Nachr. 161 (1993) 107-154.

[16] S. Hassi, A. Luger, *Generalized zeros and poles N_κ-functions: on the underlying spectral structure*, Methods of Functional Analysis and Topology Vol. 12 (2006), no. 2, pp. 131-150.

[17] S. Hassi, H.S.V. de Snoo, and H. Woracek, *Some interpolation problems of NevanlinnaPick type*, Oper. Theory Adv. Appl.106 (1998), 201-216.

[18] I. S. Iohvidov, M. G. Krein, H. Langer, *Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric*, Akademie-Verlag, Berlin, 1982.

[19] M. G. Krein and H. Langer, *Über die Q-Funktion eines π-hermiteschen Operatos im Raume* Π_κ, Acta Sci. Math. 34, 190-230 (1973).

[20] M. G. Krein and H. Langer, *Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume Π_κ zusammenhangen*, I. Einige Funktionenklassen und ihre Darstellungen, Math. Nachr. 77, 187-236 (1977).

[21] M. G. Krein and H. Langer, *On defect subspaces and generalized resolvents of Hermitian operator acting in Pontryagyn space*, Functional. Anal. i. Prilozhen 5, No. 3 (1971), 54-69; Engl. transl. in Functional Anal. Appl. 5 (1971).

[22] A Luger, *A factorization of regular generalized Nevanlinna functions*, Integr. Equ. Oper. Theory 43 (2002) 326-345.

[23] A. Luger, *Generalized Nevanlinna Functions: Operator Representations, Asymptotic Behavior*, In book: Operator Theory, Chapter 15, pp.345-371 (2014); DOI: 10.1007/978-3-0348-0667-1_35

[24] P. Sorjonen, *On linear relations in an indefinite inner product space*, Annales Academiae Scientiarum Fennica, Series A. I. Mathematica, Vol.4, 1978/1979, 169-192

(Received: November 29, 2023)
(Revised: February 26, 2024)

Muhamed Borogovac
Boston Mutual Life
Actuarial Department
120 Royall St. Canton, MA 02021
USA
e-mail: *muhamed.borogovac@gmail.com*