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THE FIRST REGULARIZED TRACE OF THE STURM-LIOUVILLE OPERATOR WITH ROBIN BOUNDARY CONDITIONS

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Dedicated to our dear professor Mirjana Vuković for her jubilee

ABSTRACT. This paper deals with the boundary value problem for the operator Sturm-Liouville type $D^2 = D^2(h,H,q_1,q_2,\tau,\varphi)$ generated by

$$-y''(x)+\sum_{i=1}^{2} q_i(x)y(x-i\tau) = \lambda y(x),\, x\in[0,\pi]$$

$$y'(0)-hy(0)=0,\, y'(\pi)+Hy(\pi)=0$$

where $\frac{\pi}{3}\le\tau<\frac{\pi}{2}$, $h,H\in R\setminus\{0\}$ and λ is a spectral parameter. We assume that q_i, $i=1,2$ are real-valued potential functions from $L_2[0,\pi]$. We establish a formula for the first regularized trace of this operator.

1. INTRODUCTION

In addition to spectrum, decomposition and inverse problems, determining the trace of an operator is one of the basic tasks in the spectral theory of differential operators (see: [12]). The formulas for the trace of the operator can be used to find the first eigenvalues of the operator when solving the inverse spectral problems, as well as to determine the spectral function of the operator. The role of the first regularized trace is particularly important in solving inverse spectral problems for operators generated by differential equations with delays (see: [4], [6], [7], [20]). It has been showed that the first regularized trace leads to the answer to the question about sufficient conditions for two sets of numbers to be eigenvalues of a Sturm-Liuville type operator with delays (see: [1], [2], [8], [9], [10], [13], [14], [15], [16], [17], [18], [19]). These conditions for the classic Sturm-Liuville operator are given in [5].

2. THE CONSTRUCTION OF THE SOLUTION OF THE BOUNDARY VALUE PROBLEM

We deal with the boundary value problem:

$$-y''(x)+q_1(x)y(x-\tau)+q_2(x)y(x-2\tau)=\lambda y(x), x\in(0,\pi] \tag{2.1}$$

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$$y(x-2\tau) \equiv y(0)\varphi(x-2\tau), x \in [0,2\tau], \varphi(0) = 1 \tag{2.2}$$

$$y'(0) - hy(0) = 0 \tag{2.3}$$

$$y'(\pi) + Hy(\pi) = 0 \tag{2.4}$$

where $q_1, q_2 \in L_2[0,\pi]$ and $\frac{\pi}{3} \leq \tau < \frac{\pi}{2}$, $h, H \in R \setminus \{0\}$ and λ is a spectral parameter. Solving the equation (2.1) with conditions (2.3) by the method of variation of constants, we obtain the integral equation

$$\begin{aligned} y(x,z) = \cos xz + \frac{h}{z}\sin xz + \frac{1}{z}\int_0^x q_1(t)\sin z(x-t)y(t-\tau,z)dt \\ + \frac{1}{z}\int_0^x q_2(t)\sin z(x-t)y(t-2\tau,z)dt \end{aligned} \tag{2.5}$$

where $z^2 = \lambda$.

By the method of steps it can be easily verified that the solution of integral equation (2.5) on $(2\tau,\pi]$ is

$$\begin{aligned} y(x,z) = \cos xz + \frac{h}{z}\sin xz + \frac{1}{z}\left[\sum_{k=1}^{2} a_s^{(\tilde{k})}(k\tau,x,z) + \sum_{k=1}^{2} a_{sc}^{(k)}(k\tau,x,z)\right] \\ + \frac{1}{z^2}\left[a_{s^2}^{(1,\tilde{1})}(\tau,2\tau,x,z) + a_{s^2}^{(1,\tilde{2})}(\tau,x,z) + a_{s^2}^{(1,\tilde{1})}(2\tau,x,z)\right] \\ + \frac{1}{z^2}\left[a_{s^2c}^{(1^2)}(2\tau,x,z) + h\sum_{k=1}^{2} a_{s^2}^{(k)}(k\tau,x,z) + \sum_{k=1}^{2} a_{s^2}^{(2,\tilde{k})}(2\tau,x,z)\right] \\ + \frac{1}{z^3}\left[ha_{s^3}^{(1^2)}(2\tau,x,z) + \sum_{k=1}^{2} a_{s^3}^{(1^2,\tilde{k})}(2\tau,x,z)\right] \end{aligned} \tag{2.6}$$

where

$$\tilde{q}_1(t) = q_1(t)\varphi(t-\tau), t \in [0,\tau],$$
$$\tilde{q}_2(t) = q_2(t)\varphi(t-2\tau), t \in [0,2\tau],$$

$$a_s^{(\tilde{1})}(\tau,x,z) = \begin{cases} \int_0^x \tilde{q}_1(t)\sin z(x-t)dt, & x \in [0,\tau] \\ \int_0^\tau \tilde{q}_1(t)\sin z(x-t)dt, & x \in (\tau,2\tau] \end{cases}$$

$$a_s^{(\tilde{2})}(2\tau,x,z) = \int_0^x \tilde{q}_2(t)\sin z(x-t)dt, \quad x \in [0,2\tau],$$

$$a_{sc}^{(k)}(k\tau,x,z) = \int_{k\tau}^x q_k(t)\sin z(x-t)\cos z(t-k\tau)dt, \ k = 1,2,$$

$$a_{s^2}^{(k)}(k\tau,x,z) = \int_{k\tau}^x q_k(t)\sin z(x-t)\sin z(t-k\tau)dt, \ k = 1,2,$$

$$a_{s^2}^{(1,\tilde{2})}(\tau,x,z)=\int_{\tau}^{x}q_1(t_1)\int_{0}^{t_1-\tau}\tilde{q}_2(t_2)\sin z(x-t_1)\sin z(t_1-\tau-t_2)dt_2dt_1,$$

$$a_{s^2}^{(1,\tilde{1})}(\tau,2\tau,x,z)=\int_{\tau}^{2\tau}q_1(t_1)\int_{0}^{t_1-\tau}\tilde{q}_1(t_2)\sin z(x-t_1)\sin z(t_1-\tau-t_2)dt_2dt_1,$$

$$a_{s^2}^{(1,\tilde{1})}(2\tau,x,z)=\int_{2\tau}^{x}q_1(t_1)\int_{0}^{\tau}\tilde{q}_1(t_2)\sin z(x-t_1)\sin z(t_1-\tau-t_2)dt_2dt_1,$$

$$a_{s^2c}^{(1^2)}(2\tau,x,z)=\int_{2\tau}^{x}q_1(t_1)\int_{\tau}^{t_1-\tau}q_1(t_2)\sin z(x-t_1)\sin z(t_1-\tau-t_2)\cos z(t_2-\tau)dt_2dt_1,$$

$$a_{s^3}^{(1^2)}(2\tau,x,z)=\int_{2\tau}^{x}q_1(t_1)\int_{\tau}^{t_1-\tau}q_1(t_2)\sin z(x-t_1)\sin z(t_1-\tau-t_2)\sin z(t_2-\tau)dt_2dt_1,$$

$$a_{s^3}^{(1^2,\tilde{k})}(2\tau,x,z)=$$

$$\int_{2\tau}^{x}q_1(t_1)\int_{\tau}^{t_1-\tau}q_1(t_2)\int_{0}^{t_2-\tau}\tilde{q}_k(t_3)\sin z(x-t_1)\sin z(t_1-\tau-t_2)\sin z(t_2-\tau-t_3)dt_3dt_2dt_1,$$

$$a_{s^2}^{(2,\tilde{k})}(2\tau,x,z)=\int_{2\tau}^{x}q_2(t_1)\int_{0}^{t_1-2\tau}\tilde{q}_k(t_2)\sin z(x-t_1)\sin z(t_1-2\tau-t_2)dt_2dt_1,k=1,2.$$

3. Characteristic function. Asymptotic properties of the zeros of the characteristic function

Denote

$$F(\tau,z)=y'(\pi,z)+Hy(\pi,z).^{1}$$

For $x=\pi$ from (2.6) we get the characteristic function

$$\begin{aligned}F(\tau,z)=&\left(-z+\frac{hH}{z}\right)\sin\pi z+(h+H)\cos\pi z+\sum_{k=1}^{2}a_c^{(\tilde{k})}(k\tau,z)+\sum_{k=1}^{2}a_{c^2}^{(k)}(k\tau,z)\\&+\frac{1}{z}\left[H\sum_{k=1}^{2}a_s^{(\tilde{k})}(k\tau,z)+H\sum_{k=1}^{2}a_{sc}^{(k)}(k\tau,z)+h\sum_{k=1}^{2}a_{cs}^{(k)}(k\tau,z)+\sum_{k=1}^{2}a_{cs}^{(2,\tilde{k})}(2\tau,z)\right]\\&+\frac{1}{z}\left[a_{cs}^{(1,\tilde{1})}(\tau,2\tau,z)+a_{cs}^{(1,\tilde{2})}(\tau,z)+a_{cs}^{(1,\tilde{1})}(2\tau,z)+a_{csc}^{(1^2)}(2\tau,z)\right]\\&+\frac{1}{z^2}\left[ha_{cs^2}^{(1^2)}(2\tau,z)+\sum_{k=1}^{2}a_{cs^2}^{(1^2,\tilde{k})}(2\tau,z)+Hh\sum_{k=1}^{2}a_{s^2}^{(k)}(k\tau,z)+H\sum_{k=1}^{2}a_{s^2}^{(2,\tilde{k})}(2\tau,z)\right]\end{aligned}$$

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[1]Below, instead of the argument $a(\pi,\tau,z)$ we write argument $a(\tau,z)$

$$+\frac{1}{z^2}\left[ha_{cs^2}^{(1^2)}(2\tau,z)+\sum_{k=1}^{2}a_{cs^2}^{(1^2,\tilde{k})}(2\tau,z)+Hh\sum_{k=1}^{2}a_{s^2}^{(k)}(k\tau,z)+H\sum_{k=1}^{2}a_{s^2}^{(2,\tilde{k})}(2\tau,z)\right]$$
$$+\frac{H}{z^2}\left[a_{s^2}^{(1,\tilde{1})}(\tau,2\tau,z)+a_{s^2}^{(1,\tilde{2})}(\tau,z)+a_{s^2}^{(1,\tilde{1})}(2\tau,z)+a_{s^2c}^{(1^2)}(2\tau,z)\right]$$
$$+\frac{H}{z^3}\left[\sum_{k=1}^{2}a_{s^3}^{(1^2,\tilde{k})}(2\tau,z)+ha_{s^3}^{(1^2)}(2\tau,z)\right] \tag{3.1}$$

where for k=1,2

$$a_c^{(\tilde{k})}(k\tau,z)=\int_0^{k\tau}\tilde{q}_k(t)\cos z(\pi-t)dt,$$
$$a_{c^2}^{(k)}(k\tau,z)=\int_{k\tau}^{\pi}q_k(t)\cos z(\pi-t)\cos z(t-k\tau)dt,$$
$$a_{cs}^{(k)}(k\tau,z)=\int_{k\tau}^{\pi}q_k(t)\cos z(\pi-t)\sin z(t-k\tau)dt,$$

$$a_{cs}^{(2,\tilde{k})}(2\tau,z)=\int_{2\tau}^{\pi}q_2(t_1)\int_0^{t_1-2\tau}\tilde{q}_k(t_2)\cos z(\pi-t_1)\sin z(t_1-2\tau-t_2)dt_2dt_1,$$

$$a_{cs^2}^{(1^2,\tilde{k})}(2\tau,z)=$$
$$\int_{2\tau}^{\pi}q_1(t_1)\int_{\tau}^{t_1-\tau}q_1(t_2)\int_0^{t_2-\tau}\tilde{q}_k(t_3)\cos z(\pi-t_1)\sin z(t_1-\tau-t_2)\sin z(t_2-\tau-t_3)dt_3dt_2dt_1,$$

and

$$a_{csc}^{(1^2)}(2\tau,z)=\int_{2\tau}^{\pi}q_1(t_1)\int_{\tau}^{t_1-\tau}q_1(t_2)\cos z(\pi-t_1)\sin z(t_1-\tau-t_2)\cos z(t_2-\tau)dt_2dt_1,$$
$$a_{cs^2}^{(1^2)}(2\tau,z)=\int_{2\tau}^{\pi}q_1(t_1)\int_{\tau}^{t_1-\tau}q_1(t_2)\cos z(\pi-t_1)\sin z(t_1-\tau-t_2)\sin z(t_2-\tau)dt_2dt_1,$$

$$a_{cs}^{(1,\tilde{1})}(2\tau,z)=\int_{2\tau}^{\pi}q_1(t_1)\int_0^{\tau}\tilde{q}_1(t_2)\cos z(\pi-t_1)\sin z(t_1-\tau-t_2)dt_2dt_1,$$
$$a_{cs}^{(1,\tilde{2})}(\tau,z)=\int_{\tau}^{\pi}q_1(t_1)\int_0^{t_1-\tau}\tilde{q}_2(t_2)\cos z(\pi-t_1)\sin z(t_1-\tau-t_2)dt_2dt_1,$$
$$a_{cs}^{(1,\tilde{1})}(\tau,2\tau,z)=\int_{\tau}^{2\tau}q_1(t_1)\int_0^{t_1-\tau}\tilde{q}_1(t_2)\cos z(\pi-t_1)\sin z(t_1-\tau-t_2)dt_2dt_1.$$

It is known that eigenvalues λ_n of the operator D^2 are squares of zeros of the characteristic function. It is also known that zeros of the characteristic function have the form

$$z_n = n + \varsigma_n, \quad \varsigma_n \in l_2.$$

Since our primary goal is to find the first regularized trace of the operator D^2, it is sufficient to find an asymptotic decomposition of the characteristic function's zeros z_n in the form (see [3])

$$z_n(\tau) = n + \frac{C_1(n,\tau)}{n} + \frac{C_2(n,\tau)}{n^2} + o\left(\frac{C_2(n,\tau)}{n^2}\right)^2, \quad (n \to \infty). \tag{3.2}$$

From (3.1), we get the asymptotic of the characteristic function in the form

$$\begin{aligned} F(\tau,z) = {} & \left(-z+\frac{hH}{z}\right)\sin\pi z+(h+H)\cos\pi z+\sum_{k=1}^{2} a_c^{(\tilde{k})}(k\tau,z)+\sum_{k=1}^{2} a_{c^2}^{(k)}(k\tau,z) \\ & +\frac{1}{z}\left[H\sum_{k=1}^{2} a_s^{(\tilde{k})}(k\tau,z)+H\sum_{k=1}^{2} a_{sc}^{(k)}(k\tau,z)+h\sum_{k=1}^{2} a_{cs}^{(k)}(k\tau,z)+\sum_{k=1}^{2} a_{cs}^{(2,\tilde{k})}(2\tau,z)\right] \\ & +\frac{1}{z}\left[a_{cs}^{(1,\tilde{1})}(\tau,2\tau,z)+a_{cs}^{(1,\tilde{2})}(\tau,z)+a_{cs}^{(1,\tilde{1})}(2\tau,z)+a_{csc}^{(1^2)}(2\tau,z)\right] \\ & +O\left(\frac{a_{s^2}^{(1,\tilde{1})}(2\tau,z)}{z^2}\right), |z| \to \infty. \end{aligned} \tag{3.3}$$

In order to determine the first regularized trace, we should to transform the characteristic function (3.3).

For this purpose we introduce the next function

$$\breve{q}^{(1)}(\tau,\theta) = q_1(2\theta)\varphi(2\theta-\tau), \theta \in [0,\tau/2],$$

then we get

$$a_c^{(\tilde{1})}(\tau,z) = 2\int_0^{\frac{\tau}{2}} \breve{q}^{(1)}(t,\theta)\cos z(\pi-2\theta)d\theta = 2\breve{a}^{(1)}(\tau,z).$$

Also, for

$$\breve{q}^{(2)}(\tau,\theta) = q_2(2\theta)\varphi(2\theta-2\tau), \theta \in [0,\tau]$$

we get

$$a_c^{(\tilde{2})}(2\tau,z) = 2\int_0^{\tau} \breve{q}^{(2)}(\tau,\theta)\cos z(\pi-2\theta)d\theta = 2\breve{a}^{(2)}(\tau,z).$$

Then for $k = 1,2$ we obtain

$$\breve{a}^{(k)}(\tau,z) = \int_0^{k\tau/2} \breve{q}^{(k)}(\tau,\theta)\cos z(\pi-2\theta)d\theta. \tag{3.4}$$

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[2]Below, instead of the argument $C_i(n,\tau)$ we write argument C_i, $i = 1,2$

Introduce the following notation:

$$J_1^{(k)}(\tau)=\int\limits_{k\tau}^{\pi} q_k(t)dt, \quad \hat{q}_k(\tau,\theta)=q_k(\theta+\frac{k\tau}{2}),\ k=1,2,$$

$$J_2^{(1)}(\tau)=\int\limits_{2\tau}^{\pi} q_1(t_1)\int\limits_{\tau}^{t_1-\tau} q_1(t_2)dt_2dt_1.$$

Then we have

$$a_{c^2}^{(k)}(k\tau,z)=\frac{J_1^{(k)}(\tau)}{2}\cos z(\pi-k\tau)+\frac{1}{2}\hat{a}^{(k)}(\tau,z) \tag{3.5}$$

$$a_{sc}^{(k)}(k\tau,z)=\frac{J_1^{(k)}(\tau)}{2}\sin z(\pi-k\tau)+\frac{1}{2}\hat{b}^{(k)}(\tau,z) \tag{3.6}$$

$$a_{cs}^{(k)}(k\tau,z)=\frac{J_1^{(k)}(\tau)}{2}\sin z(\pi-k\tau)-\frac{1}{2}\hat{b}^{(k)}(\tau,z) \tag{3.7}$$

$$a_{s}^{(\tilde{k})}(k\tau,z)=2\int\limits_{0}^{k\tau/2} \breve{q}^{(k)}(\tau,\theta)\sin z(\pi-2\theta)d\theta=2\breve{b}^{(k)}(\tau,z) \tag{3.8}$$

$$a_{s^2}^{(k)}(k\tau,z)=-\frac{J_1^{(k)}(\tau)}{2}\cos z(\pi-k\tau)+\frac{1}{2}\hat{a}^{(k)}(\tau,z) \tag{3.9}$$

$$a_{csc}^{(1^2)}(2\tau,z)=\frac{J_2^{(1)}(\tau)}{4}\sin z(\pi-2\tau)+\frac{1}{4}b^{(1)}(\tau,z) \tag{3.10}$$

where for k=1,2

$$\hat{a}^{(k)}(\tau,z)=\int\limits_{k\tau/2}^{\pi-k\tau/2} \hat{q}_k(\tau,\theta)\cos z(\pi-2\theta)d\theta,$$

$$\breve{b}^{(k)}(\tau,z)=\int\limits_{0}^{k\tau/2} \breve{q}^{(k)}(\tau,\theta)\sin z(\pi-2\theta)d\theta,$$

$$\hat{b}^{(k)}(\tau,z)=\int\limits_{k\tau/2}^{\pi-k\tau/2} \hat{q}_k(\tau,\theta)\sin z(\pi-2\theta)d\theta.$$

To simplify further consideration we define the so called transitional functions Q as follows

$$Q^{(1)}(\tau,\theta)=\begin{cases} -q_1(\theta+\tau)\int\limits_{\tau}^{\theta} q_1(t)dt-\int\limits_{\theta+\tau}^{\pi} q_1(t)q_1(t-\theta)dt \\ \qquad\qquad +q_1(\theta)\int\limits_{\theta+\tau}^{\pi} q_1(t)dt,\ \theta\in[\tau,\pi-\tau] \\ 0, \qquad \theta\in[0,\tau)\cup(\pi-\tau,\pi] \end{cases}$$

$$Q^{(1,\tilde{1})}(\tau,\theta)=\begin{cases}\tilde{q}_1(2\theta-\tau)\int\limits_{2\tau}^{\pi}q_1(t)dt, & \theta\in[\tau/2,\tau]\\ -\int\limits_{2\tau}^{\theta+\tau}q_1(t)\tilde{q}_1(2t-2\theta-\tau)dt, & \theta\in(\tau,3\tau/2]\\ -\int\limits_{2\tau}^{\pi}q_1(t)\tilde{q}_1(2t-2\theta-\tau)dt, & \theta\in(3\tau/2,\pi-\tau]\\ -\int\limits_{\theta+\tau/2}^{\pi}q_1(t)\tilde{q}_1(2t-2\theta-\tau)dt, & \theta\in(\pi-\tau,\pi-\tau/2]\\ 0, & \theta\in[0,\tau/2)\cup(\pi-\tau/2,\pi]\end{cases}$$

and for $k=1,2$

$$Q^{(1,\tilde{k})*}(\tau,\theta)=\begin{cases}\tilde{q}_k(2\theta-\tau)\int\limits_{2\theta}^{2\tau}q_1(t)dt-\int\limits_{\theta+\tau/2}^{2\theta}q_1(t)\tilde{q}_k(2t-2\theta-\tau)dt, & \theta\in[\tau/2,\tau]\\ -\int\limits_{\theta+\tau/2}^{2\tau}q_1(t)\tilde{q}_k(2t-2\theta-\tau)dt, & \theta\in(\tau,3\tau/2]\\ 0, & \theta\in[0,\tau/2)\cup(3\tau/2,\pi]\end{cases}$$

$$Q^{(2,\tilde{k})}(\tau,\theta)=\begin{cases}\tilde{q}_k(2\theta+2\tau)\int\limits_{2\theta}^{\pi}q_2(t)dt-\int\limits_{\theta+\tau}^{2\theta}q_2(t)\tilde{q}_k(2t-2\theta-2\tau)dt, & \theta\in[\tau,\pi/2]\\ -\int\limits_{\theta+\tau}^{\pi}q_2(t)\tilde{q}_k(2t-2\theta-2\tau)dt, & \theta\in(\pi/2,\pi-\tau]\\ 0, & \theta\in[0,\tau)\cup(\pi-\tau,\pi]\end{cases}$$

One can easily show that the folowing relations hold

$$a_{cs}^{(1,\tilde{1})}(2\tau,z)=\int\limits_{\tau/2}^{\pi-\tau/2}Q^{(1,\tilde{1})}(\tau,\theta)\sin z(\pi-2\theta)d\theta=b^{(1,\tilde{1})}(\tau,z)\qquad(3.11)$$

$$a_{cs}^{(1,\tilde{1})}(\tau,2\tau,z)=\int\limits_{\tau/2}^{3\tau/2}Q^{(1,\tilde{1})*}(\tau,\theta)\sin z(\pi-2\theta)d\theta=b^{(1,\tilde{1})*}(\tau,z)\qquad(3.12)$$

$$a_{s^2}^{(1,\tilde{1})}(2\tau,z)=-\int\limits_{\tau/2}^{3\tau/2}Q^{(1,\tilde{1})*}(\tau,\theta)\cos z(\pi-2\theta)d\theta=-a^{(1,\tilde{1})*}(\tau,z)\qquad(3.13)$$

$$a_{cs}^{(2,\tilde{k})}(2\tau,z)=\int\limits_{\tau}^{\pi-\tau}Q^{(2,\tilde{k})}(\tau,\theta)\sin z(\pi-2\theta)d\theta=b^{(2,\tilde{k})}(\tau,z)\qquad(3.14)$$

where for k=1,2

$$b^{(1,\tilde{k})*}(\tau,z)=\int_0^\pi Q^{(1,\tilde{k})*}(\tau,\theta)\sin z(\pi-2\theta)d\theta,$$
$$b^{(2,\tilde{k})}(\tau,z)=\int_0^\pi Q^{(2,\tilde{k})}(\tau,\theta)\sin z(\pi-2\theta)d\theta,$$

and

$$b^{(1,\tilde{1})}(\tau,z)=\int_0^\pi Q^{(1,\tilde{1})}(\tau,\theta)\sin z(\pi-2\theta)d\theta,$$
$$a^{(1,\tilde{1})*}(\tau,z)=\int_0^\pi Q^{(1,\tilde{1})*}(\tau,\theta)\cos z(\pi-2\theta)d\theta,$$
$$b^{(1)}(\tau,z)=\int_0^\pi Q^{(1)}(\tau,\theta)\sin z(\pi-2\theta)d\theta.$$

Using aforementioned relations (3.4) – (3.14), we can rewrite characteristic functions (3.3) as follows

$$\begin{aligned}F(\tau,z)=&\left(-z+\frac{hH}{z}\right)\sin\pi z+(h+H)\cos\pi z+\sum_{k=1}^{2}\frac{J_1^{(k)}(\tau)}{2}\cos z(\pi-k\tau)\\&+\frac{H+h}{2z}\sum_{k=1}^{2}J_1^{(k)}(\tau)\sin z(\pi-k\tau)+\frac{J_2^{(1)}(\tau)}{4z}\sin z(\pi-2\tau)+2\sum_{k=1}^{2}\breve{a}^{(k)}(\tau,z)\\&+\frac{1}{2}\sum_{k=1}^{2}\hat{a}^{(k)}(\tau,z)+\frac{2H}{z}\sum_{k=1}^{2}\breve{b}^{(k)}(\tau,z)+\frac{H-h}{2z}\sum_{k=1}^{2}\hat{b}^{(k)}(\tau,z)+\frac{1}{z}b^{(1,\tilde{1})}(\tau,z)\\&+\frac{1}{4z}b^{(1)}(\tau,z)+\frac{1}{z}\sum_{k=1}^{2}\left[b^{(1,\tilde{k})*}(\tau,z)+b^{(2,\tilde{k})}(\tau,z)\right]+O\left(\frac{a^{(1,\tilde{1})*}(\tau,z)}{z^2}\right),\ |z|\to\infty.\end{aligned}\tag{3.15}$$

Let us introduce the following notation. For k=1,2

$$\breve{a}_{2n}^{(k)}(\tau)=\frac{2}{\pi}\int_0^{k\tau/2}\breve{q}^{(k)}(\tau,\theta)\cos 2n\theta d\theta;\quad \breve{b}_{2n}^{(k)}(\tau)=\frac{2}{\pi}\int_0^{k\tau/2}\breve{q}^{(k)}(\tau,\theta)\sin 2n\theta d\theta,$$
$$\breve{b}_{2n}^{(k)*}(\tau)=\frac{2}{\pi}\int_0^{k\tau/2}\theta\breve{q}^{(k)}(\tau,\theta)\sin 2n\theta d\theta;\quad \hat{a}_{2n}^{(k)}(\tau)=\frac{2}{\pi}\int_{k\tau/2}^{\pi-k\tau/2}\hat{q}_k(\tau,\theta)\cos 2n\theta d\theta,$$
$$\hat{b}_{2n}^{(k)}(\tau)=\frac{2}{\pi}\int_{k\tau/2}^{\pi-k\tau/2}\hat{q}_k(\tau,\theta)\sin 2n\theta d\theta;\quad \hat{b}_{2n}^{(k)*}(\tau)=\frac{2}{\pi}\int_{k\tau/2}^{\pi-k\tau/2}\theta\hat{q}_k(\tau,\theta)\sin 2n\theta d\theta,$$
$$b_{2n}^{(1,\tilde{k})*}(\tau)=\frac{2}{\pi}\int_0^{\pi}Q^{(1,\tilde{k})*}(\tau,\theta)\sin 2n\theta d\theta;\quad b_{2n}^{(2,\tilde{k})}(\tau)=\frac{2}{\pi}\int_0^{\pi}Q^{(2,\tilde{k})}(\tau,\theta)\sin 2n\theta d\theta,$$

and

$$b_{2n}^{(1,\tilde{1})}(\tau) = \frac{2}{\pi}\int_0^\pi Q^{(1,\tilde{1})}(\tau,\theta)\sin 2n\theta d\theta;\;\; b_{2n}^{(1)}(\tau) = \frac{2}{\pi}\int_0^\pi Q^{(1)}(\tau,\theta)\sin 2n\theta d\theta.$$

Now we can prove the theorem about zeros of the characteristic function.

Theorem 3.1. *If $q_j(x) \in L_2[0,\pi]$, $j = 1,2$ then, zeros $z_n(\tau)$, $n \in N$ of the function* (3.15) *have an asymptotics shape*

$$z_n(\tau) = n + \frac{C_1}{n} + \frac{C_2}{n^2} + o\left(\frac{C_2}{n^2}\right), (n \to \infty)$$

where

$$C_1 = \frac{H+h}{\pi} + \frac{1}{2\pi}\sum_{k=1}^{2} J_1^{(k)}(\tau)\cos kn\tau + \alpha_{2n}(\tau), \tag{3.16}$$

$$\alpha_{2n}(\tau) = o(1),\; n \to \infty,$$

$$\begin{aligned} C_2 = &\left[\frac{(h+H)(-\tau)}{2\pi^2}J_1^{(1)}(\tau) - \frac{\tau}{8\pi^2}J_1^{(1)}(\tau)J_1^{(2)}(\tau)\right]\sin n\tau \\ &+ \left[\frac{(h+H)(-2\tau)}{2\pi^2}J_1^{(2)}(\tau) + \frac{\pi-\tau}{8\pi^2}(J_1^{(1)}(\tau))^2 - \frac{J_2^{(2)}(\tau)}{4\pi}\right]\sin 2n\tau \\ &+ \frac{2\pi-3\tau}{8\pi^2}J_1^{(1)}(\tau)J_1^{(2)}(\tau)\sin 3n\tau + \frac{\pi-2\tau}{8\pi^2}(J_1^{(2)}(\tau))^2\sin 4n\tau + \sigma_{2n}(\tau), \end{aligned} \tag{3.17}$$

$$\sigma_{2n}(\tau) = o(1),\; n \to \infty.$$

Proof. Put (3.2) into the equation (3.15). Let us notice that the following asymptotic formulas hold:

$$\sin\pi z_n(\tau) = (-1)^n\left[\frac{\pi C_1}{n} + \frac{\pi C_2}{n^2} + O\left(\frac{C_1}{n^3}\right)\right],$$

$$\frac{1}{z_n(\tau)} = \frac{1}{n} + O\left(\frac{C_2}{n^3}\right), \cos\pi z_n(\tau) = (-1)^n\left[1 + O\left(\frac{C_1}{n^3}\right)\right],$$

$$\cos(\pi - k\tau)z_n(\tau) = (-1)^n\left[\cos kn\tau + \frac{\pi - k\tau}{n}C_1\sin kn\tau\right] + O\left(\frac{1}{n^2}\right),$$

$$\sin(\pi - k\tau)z_n(\tau) = (-1)^{n+1}\sin kn\tau + O\left(\frac{1}{n}\right),$$

$$\breve{a}^{(k)}(\tau, z_n(\tau)) = (-1)^n\frac{\pi}{2}\left[\breve{a}_{2n}^{(k)}(\tau) + \frac{\pi C_1}{n}\breve{b}_{2n}^{(k)}(\tau) - \frac{2C_1}{n}\breve{b}_{2n}^{(k)*}(\tau)\right],$$

$$\hat{a}^{(k)}(\tau, z_n(\tau)) = (-1)^n\frac{\pi}{2}\left[\hat{a}_{2n}^{(k)}(\tau) + \frac{\pi C_1}{n}\hat{b}_{2n}^{(k)}(\tau) - \frac{2C_1}{n}\hat{b}_{2n}^{(k)*}(\tau)\right],$$

$$
\begin{aligned}
\breve{b}^{(k)}(\tau, z_n(\tau)) &= (-1)^{n+1}\frac{\pi}{2}\breve{b}_{2n}^{(k)}(\tau) + o\left(\frac{1}{n}\right),\\
\hat{b}^{(k)}(\tau, z_n(\tau)) &= (-1)^{n+1}\frac{\pi}{2}\hat{b}_{2n}^{(k)}(\tau) + o\left(\frac{1}{n}\right),\\
b^{(1,\tilde{k})*}(\tau, z_n(\tau)) &= (-1)^{n+1}\frac{\pi}{2}b_{2n}^{(1,\tilde{k})*}(\tau) + o\left(\frac{1}{n}\right),\\
b^{(2,\tilde{k})}(\tau, z_n(\tau)) &= (-1)^{n+1}\frac{\pi}{2}b_{2n}^{(2,\tilde{k})}(\tau) + o\left(\frac{1}{n}\right),\\
b^{(1,\tilde{1})}(\tau, z_n(\tau)) &= (-1)^{n+1}\frac{\pi}{2}b_{2n}^{(1,\tilde{1})}(\tau) + o\left(\frac{1}{n}\right),\\
b^{(1)}(\tau, z_n(\tau)) &= (-1)^{n+1}\frac{\pi}{2}b_{2n}^{(1)}(\tau) + o\left(\frac{1}{n}\right).
\end{aligned} \tag{3.18}
$$

Inserting asymptotic relations (3.18) in the equation $F(\tau, z_n) = 0$, we get

$$
\begin{aligned}
&-\pi\left[C_1 + \frac{C_2}{n}\right] + h + H + \sum_{k=1}^{2}\frac{J_1^{(k)}(\tau)}{2}\left[\cos kn\tau + \frac{\pi - k\tau}{n}C_1 \sin kn\tau\right]\\
&-\frac{J_2^{(2)}(\tau)}{4n}\sin 2n\tau - \frac{H+h}{2n}\sum_{k=1}^{2}J_1^{(k)}(\tau)\sin kn\tau - \frac{H\pi}{n}\sum_{k=1}^{2}\breve{b}_{2n}^{(k)}(\tau)\\
&+\pi\sum_{k=1}^{2}[\breve{a}_{2n}^{(k)}(\tau) + \pi\frac{C_1}{n}\breve{b}_{2n}^{(k)}(\tau) - \frac{2C_1}{n}\breve{b}_{2n}^{(k)*}(\tau)] + \frac{(h-H)\pi}{4n}\sum_{k=1}^{2}\hat{b}_{2n}^{(k)}(\tau)\\
&+\frac{\pi}{4}\sum_{k=1}^{2}[\hat{a}_{2n}^{(k)}(\tau) + \pi\frac{C_1}{n}\hat{b}_{2n}^{(k)}(\tau) - \frac{2C_1}{n}\hat{b}_{2n}^{(k)*}(\tau)] - \frac{\pi}{2n}\sum_{k=1}^{2}b_{2n}^{(1,\tilde{k})*}(\tau)\\
&-\frac{\pi}{2n}b_{2n}^{(1,\tilde{1})}(\tau) - \frac{\pi}{8n}b_{2n}^{(1)}(\tau) - \frac{\pi}{2n}\sum_{k=1}^{2}b_{2n}^{(2,\tilde{k})}(\tau) + o\left(\frac{1}{n^2}\right) = 0,
\end{aligned}
$$

and grouping expression by degrees, then we obtain:

$$
C_1 = \frac{H+h}{\pi} + \frac{1}{2\pi}\sum_{k=1}^{2}J_1^{(k)}(\tau)\cos kn\tau + \alpha_{2n}(\tau),
$$

$$
\alpha_{2n}(\tau) = \sum_{k=1}^{2}\breve{a}_{2n}^{(k)}(\tau) + \frac{1}{4}\sum_{k=1}^{2}\hat{a}_{2n}^{(k)}(\tau),
$$

$$
\begin{aligned}
C_2 = &\left[\frac{-(H+h)\tau}{2\pi^2}J_1^{(1)}(\tau) - \frac{\tau}{8\pi^2}J_1^{(1)}(\tau)J_1^{(2)}(\tau)\right]\sin n\tau\\
&+\left[\frac{-(H+h)2\tau}{2\pi^2}J_1^{(2)}(\tau) + \frac{\pi-\tau}{8\pi^2}(J_1^{(1)}(\tau))^2 - \frac{J_2^{(2)}(\tau)}{4\pi}\right]\sin 2n\tau\\
&+\frac{2\pi-3\tau}{8\pi^2}J_1^{(1)}(\tau)J_1^{(2)}(\tau)\sin 3n\tau + \frac{\pi-2\tau}{8\pi^2}(J_1^{(2)}(\tau))^2\sin 4n\tau + \sigma_{2n}(\tau),
\end{aligned}
$$

$$\sigma_{2n}(\tau) = \frac{\alpha_{2n}(\tau)}{4\pi^2}\sum_{k=1}^{2}(\pi-k\tau)J_1^{(k)}(\tau)\sin kn\tau - \frac{1}{2}\sum_{k=1}^{2} b_{2n}^{(2,\tilde{k})}(\tau) - \frac{1}{2}\sum_{k=1}^{2} b_{2n}^{(1,\tilde{k})^*}(\tau)$$
$$+\left[h+\pi\alpha_{2n}(\tau)+\frac{1}{2}\sum_{k=1}^{2}J_1^{(k)}(\tau)\cos kn\tau\right]\sum_{k=1}^{2}\breve{b}_{2n}^{(k)}(\tau) - \frac{1}{8}b_{2n}^{(1)}(\tau)$$
$$+\left[-\frac{2(H+h)}{\pi}-2\alpha_{2n}(\tau)-\frac{1}{\pi}\sum_{k=1}^{2}J_1^{(k)}(\tau)\cos kn\tau\right]\sum_{k=1}^{2}\breve{b}_{2n}^{(k)^*}(\tau)$$
$$+\left[\frac{h(\pi-1)-H}{2\pi}+\frac{\pi-2}{4}\alpha_{2n}(\tau)-\frac{\pi-2}{8\pi}\sum_{k=1}^{2}J_1^{(k)}(\tau)\cos kn\tau\right]\sum_{k=1}^{2}\hat{b}_{2n}^{(k)}(\tau),$$

where

$$\alpha_{2n}(\tau)=o(1),\ \sigma_{2n}(\tau)=o(1),\ n\to\infty.$$

This proves theorem 3.1. □

4. MAIN RESULTS

From $\lambda_n = z_n^2$ and (3.2) we have

$$\lambda_n = n^2 + 2C_1 + \frac{2C_2}{n} + o\left(\frac{C_2}{n}\right).$$

Definition 4.1. *The sum of the series*

$$s_1(\tau) = \sum_{n=1}^{\infty}\left(\lambda_n - n^2 - \frac{2(H+h)}{\pi} - \frac{1}{\pi}\sum_{k=1}^{2}J_1^{(k)}(\tau)\cos kn\tau - \alpha_{2n}(\tau)\right)$$

is called the regularized trace of first order of the operator D^2.

Remark 4.1. Since

$$\lambda_n - n^2 - \frac{2(H+h)}{\pi} - \frac{1}{\pi}\sum_{k=1}^{2}J_1^{(k)}(\tau)\cos kn\tau - \alpha_{2n}(\tau) = o\left(\frac{C_2}{n}\right)$$

where C_2 is given by (3.17), the series

$$\sum_{n=1}^{\infty}\left(\lambda_n - n^2 - \frac{2(H+h)}{\pi} - \frac{1}{\pi}\sum_{k=1}^{2}J_1^{(k)}(\tau)\cos kn\tau - \alpha_{2n}(\tau)\right)$$

converges, so the trace $s_1(\tau)$ is well defined.

From (3.15) one can easily see that $F(\tau,z)$ is an entire, even function with unit growth with respect to variable z, and then, using Hadamard's factorization theorem it can be written in the form

$$F(\tau,z) = (\lambda_0(\tau)-z^2)\frac{\sin\pi z}{z}\prod_{n=1}^{\infty}\left(1-\frac{n^2-\lambda_n(\tau)}{n^2-z^2}\right). \tag{4.1}$$

We will now study the asymptotic behavior of function $F(\tau,z)$ for $z=-iy,\ y\to+\infty$. We have

$$F(\tau,-iy)=(\lambda_0(\tau)+y^2)\frac{\sinh\pi y}{y}\prod_{n=1}^{\infty}\left(1-\frac{n^2-\lambda_n(\tau)}{n^2+y^2}\right). \tag{4.2}$$

Denote

$$\Phi(\tau,y)=\prod_{n=1}^{\infty}\left(1-\frac{n^2-\lambda_n(\tau)}{n^2+y^2}\right).$$

One can show that (see [5])

$$\ln\Phi(\tau,y)=-\sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\left(\frac{n^2-\lambda_n(\tau)}{n^2+y^2}\right)^k. \tag{4.3}$$

Lemma 4.1. (see [5]) *If* $|n^2-\lambda_n(\tau)|\le a,$ *then*

$$\sum_{n=1}^{\infty}\frac{|n^2-\lambda_n(\tau)|^k}{(n^2+y^2)^k}\le\frac{\pi a^k}{2y^{2k-1}},\ (\forall k).$$

Based on the lemma 4.1 we can evaluate all sums in (4.3), except the first one, i.e. we have

$$\sum_{k=2}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\frac{|n^2-\lambda_n(\tau)|^k}{(n^2+y^2)^k}\le\frac{\pi}{2}\sum_{k=2}^{\infty}\frac{a^k}{y^{2k-1}}<\frac{\pi a^2}{2y^3}.$$

For $k=1$ we have

$$-\sum_{n=1}^{\infty}\frac{n^2-\lambda_n(\tau)}{n^2+y^2}=\sum_{n=1}^{\infty}\frac{2C_1}{y^2+n^2}+\frac{1}{y^2}\sum_{n=1}^{\infty}\left(\lambda_n(\tau)-n^2-2C_1\right)$$
$$-\frac{1}{y^2}\sum_{n=1}^{\infty}\frac{\left(\lambda_n(\tau)-n^2-2C_1\right)n^2}{y^2+n^2}.$$

For further assessments, we use (3.17). One can show that $C_2(n,\tau)$ has the form

$$C_2(n,\tau)=\sum_{k=1}^{4}\xi_k(\tau)\sin kn\tau+\sigma_{2n}(\tau)$$

where

$$\xi_1(\tau)=\frac{-(H+h)\tau}{2\pi^2}J_1^{(1)}(\tau)-\frac{\tau}{8\pi^2}J_1^{(1)}(\tau)J_1^{(2)}(\tau),$$
$$\xi_2(\tau)=\frac{-(H+h)2\tau}{2\pi^2}J_1^{(2)}(\tau)+\frac{\pi-\tau}{8\pi^2}(J_1^{(1)}(\tau))^2-\frac{J_2^{(2)}(\tau)}{4\pi},$$
$$\xi_3(\tau)=\frac{2\pi-3\tau}{8\pi^2}J_1^{(1)}(\tau)J_1^{(2)}(\tau),\ \xi_4(\tau)=\frac{\pi-2\tau}{8\pi^2}(J_1^{(2)}(\tau))^2.$$

Since

$$\frac{2}{n}C_2(n,\tau)=\frac{2}{n}\sum_{k=1}^{4}\xi_k(\tau)\sin kn\tau+o\left(\frac{\sigma_{2n}(\tau)}{n}\right),$$

it follows that the series

$$\sum_{n=1}^{\infty}\frac{\left(\lambda_n(\tau)-n^2-2C_1\right)n^2}{y^2+n^2}$$

behaves like the series

$$\sum_{n=1}^{\infty}\frac{2n}{y^2+n^2}\left[\sum_{k=1}^{4}\xi_k(\tau)\sin kn\tau+\sigma_{2n}(\tau)\right].$$

It is well known (see [11]) that

$$\sum_{n=1}^{\infty}\frac{n\sin nx}{n^2+y^2}=\frac{\pi}{2}\frac{\sinh(\pi-x)y}{\sinh\pi y},\ 0<x<2\pi \quad \text{holds.}$$

In our case since $\frac{\pi}{3}\leq\tau<\frac{\pi}{2}$ we have $0<k\tau<2\pi,\ k=1,2,3,4.$
Therefore,

$$2\sum_{k=1}^{4}\xi_k(\tau)\sum_{n=1}^{\infty}\frac{n\sin kn\tau}{n^2+y^2}=2\frac{\pi}{2}\sum_{k=1}^{4}\xi_k(\tau)\frac{\sinh(\pi-k\tau)y}{\sinh\pi y}=o\left(\frac{1}{y^m}\right),\ \forall m\in N,\ y\to\infty.$$

This evaluation also is valid for the series $\sum_{n=1}^{\infty}\frac{n\sigma_{2n}(\tau)}{n^2+y^2}$ because the $\sigma_{2n}(\tau)=o(1),\ n\to\infty.$

Further, based on (3.16) we have

$$2\sum_{n=1}^{\infty}\frac{C_1(n,\tau)}{n^2+y^2}=\frac{2}{\pi}(h+H)\sum_{n=1}^{\infty}\frac{1}{n^2+y^2}+\frac{1}{\pi}\sum_{k=1}^{2}J_1^{(k)}(\tau)\sum_{n=1}^{\infty}\frac{\cos kn\tau}{n^2+y^2}$$
$$+\sum_{k=1}^{2}\left[\sum_{n=1}^{\infty}\frac{2\breve{a}_{2n}^{(k)}(\tau)}{n^2+y^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\hat{a}_{2n}^{(k)}(\tau)}{n^2+y^2}\right].$$

It is known that (see [11])

$$\sum_{n=1}^{\infty}\frac{1}{y^2+n^2}=\frac{\pi}{2y}\frac{\cosh\pi y}{\sinh\pi y}-\frac{1}{2y^2},$$
$$\sum_{n=1}^{\infty}\frac{\cos kn\tau}{y^2+n^2}=\frac{\pi}{2y}\frac{\cosh(\pi-k\tau)y}{\sinh\pi y}-\frac{1}{2y^2},\ k=1,2.$$

Therefore

$$\sum_{n=1}^{\infty}\frac{\breve{a}_{2n}^{(1)}(\tau)}{y^2+n^2}=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{y^2+n^2}\int_0^{\tau/2}\breve{q}^{(1)}(\tau,\theta)\cos 2n\theta d\theta.$$

Since

$$\left|\frac{1}{y^2+n^2}\int_0^{\tau/2}\breve{q}^{(1)}(\tau,\theta)\cos 2n\theta d\theta\right|\leq$$
$$\leq\frac{1}{y^2+n^2}\int_0^{\tau/2}|\breve{q}^{(1)}(\tau,\theta)|d\theta\leq\frac{\|\breve{q}^{(1)}(\tau,\theta)\|_{L_1[0,\pi]}}{n^2+y^2},\ \forall\theta\in[0,\tau/2],\ \forall n\in N$$

and arbitrary $y\in R^+$, we conclude that the series

$$\sum_{n=1}^{\infty}\frac{1}{y^2+n^2}\int_0^{\tau/2}\breve{q}^{(1)}(\tau,\theta)\cos 2n\theta d\theta$$

converges uniformly for $\theta\in[0,\tau/2]$. Therefore we have

$$\sum_{n=1}^{\infty}\frac{\breve{a}_{2n}^{(1)}(\tau)}{y^2+n^2}=\frac{2}{\pi}\int_0^{\tau/2}\breve{q}^{(1)}(\tau,\theta)\sum_{n=1}^{\infty}\frac{\cos 2n\theta}{y^2+n^2}d\theta$$
$$=\frac{1}{y}\int_0^{\tau/2}\breve{q}^{(1)}(\tau,\theta)\frac{\cosh(\pi-2\theta)y}{\sinh\pi y}d\theta-\frac{1}{\pi y^2}\int_0^{\tau/2}\breve{q}^{(1)}(\tau,\theta)d\theta.$$

Taking into account that

$$\frac{\cosh(\pi-2\theta)y}{\sinh\pi y}\sim e^{-2\theta y},\ y\to\infty,\theta\in(0,\tau/2]$$

we conclude that $\int_0^{\tau/2}|\breve{q}^{(1)}(\tau,\theta)|\frac{\cosh(\pi-2\theta)y}{\sinh\pi y}d\theta$ tends to zero exponentially. In this way, it has been shown that

$$\sum_{n=1}^{\infty}\frac{\breve{a}_{2n}^{(1)}(\tau)}{y^2+n^2}=-\frac{1}{\pi y^2}\int_0^{\tau/2}\breve{q}^{(1)}(\tau,\theta)d\theta+O\left(\frac{1}{y^3}\right),\ y\to\infty.$$

In a similar way we obtain

$$\sum_{n=1}^{\infty}\frac{\breve{a}_{2n}^{(2)}(\tau)}{y^2+n^2}=-\frac{1}{\pi y^2}\int_0^{\tau}\breve{q}^{(2)}(\tau,\theta)d\theta+O\left(\frac{1}{y^3}\right),\ y\to\infty$$

and

$$\sum_{n=1}^{\infty}\frac{\hat{a}_{2n}^{(k)}(\tau)}{y^2+n^2}=-\frac{1}{\pi y^2}\int_{k\tau/2}^{\pi-k\tau/2}\hat{q}_{(k)}(\tau,\theta)d\theta+O\left(\frac{1}{y^3}\right),\ y\to\infty,\ k=1,2.$$

Therefore

$$2\sum_{n=1}^{\infty}\frac{C_1(n,\tau)}{n^2+y^2}=\frac{h+H}{y}+\frac{1}{y^2}\left[-\frac{h+H}{\pi}-\frac{1}{2\pi}\sum_{k=1}^{2}J_1^{(k)}(\tau)-\frac{2}{\pi}\sum_{k=1}^{2}\breve{J}_1^{(k)}(\tau)\right]$$
$$-\frac{1}{2\pi y^2}\sum_{k=1}^{2}\hat{J}_1^{(k)}(\tau)+o\left(\frac{1}{y^2}\right).$$

where

$$\check{J}_1^{(k)} = \int_0^{k\tau/2} \check{q}^{(k)}(\tau,\theta)d\theta,\ \ \hat{J}_1^{(k)} = \int_{k\tau/2}^{\pi-k\tau/2} \hat{q}_{(k)}(\tau,\theta)d\theta,\ k=1,2.$$

Then we obtain

$$\ln\Phi(\tau,y) = \frac{\Delta_0}{y} + \frac{\Delta_1(\tau)+s_1(\tau)}{y^2} + o\left(\frac{1}{y^2}\right)$$

where

$$\Delta_0 = h+H,\ \Delta_1(\tau) = -\frac{h+H}{\pi} - \frac{1}{2\pi}\sum_{k=1}^{2} J_1^{(k)}(\tau) - \frac{2}{\pi}\sum_{k=1}^{2}\check{J}_1^{(k)}(\tau) - \frac{1}{2\pi}\sum_{k=1}^{2}\hat{J}_1^{(k)}(\tau),$$

i.e.

$$\Phi(\tau,y) = 1 + \frac{\Delta_0}{y} + \frac{\Delta_1(\tau)+s_1(\tau)}{y^2} + \frac{\Delta_0^2}{2y^2} + O\left(\frac{1}{y^3}\right). \tag{4.4}$$

From (4.2) and (4.4) we obtain

$$F(\tau,-iy) = y\sinh\pi y\left\{1+\frac{\Delta_0}{y}+\frac{1}{y^2}\left[\lambda_0(\tau)+s_1(\tau)\Delta_1(\tau)+\frac{1}{2}\Delta_0^2\right]\right\}+O\left(\frac{1}{y^3}\right). \tag{4.5}$$

From the other side, we can determine asymptotic formulas directly form (3.15). We have

$$F(\tau,-iy) = y\sinh\pi y\left[1+\frac{h+H}{y}+\frac{hH}{y^2}+O\left(\frac{1}{y^3}\right)\right]. \tag{4.6}$$

Now, from (4.5) and (4.6) we obtain

$$s_1(\tau) = hH - \lambda_0(\tau) - \Delta_1(\tau) - \frac{1}{2}\lambda_0^2(\tau)$$

i.e.

$$s_1(\tau) = \frac{H+h}{\pi}+\frac{1}{2}(H^2+h^2)-\lambda_0(\tau)+\frac{1}{2\pi}\sum_{k=1}^{2}\left(J1^{(k)}(\tau)+4\check{J}_1^{(k)}(\tau)+\hat{J}_1^{(k)}(\tau)\right). \tag{4.7}$$

Formula (4.7) represents the first regularized trace of the operator D^2. In this way, the following theorem has been proved.

Theorem 4.1. *If $q_j(x) \in L_2[0,\pi]$, $j=1,2$, the first regularized trace of the operator D^2 has the form*

$$s_1(\tau) = \frac{H+h}{\pi} + \frac{1}{2}(H^2+h^2) - \lambda_0(\tau) + \frac{1}{2\pi}\sum_{k=1}^{2}\left(J_1^{(k)}(\tau)+4\check{J}_1^{(k)}(\tau)+\hat{J}_1^{(k)}(\tau)\right).$$

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