SARAJEVO JOURNAL OF MATHEMATICS
Vol.15 (28), No.1, (2019), 67–79

DOI: 10.5644/SJM.15.01.07

ON THE STUDY OF AN INITIAL VALUE PROBLEM FOR A SECOND ORDER DIFFERENTIAL EQUATION SET IN A SINGULAR CYLINDRICAL DOMAIN

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Abstract. We will investigate an initial value problem for a second order differential equation set in a singular cylindrical domain Π. The existence, uniqueness and maximal regularity results are obtained for the classical solutions by using the semigroup theory. The study is performed in the framework of little Hölder space $h^{2\sigma}(\Pi)$ with $0 < 2\sigma < 1$.

1. Introduction

Let Π the unbounded cylindrical domain defined by

$$\Pi = \mathbb{R}^+ \times \Omega, \tag{1.1}$$

where Ω is the conical domain given by

$$\Omega = \left\{ (x,y,z) \in \mathbb{R}^3 : \sqrt{x^2+y^2} \leq \varphi(z), \ 0 \leq z \leq 1 \right\}.$$

Here, $\varphi : (0,1] \to \mathbb{R}^+$ is a locally Lipschitz function such that

$$\varphi(0) = \varphi'(0) = 0.$$

In the cylinder Π, we consider the following boundary value problem for second-order differential equation

$$\begin{array}{l} \partial_t^2 u + \Delta u = h, \\ u|_{\{0\}\times\Omega} = \partial_t u|_{\{0\}\times\Omega}\,, \\ \lim\limits_{t\to+\infty} u|_{\Omega} = 0, \\ u|_{\mathbb{R}^+\times\partial\Omega} = 0. \end{array} \tag{1.2}$$

Denote by $BUC^k([0,+\infty[\,; C(\overline{\Omega}))$, $k \in \mathbb{N}$ is the space the space consisted of vector-valued functions with uniformly continuous and bounded derivatives up to order k in $[0,+\infty[$. Our study is performed in the little Hölder space

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2010 *Mathematics Subject Classification.* 34G10, 34K10,12H2O, 35J25, 44A45.

Key words and phrases. Fractional powers of linear operators, Analytic Semigroup, Abstract differential equation, Little Hölder space, Interpolation spaces.

$$h^{2\sigma}\left(\overline{\Pi}\right), \ \sigma \in (0,1/2),$$

which can be identified to

$$h^{2\sigma}([0,+\infty[\,;C(\overline{\Omega})) \cap L^{\infty}(]0,+\infty[\,;h^{2\sigma}(\overline{\Omega})), \ \sigma \in (0,1/2),$$

where $h^{2\sigma}([0,+\infty[\,;C(\overline{\Omega}))$ is the space defined by

$$\left\{ \phi \in BUC([0,+\infty[\,;C(\overline{\Omega})) : \lim_{\varepsilon \to 0^+} \sup_{0<|t-t'|\leq \varepsilon} \frac{\|\phi(t)-\phi(t')\|_{C(\overline{\Omega})}}{|t-t'|^{2\sigma}} = 0 \right\},$$

while $L^{\infty}(]0,+\infty[\,;h^{2\sigma}(\overline{\Omega}))$ is the space consisting of all Bochner measurable functions $\phi :]0,+\infty[\to h^{2\sigma}\left(\overline{\Omega}\right)$ satisfying

$$\sup_{t>0} \|\phi(t)\|_{h^{2\sigma}(\overline{\Omega})} < \infty.$$

We assume also that

$$h|_{\{0\}\times\Omega} = 0, \qquad h|_{[0,+\infty[\times\partial\Omega} = 0.$$

In literature, several works were concerned with the study of boundary value problems in cylindrical domains with nonsmooth base. A great number of these papers dealt with L^p setting of such problems. For some illustrative examples, please see [4], [5], [6] and the references therein. These results obtained are mainly based on the use of classical arguments such us the variational method or the potential theory. In our situation, the difficulties encountered are related to the nature of the functional framework of the little Hölder spaces and the presence of the cusp point on the boundary of our cylinder Π. In fact, it is well known that above cited techniques can be hardly applied and seems not adapted to our functional framework.

At this level, it is important to note that this work can be viewed in some sense as a continuation of [2], where the study of the same problem with a non-zero spectral parameter was investigated. The commutative version of the well known sum's operator theory was successfully used to establish some Hölder continuous regularity results. Unfortunately, in our situation this method fails. For this reason, we opt for the use of the abstract differential equation theory combined with the domain's approximation method as in [10]. These powerful tools allows us not only to prove the existence of the solution near the singular part of the boundary but also provides us some regularity results of such solution. The main result of this paper is formulated as follows:

Theorem 1.1. *Let $h \in h^{2\sigma}(\overline{\Pi})$, with $\sigma \in (0,1/2)$ satisfying*

$$h|_{\{0\}\times\Omega} = 0, \qquad h|_{[0,+\infty[\times\partial\Omega} = 0.$$

Then, Problem (1.2) has a unique strict solution u such that

$$\partial_t^2 u, \ \triangle u \in h_w^{2\sigma}(\overline{\Pi}),$$

with

$$h_w^{2\sigma}(\overline{\Pi}) = \left\{ h \in h^{2\sigma}(\overline{\Pi}) : (\varphi(z))^{2\sigma} h^{2\sigma}(\overline{\Pi}) \in h^{2\sigma}(\overline{\Pi}) \right\}.$$

2. Change of variables

We consider the following change of variables

$$\begin{aligned} T: \quad & \Pi \to Q, \\ & (t,x,y,z) \mapsto (t,\xi,\eta,z) = \left(t, \frac{x}{\varphi(z)}, \frac{y}{\varphi(z)}, z \right), \end{aligned} \tag{2.1}$$

with

$$\begin{cases} Q = \mathbb{R}^+ \times \Omega', \\ \Omega' = [0,1] \times D(0,1), \end{cases}$$

and

$$D(0,1) = \left\{ (\xi,\eta) \in \mathbb{R}^2 : \xi^2 + \eta^2 \le 1 \right\}.$$

In the new cylinder Q, the problem (1.2) can be written as

$$\begin{aligned} & \partial_t^2 v + \widetilde{\Delta} v = f, \\ & v|_{\{0\}\times\Omega'} = \partial_t v|_{\{0\}\times\Omega'}, \\ & \lim_{t\to+\infty} v|_{\Omega'} = 0, \\ & v|_{\mathbb{R}}^{+} \times \partial\Omega' = 0, \end{aligned} \tag{2.2}$$

where $\widetilde{\Delta}$ is the transformed Laplace operator given by

$$\widetilde{\Delta} = \partial_z^2 + \frac{1}{\varphi^2(z)} \Delta_{(\xi,\eta)} + \frac{\varphi'(z)}{\varphi(z)} \{ \xi\partial_\xi + \eta\partial_\eta \}$$

and

$$\begin{cases} u(t,x) = v(t,\xi) \\ h(t,x) = f(t,\xi). \end{cases}$$

Here, we shall denote by (ξ,η,z) a generic point of $\mathbb{R}^3$ with respect to the new change of variables (2.1).

Note here that due to the presence of the singular point $\{0\}$, the change of variables (2.1) leads to the study of a transformed equation containing singular coefficients. As in [10], our strategy consists to approximate the cylinder Π by a sequence of regular subdomains. For this end, we consider the following natural change of variables given by

$$\begin{aligned} T_n: \quad & \Pi \to Q_n \\ & (t,x,y,z) \mapsto (t,\xi,\eta,z_n) = \left(t, \frac{x}{\varphi(z_n)}, \frac{y}{\varphi(z_n)}, z_n \right) \end{aligned}$$

with

$$\begin{cases} Q_n = \mathbb{R}^+ \times \Omega'_n \\ \Omega'_n = [z_n, 1] \times D(0,1). \end{cases}$$

Here, $(z_n)_{n\in\mathbb{N}}$ is a decreasing sequence such that

$$\begin{cases} 0 < z_n \leq 1, \\ \text{and} \\ \lim\limits_{n\to+\infty} z_n = 0. \end{cases}$$

Set

$$\begin{cases} v_n := v|_{Q_n}, \\ f_n := f|_{Q_n}, \\ \varphi_n := \varphi_n(z) = \varphi|_{Q_n}. \end{cases}$$

Now, in the regular domains Q_n, we consider the following approximate problems

$$\begin{aligned} &\partial_t^2 v_n + \widetilde{\Delta} v = f_n, \\ &v_n|_{\{0\}\times\Omega_n'} = \partial_t v_n|_{\{0\}\times\Omega_n'}, \\ &\lim_{t\to+\infty} v_n|_{\Omega_n'}, \\ &v_n|_{\mathbb{R}^+\times\partial\Omega_n'}. \end{aligned} \tag{2.3}$$

Before studying these problems, it is necessary to clarify the relationship between the change of variables (2.1) and functional framework of little Hölder spaces

Lemma 2.1. *Let $\sigma \in (0, 1/2)$. Then*

(1) $h \in h^{2\sigma}(\overline{\Pi}) \Rightarrow f \in h^{2\sigma}(\overline{Q})$.

(2) $f \in h^{2\sigma}(\overline{Q}) \Rightarrow h \in h_w^{2\sigma}(\overline{\Pi})$ *with*

$$h_w^{2\sigma}(\overline{\Pi}) = \left\{ h \in h^{2\sigma}(\overline{\Pi}) : (\varphi(z))^{2\sigma} h^{2\sigma}(\overline{\Pi}) \in h^{2\sigma}(\overline{\Pi}) \right\}.$$

Proof. See Proposition 3.1 in [3]. □

Remark 2.2. We note here that *any function* of $h^{2\sigma}(\Pi)$, *can be extended to a function of* $h^{2\sigma}(\overline{\Pi})$. *This is why we shall write in the sequel* $h^{2\sigma}(\Pi)$ *or* $h^{2\sigma}(\overline{\Pi})$.

3. A COMPLETE STUDY OF THE PROBLEM (2.3)

3.1. Statement of the abstract version of problem (2.3).

As a first classical step, let us introduce the following vector-valued functions:

$$\begin{aligned} v_n &: [0, +\infty[\to C_0(\Omega_n');\ t \longrightarrow v_n(t); \quad v_n(t)(\xi) = v_n(t, \xi), \\ f_n &: [0, +\infty[\to C_0(\Omega_n');\ t \longrightarrow f_n(t); \quad f_n(t)(\xi) = f_n(t, \xi), \end{aligned}$$

where

$$C_0(\Omega_n') = \left\{ \phi \in C(\Omega_n') : \phi|_{\partial\Omega_n'} = 0 \right\}.$$

Consequently, the transformed problem (2.3) can be viewed as an abstract differential equation set on unbounded domain given by

$$\begin{aligned} &v_n''(t) + \frac{1}{\varphi_n^2} A_n v_n(t) = f_n(t), \quad t \geq 0, \\ &v_n(0) = v_n'(0), \\ &v_n(+\infty) = 0. \end{aligned} \tag{3.1}$$

Here $(A_n, D(A_n))$ are the linear operators defined by

$$\begin{cases} D(A_n) = \{v_n \in BUC(\mathbb{R}^+; C_0(\Omega_n')) : v_n(t) \in D(L_n)\}, \\ (A_n v)(t) = L_n(v_n(t)), \end{cases} \tag{3.2}$$

where

$$\begin{cases} D(L_n) = \left\{\phi \in W^{2,p}(\Omega_n') \cap C_0(\Omega_n'),\ p > 3 : L_n(\phi) \in C_0(\Omega_n')\right\}, \\ (L_n \phi)(\xi) = \left\{\varphi_n^2 \partial_z^2 + \Delta_{(\xi,\eta)} + \varphi_n \varphi_n' \xi \partial_\xi + \varphi_n \varphi_n' \eta \partial_\eta\right\} \phi. \end{cases} \tag{3.3}$$

In order to describe some spectral properties of the above operators, we recall the following definition:

Definition 3.1. The linear operator H is called a pseudo-sectorial operator if

1. $\rho(H)$, the resolvent set of H, contains a sector
$$S_\theta := \{\lambda \in \mathbb{C}^* : |\arg(\lambda)| < \theta\},$$
with $\theta > 0$.
2. There exists $M > 0$ such that
$$\sup_{\lambda \in \overline{S}_\theta} \left\| \lambda (H - \lambda I)^{-1} \right\|_{L(X)} \leqslant M. \tag{3.4}$$

Remark 3.2. Henceforward, the letter M will be used to denote different constants encountered in our formulas.

Our first results are concerned with the closed operator A_n defined by (3.2):

Lemma 3.3. *The closed linear operator* $\frac{1}{\varphi_n^2} A_n$ *is a pseudo-sectorial operator.*

Proof. Observe here that the linear operator $(L_n, D(L_n))$ defined by (3.3) is an elliptic operator with smooth and bounded coefficients on Q_n. Then, it follows from Theorem 1 in [12] that L_n is a pseudo-sectorial operator. More precisely, there exists a sector of the form

$$S_{\delta+\pi/2} := \{\lambda \in \mathbb{C}^* : |\arg(\lambda)| < \delta + \pi/2\}, \tag{3.5}$$

with $\delta > 0$ such that

$$\rho(L_n) \supset S_{\delta+\pi/2} \text{ and } \exists\ M > 0 : \forall \lambda \in S_{\delta+\pi/2} \qquad \left\| (L_n - \lambda I)^{-1} \right\|_{L(E)} \leq \frac{M}{|\lambda|}.$$

On the other hand, it is easy to see that

$$\left\| \left(\frac{1}{\varphi_n^2} A_n - \lambda \right)^{-1} \right\|_{L(C(\Omega'_n))} \leq \frac{M}{|\lambda|}. \qquad \square$$

Remark 3.4. In virtue of the above lemma, it is possible to make use of square roots

$$-\left(-\frac{1}{\varphi_n^2} A_n \right)^{1/2}.$$

which is well defined. Furthermore, this operator is a pseudo-sectorial one. This says exactly that the closed operator

$$-\left(-\frac{1}{\varphi_n^2} A_n \right)^{1/2},$$

is the infinitesimal generator of a generalized analytic semigroup

$$\left(\exp\left(\frac{B_n}{\varphi_n} t \right) \varphi \right)_{t>0},$$

where

$$B_n = -\left(-A_n \right)^{1/2}.$$

Thus, one has for all $t > 0$ and $\varphi \in C_0(\Omega'_n)$,

$$\exp\left(\frac{B_n}{\varphi_n} t \right) \varphi = \frac{1}{2i\pi} \int_\gamma e^{zt} (\frac{B_n}{\varphi_n} - zI)^{-1} \varphi dz,$$

with γ is the sectorial boundary curve of $S_{\delta+\pi/2}$ oriented from $\infty e^{i(\delta+\pi/2)}$ to $\infty e^{-i(\delta+\pi/2)}$.

Note here that is a "generalized analytic semigroup" in the sense that B_n is not supposed to be densely defined and so $\left(\exp\left(\frac{B_n}{\varphi_n} t \right) \varphi \right)_{t>0}$ is not supposed to be a strongly continuous semi-group; for more details about the spectral properties of $\left(\exp\left(\frac{B_n}{\varphi_n} t \right) \varphi \right)_{t>0}$, see E. Sinestrari [11] and A. Lunardi [8].

In order to give a complete study of problem (3.1), we use a technique which is essentially based on the analytic semigroup's theory. We need also to introduce some useful real interpolation spaces between $D\left(A_n\right)$ and $C_0(\overline{\Omega'_n})$ given by

$$D_{A_n}\left(\sigma\right) = \left\{ \phi \in C_0(\Omega'_n) : \lim_{r \to +\infty} \left\| r^\sigma A_n \left(A_n - rI\right)^{-1} \phi \right\|_{C_0(\Omega'_n)} = 0 \right\},$$

where $\sigma \in (0, 1/2)$; more details about these Banach spaces are given in [11].

Remark 3.5. In our situation, one has exactly

$$D_{A_n}\left(\sigma\right) = D_{(A_n/\varphi_n^2)}\left(\sigma\right) = h_0^{2\sigma}\left(\Omega'_n\right),\ \sigma \in (0, 1/2).$$

3.2. **Regularity results for the problem (3.1).** The investigation of problem (3.1) implies the use of well known Krein's reduction order method; more details about this method are given in the pioneering work of [7]. We know that, for a fixed $T>0$, an arbitrary solution of (3.1) has the form

$$u(t)=C_1\exp\left(\frac{B_n}{\varphi_n}t\right)+C_2\exp\left(\frac{B_n}{\varphi_n}(T-t)\right)+\mathcal{I}(t)+\mathcal{J}(t),\quad 0\leq t\leq T,$$

where

$$\begin{aligned}\mathcal{I}(t)=&\frac{1}{2}\int_0^t\exp\left(\frac{B}{\varphi_n}(t-s)\right)B^{-1}f(s)ds,\\ \mathcal{J}(t)=&\frac{1}{2}\int_t^T\exp\left(\frac{B}{\varphi_n}(s-t)\right)B^{-1}f(s)ds,\end{aligned}$$

and the constant C_1 and C_2 are uniquely determined via the boundary condition. A direct computation shows that for $T\to+\infty$, the solution of (3.1) is uniquely given formally by

$$\begin{aligned}v_n(t)=&-\frac{B_n^{-1}}{2}\mathcal{L}_1(t,\varphi_nf_n)+\frac{B_n^{-1}}{2}\mathcal{L}_2(t,\varphi_nf_n)\\ &+\frac{B_n^{-1}}{2}\mathcal{L}_3\left(t,\left(\frac{B_n}{\varphi_n}-I\right)\left(\frac{B_n}{\varphi_n}+I\right)^{-1}f_n\right),\end{aligned}\tag{3.6}$$

where

$$\begin{cases}\mathcal{L}_1(t,f_n)=\int\limits_0^t\exp\left(\frac{B_n}{\varphi_n}(s-t)\right)f_n(s)ds,\\ \mathcal{L}_2(t,f_n)=\int\limits_t^{+\infty}\exp\left(\frac{B_n}{\varphi_n}(t-s)\right)f_n(s)ds,\\ \mathcal{L}_3(t,f_n)=\int\limits_0^{+\infty}\exp\left(\frac{B_n}{\varphi_n}(s+t)\right)f_n(s)ds.\end{cases}$$

In fact, a direct computation shows that

$$\begin{aligned}v_n''(t)=&+\frac{1}{2}\frac{B_n}{\varphi_n}\left(\int_0^t\exp\left(\frac{B_n}{\varphi_n}(s-t)\right)f_n(s)ds\right)\\ &+\frac{1}{2}\frac{B_n}{\varphi_n}\left(\int_t^{+\infty}\exp\left(\frac{B_n}{\varphi_n}(t-s)\right)f_n(s)ds\right)+f_n(t)\\ &+\frac{1}{2}\frac{B_n}{\varphi_n}\left(\frac{B_n}{\varphi_n}-I\right)\left(\frac{B_n}{\varphi_n}+I\right)^{-1}\left(\int_0^{+\infty}\exp\left(\frac{B_n}{\varphi_n}(s+t)\right)f_n(s)ds\right),\end{aligned}$$

and

$$\begin{aligned}\left(\frac{B_n}{\varphi_n}\right)^2 v_n(t) &= \frac{1}{2}\left(\frac{B_n}{\varphi_n}\int_0^t \exp\left(\frac{B_n}{\varphi_n}(s-t)\right) f_n(s)ds\right)\\ &+\frac{B_n}{2\varphi_n}\left(\int_t^{+\infty} \exp\left(\frac{B_n}{\varphi_n}(t-s)\right) f_n(s)ds\right)\\ &+\frac{1}{2}\frac{B_n}{\varphi_n}\left(\frac{B_n}{\varphi_n}-I\right)\left(\frac{B_n}{\varphi_n}+I\right)^{-1}\left(\int_0^{+\infty} \exp\left(\frac{B_n}{\varphi_n}(s+t)\right) f_n(s)ds\right).\end{aligned}$$

Keeping in mind that

$$B_n^2 = -A,$$

a direct computation shows that

$$v_n''(t) + \frac{1}{(\varphi_n)^2} A v_n(t) = f_n(t).$$

By the same way, we see that the solution v_n satisfies the boundary conditions associated to our problem. To prove that $v_n(+\infty) = 0$, it suffices to take into account the boundedness of the semigroup $\left(\exp\left(\frac{B_n}{\varphi_n}t\right)\varphi\right)_{t>0}$ and the conditions $f_n(+\infty) = 0$ as in [3].

Remark 3.6. The uniqueness of solution is easy to check. In fact, thanks to the boundary conditions, we see that $v_n = 0$ is a unique solution of the problem

$$\begin{aligned}&v_n''(t) + \frac{1}{\varphi_n^2} A_n v_n(t) = 0, \quad t \geq 0,\\ &v_n(0) = v_n'(0),\\ &v_n(+\infty) = 0.\end{aligned}$$

The following results are handled by adapting the same techniques delivered in the proof of Proposition 4.1 and Proposition 5.2 in [3].

Lemma 3.7. *Let* $f_n \in h^{2\sigma}\left(\mathbb{R}^+; C_0(\overline{\Omega_n'})\right)$, *with*

$$\sigma \in (0, 1/2),$$

satisfying

$$f_n(+\infty) = 0, \ f_n(0) = 0.$$

Then, Problem (3.1) has a unique solution

$$v_n \in BUC^2(\mathbb{R}^+; C_0(\Omega_n')) \cap BUC(\mathbb{R}^+; D(A_n)),$$

satisfying

1. $v_n''(.) \in h^{2\sigma}(\mathbb{R}^+; C_0(\Omega_n'))),$
2. $\frac{1}{\varphi_n^2} A_n v_n(.) \in h^{2\sigma}(\mathbb{R}^+; C_0(\Omega_n'))).$

Lemma 3.8. *Let* $f_n \in BUC(\mathbb{R}^+, C_0(\Omega_n')) \cap L^\infty(\mathbb{R}^+; D_{A_n}(\sigma))$, *with*

$$\sigma \in (0, 1/2),$$

such that

$$f_n(+\infty) = 0, \; f_n(0) = 0.$$

Then, Problem (3.1) has a unique solution

$$v_n \in W^{2,\infty}\left(\mathbb{R}^+; C_0(\Omega_n')\right) \cap L^\infty(\mathbb{R}^+; D(A_n)),$$

satisfying

1. $v_n''(.) \in L^\infty(\mathbb{R}^+; D_{A_n}(\sigma))$.
2. $\dfrac{1}{\varphi_n^2} A_n v_n(.) \in L^\infty(\mathbb{R}^+; D_{A_n}(\sigma))$.

Our main result concerning the problem (2.3) is given by

Proposition 3.9. *Let* $f_n \in h^{2\sigma}(Q_n)$, *with* $\sigma \in (0, 1/2)$ *such that*

$$f_n(+\infty) = 0, \; f_n(0) = 0. \tag{3.7}$$

Then, there exists a unique solution

$$v_n \in BUC^2(\mathbb{R}^+; C^2(\Omega)),$$

of (2.3) such that

$$\begin{cases} \partial_t^2 v_n \in h^{2\sigma}(Q_n), \\ \left(\partial_z^2 + \dfrac{1}{\varphi_n^2}\Delta_{(\xi,\eta)} + \dfrac{\varphi_n'}{\varphi_n}\{\xi\partial_\xi + \eta\partial_\eta\}\right) v_n \in h^{2\sigma}(Q_n). \end{cases}$$

At this level, it is obvious to see that

$$\begin{cases} \|f_n\|_{BUC(\mathbb{R}^+, C_0(\Omega_n'))} \leq \|f\|_{BUC(\mathbb{R}^+, C_0(\Omega'))}, \\ \|f_n\|_{L^\infty(\mathbb{R}^+; h^{2\sigma}(\Omega_n'))} \leq \|f\|_{L^\infty(\mathbb{R}^+; h^{2\sigma}(\Omega'))}, \\ \|f_n\|_{h^{2\sigma}(\mathbb{R}^+; C_0(\Omega_n'))} \leq \|f\|_{h^{2\sigma}(\mathbb{R}^+; C_0(\Omega'))} \end{cases}$$

as well as the following useful estimates hold true:

Proposition 3.10. *Let* $f_n \in h^{2\sigma}(Q_n)$ *with* $\sigma \in (0, 1/2)$ *such that*

$$f_n(+\infty) = 0, \; f_n(0) = 0.$$

Then, there exits constants $M > 0$ *independent of* n *such that*

1. $\|v_n(.)\|_{BUC(\mathbb{R}^+, C_0(\Omega_n'))} \leq M$,
2. $\|v_n'(.)\|_{BUC(\mathbb{R}^+, C_0(\Omega_n'))} \leq M$,
3. $\left\|\dfrac{1}{\varphi_n^2} A_n v_n(.)\right\|_{BUC(\mathbb{R}^+, C_0(\Omega_n'))} \leq M$,
4. $\left\|\dfrac{1}{\varphi_n^2} A_n v_n(.)\right\|_{L^\infty(\mathbb{R}^+; h^{2\sigma}(\Omega_n'))} \leq M$,

$$(5)\quad \left\| \frac{1}{\varphi_n^2} A_n v_n(.) \right\|_{h^{2\sigma}(\mathbb{R}^+;C_0(\Omega_n'))} \leq M.$$

Sketch of the proof. The proof of these results using the formula (3.6) leads to a cumbersome calculus. One must be careful, since the operators are not densely defined. So, we prefer the natural use of the Green's kernels. In fact, using the Dunford's calculus it can be easily seen that the unique solution v_n of (3.1) is given by the formula

$$\begin{aligned} v_n(t) &= \frac{1}{4i\pi}\int_\gamma \int_0^t \frac{\exp\left(-\frac{\sqrt{-\lambda}}{\varphi_n}(t-s)\right)}{\sqrt{-\lambda}} \left(\frac{1}{\varphi_n^2}A_n - \lambda\right)^{-1} f_n(s)dsd\lambda \\ &+ \frac{1}{4i\pi}\int_\gamma \int_t^{+\infty} \frac{\exp\left(-\frac{\sqrt{-\lambda}}{\varphi_n}(s-t)\right)}{\sqrt{-\lambda}} \left(\frac{1}{\varphi_n^2}A_n - \lambda\right)^{-1} f_n(s)dsd\lambda \\ &+ \frac{1}{4i\pi}\int_\gamma \int_0^{+\infty} \omega(\lambda) \frac{\exp\left(-\frac{\sqrt{-\lambda}}{\varphi_n}(t+s)\right)}{\sqrt{-\lambda}} \left(\frac{1}{\varphi_n^2}A_n - \lambda\right)^{-1} f_n(s)dsd\lambda, \end{aligned}$$

with

$$\omega(\lambda) = \frac{\sqrt{-\lambda}-1}{\sqrt{-\lambda}+1},$$

and $\sqrt{-\lambda}$ is the analytic determination defined by $\Re\sqrt{-\lambda} > 0$ and γ is the retrograde oriented boundary of the sector $S_{\delta+\pi/2}$ defined by (3.5).
The properties of such kernels appearing in the above formula was extensively studied in [1]. Then, it suffices to reiterate exactly the same argument used in the proof of Theorem 3.1 in the cited paper. □

Now, we are able to describe the local behavior of the solution of (2.3) in the neighborhood of the cusp base Ω', it means for $z \to 0$. As a direct consequence of the Proposition 3.10 and using a classical argument, it is possible to extract a convergent subsequence (v_{nj}) from (v_n). Denote by v the limit of (v_{nj}). Then, it is easy to see that

$$v_{nj} = v(t, \xi, \eta, z_{nj}) = v|_{Q_{nj}}.$$

with

$$\begin{cases} \lim\limits_{n\to+\infty} z_{nj} = 0, \\ \text{and} \\ \lim\limits_{n\to+\infty} Q_{nj} = Q. \end{cases}$$

This implies that

$$\begin{cases} v\left(t,\xi,\eta,z_{nj}\right) = v|_{Q_{nj}} \to v|_{z=0}\,, \\ \partial_t^2 v\left(t,\xi,\eta,z_{nj}\right) = \partial_t^2 v\big|_{Q_{nj}} \to \partial_t^2 v\big|_{z=0}\,, \\ \partial_z^2 v\left(t,\xi,\eta,z_{nj}\right) = \partial_z^2 v\big|_{Q_{nj}} \to \partial_z^2 v\big|_{z=0}\,, \\ \dfrac{1}{\varphi_n^2\left(z_{nj}\right)}\Delta_{(\xi,\eta)} v\left(t,\xi,\eta,z_{nj}\right) = \dfrac{1}{\varphi^2\left(z\right)}\Delta_{(\xi,\eta)} v\Big|_{Q_{nj}} \to \dfrac{1}{\varphi^2\left(z\right)}\Delta_{(\xi,\eta)} v\Big|_{z=0}\,, \end{cases}$$

and

$$\frac{\varphi_n'\left(z_{nj}\right)}{\varphi_n\left(z_{nj}\right)}\left\{\xi\partial_\xi + \eta\partial_\eta\right\} v\left(t,\xi,\eta,z_{nj}\right) = \frac{\varphi'\left(z\right)}{\varphi\left(z\right)}\left\{\xi\partial_\xi + \eta\partial_\eta\right\} v\Big|_{Q_{nj}}$$

converges to

$$\frac{\varphi'\left(z\right)}{\varphi\left(z\right)}\left\{\xi\partial_\xi + \eta\partial_\eta\right\} v\Big|_{z=0}.$$

Then, we conclude that

$$\begin{aligned} \widetilde{\Delta} v\left(t,\xi,\eta,z_{nj}\right) &= \left(\partial_z^2 + \frac{1}{\varphi_n^2\left(z_{nj}\right)}\Delta_{(\xi,\eta)} + \frac{\varphi_n'\left(z_{nj}\right)}{\varphi_n\left(z_{nj}\right)}\left\{\xi\partial_\xi + \eta\partial_\eta\right\}\right) v\left(t,\xi,\eta,z_{nj}\right) \\ &= \left(\partial_z^2 + \frac{1}{\varphi_n^2\left(z_{nj}\right)}\Delta_{(\xi,\eta)} + \frac{\varphi_n'\left(z_{nj}\right)}{\varphi_n\left(z_{nj}\right)}\left\{\xi\partial_\xi + \eta\partial_\eta\right\}\right) v\Big|_{Q_{nj}} \end{aligned}$$

converges to

$$\left(\partial_z^2 + \frac{1}{\varphi^2\left(z\right)}\Delta_{(\xi,\eta)} + \frac{\varphi'\left(z\right)}{\varphi\left(z\right)}\left\{\xi\partial_\xi + \eta\partial_\eta\right\}\right) v\Big|_{z=0}.$$

Summing up, we deduce that the transformed problems (2.3) admits a unique solution and one has

$$v \in BUC^2(\mathbb{R}^+; C^2(\Omega')).$$

Furthermore, near the singular part of the boundary this solution must satisfy the following properties

$$\begin{cases} \partial_t^2 v\big|_{z=0} \in h^{2\sigma}(Q), \\ \widetilde{\Delta} v\Big|_{z=0} \in h^{2\sigma}(Q). \end{cases}$$

Collecting all the previous results, we conclude that

Proposition 3.11. *Let* $f \in h^{2\sigma}(Q)$, *with* $\sigma \in (0, 1/2)$ *such that*

$$f(+\infty) = 0,\ f(0) = 0.$$

Then, there exists a unique solution

$$v \in BUC^2(\mathbb{R}^+; C^2(\Omega')),$$

of

$$\begin{array}{l} \partial_t^2 v + \widetilde{\Delta} v = f \\ v|_{\{0\}\times\Omega'} = \partial_t v|_{\{0\}\times\Omega'}, \\ \lim\limits_{t\to+\infty} v|_{\Omega'} = 0, \\ v|_{\mathbb{R}^+\times\partial\Omega'} = 0, \end{array}$$

such that

$$\partial_t^2 v \text{ and } \widetilde{\Delta} v \in h^{2\sigma}(Q),$$

The go back to the original domains is provided by the use the inverse change of variables given

$$\begin{array}{ll} T^{-1}: & Q = \mathbb{R}^+ \times \Omega' \to \Pi = \mathbb{R}^+ \times \Omega, \\ & (t,\xi,\eta,z) \mapsto (t,x,y,z) = (t,\xi\varphi(z),\eta\varphi(z),z) \end{array}$$

Finally, we note that our main result is justified by exploiting all results obtained in Proposition 3.11 and Lemma 2.1.

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(Received: November 29, 2017)
(Revised: May 24, 2019)

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