

INTRINSIC PROPERTIES OF FINSLER SPACE WITH A CUBIC CHANGE IN THE MATSUMOTO METRIC

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ABSTRACT. In this paper, we investigate a Finsler space with a cubic modification of the Matsumoto metric, given by $F = \frac{\gamma^2}{\gamma - \beta}$, where γ is a cubic metric and β is a one form metric. We identify the fundamental characteristics of this modified metric. The reducibility of the Cartan torsion tensor is a key factor, as it measures how closely a Finsler metric approximates a Riemannian metric. Specifically, if C_{ijk} vanishes, the Finsler metric becomes Riemannian. Accordingly, we analyze various forms of the Cartan torsion tensor's reducibility within the context of this cubic-changed Matsumoto metric. We also establish conditions for determining whether the Finsler space is quasi C-reducible, semi C-reducible, C-reducible and C2-like.

1. INTRODUCTION

The origins of the Matsumoto metric trace back to a correspondence between Finsler and Matsumoto in 1969. Matsumoto elaborated on this communication in detail [2], concluding that:

In astronomy, distances are measured in units of time. If we use seconds as the unit, the corresponding unit surface is a sphere with a radius of 300,000 km. Each point in our space is associated with such a sphere, defining the distance and leading to the simplest geometry—Euclidean geometry. When considering a ray of light as the shortest path in a gravitational field, the geometry of our space becomes Riemannian. In an anisotropic medium, where the speed of light varies with direction, the unit surface is no longer a sphere. Similarly, on the Earth's surface, distances are sometimes measured in terms of time, such as the time indicated by a guidepost. In this context, a unit curve measured in minutes becomes a general closed curve without a center, reflecting that we cover shorter distances uphill compared to downhill. This represents a general, though approximate, geometry. The shortest path to a destination, such as a mountain summit, would be a complex curve. Based on this concept, in 1989, Matsumoto [7] developed a precise model of a Finsler surface and

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introduced a special form of the Finsler metric which is given below:

$$L(x, y, \dot{x}, \dot{y}) = \frac{\alpha^2}{(v\alpha - w\beta)}, \quad w = \frac{g}{2}, \quad (1.1)$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ represents a Riemannian metric, $\beta = b_i(x)y^i$ is one form metric, and v and w are certain scalars. Further he generalized this metric, which is known as the Matsumoto metric, and it is given in the following form:

$$\mathbb{L}(\alpha, \beta) = \frac{\alpha^2}{\alpha - \beta}. \quad (1.2)$$

Several authors [1, 8, 15, 16] have explored the significance of the Matsumoto metric in various contexts, yielding fascinating results about Finsler spaces endowed with this metric.

Shimada's pioneering work on m -th root metrics [17] has found applications in biology as an ecological metric [2]. This theory extends Riemannian metrics directly, with the second root metric corresponding to the conventional Riemannian metric. The third and fourth root metrics are known as the cubic metric and quartic metric, respectively. Recent advancements highlight the significant impact of m -th root Finsler metrics in various fields, including physics, spacetime theory, gravitation, general relativity, and seismic ray theory.

In 1979, M. Matsumoto [10] introduced the concept of a cubic metric on a differentiable manifold with local coordinates x^i , defined by

$$L(x, y) = (a_{ijk}(x)y^i y^j y^k)^{\frac{1}{3}}$$

where x represents position coordinates, y^i represents direction coordinates, $a_{ijk}(x)$ are components of a symmetric tensor field of $(0, 3)$ -type depending on the position coordinate x . A Finsler space with this cubic metric is referred to as a cubic Finsler space. Several authors [2, 3, 9] have extensively studied m -th root metrics and cubic metrics, underscoring their importance in Finsler geometry. In 2011, Pandey and Chaubey [12] introduced the concept of a (γ, β) -metric, where $\gamma = (a_{ijk}(x)y^i y^j y^k)^{\frac{1}{3}}$ represents a cubic metric and $\beta = b_i(x)y^i$ denotes a one-form. Chaubey and et al. [4, 13, 14] conducted a detailed analysis of the (γ, β) -metric, exploring significant geometric properties of Finsler spaces equipped with this metric.

Matsumoto and Hojo [9] explored various special Finsler spaces, including C-reducible Finsler spaces, semi-C-reducible Finsler spaces, and C2-like Finsler spaces. Their work involved analyzing a specific form of the Cartan torsion tensor C_{ijk} and deriving key geometric properties. Understanding the reducibility of the Cartan torsion tensor is crucial, as it measures how much a Finsler metric deviates from being Riemannian [19]. Specifically, if C_{ijk} vanishes, the Finsler metric reduces to a Riemannian metric. Since computing the Cartan torsion in its full form can be complex, studying its reducibility is particularly valuable in Finsler geometry. Numerous authors have extensively studied C-reducible Finsler space, semi C-reducible Finsler

space, quasi C-reducible Finsler space and C2-like Finsler space for the Finsler spaces with various Finsler metrics [2, 5, 6, 9, 11, 19], contributing significantly to advancements in the field. In 2013, Shukla and Mishra [18] investigated different forms of the Cartan torsion tensor and curvature, including quasi C-reducible, semi C-reducible, C-reducible, C2-like, and S3-like properties, specifically for Finsler spaces endowed with the (γ, β) -metric.

In this paper, we propose a new Finsler metric by replacing the Riemannian metric α with a cubic Finsler metric γ in the Matsumoto metric. The resulting metric is expressed as follows:

$$F = \frac{\gamma^2}{\gamma - \beta}. \quad (1.3)$$

We refer to this new metric as the cubic changed Matsumoto metric, and the Finsler space equipped with it as a Finsler space with the cubic-changed Matsumoto metric. In the paper, we derive the fundamental tensors necessary for describing the geometric properties of this Finsler space in Propositions 2.1, 2.5, 2.6 and 2.7 respectively. Furthermore, we identified conditions under which the Finsler space with a cubic changed infinite series metric can be classified as quasi C-reducible, semi C-reducible, C-reducible, and C2-like Finsler spaces in theorems 3.2, 3.4, 3.6 and 3.8 respectively.

2. FUNDAMENTAL PROPERTIES OF FINSLER SPACE WITH THE CUBIC-CHANGED MATSUMOTO METRIC

In this section, we derive the fundamental metric tensors for the Finsler space equipped with the cubic changed Matsumoto metric.

Differentiating (1.3) partially with respect to y^i , we get

$$l_i = \dot{\partial}_i F = \frac{F_\gamma}{\gamma^2} a_i + F_\beta b_i, \quad (2.1)$$

where $a_i(x, y) = a_{ijk}(x)y^j y^k$, and F_γ and F_β are the partial derivatives of the cubic changed Matsumoto metric with respect to γ and β respectively.

Equation (2.1) can also be expressed as

$$y_i = F \dot{\partial}_i F = \frac{F F_\gamma}{\gamma^2} a_i + F F_\beta b_i. \quad (2.2)$$

Again, differentiating (2.2) with respect to y^j , we get the angular metric tensor $h_{ij} = F \dot{\partial}_i \dot{\partial}_j F$ as

$$h_{ij} = p_{-1} a_{ij} + q_0 b_i b_j + q_{-2} (a_i b_j + a_j b_i) + q_{-4} a_i a_j, \quad (2.3)$$

where, $a_{ij}(x, y) = 2a_{ijk}y^k$, $q_0 = \frac{2\gamma^4}{(\gamma-\beta)^4}$, $q_{-2} = \frac{-2\gamma\beta}{(\gamma-\beta)^4}$, $q_{-4} = \frac{6\gamma\beta-2\gamma^2-2\beta^2}{\gamma^2(\gamma-\beta)^4}$.

Owing to the homogeneity or $h_{ij}y^j = 0$, we have two identities

$$\begin{cases} p_{-1} + q_{-2}\beta + q_{-4}\gamma^3 = 0, \\ q_0\beta + q_{-2}\gamma^3 = 0. \end{cases} \quad (2.4)$$

Remark: In (2.3), the subscripts of coefficients $p_{-1}, q_0, q_{-2}, q_{-4}$ are used to indicate respective degrees of homogeneity.

Now, the fundamental metric tensor $g_{ij}(x, y)$ for a Finsler metric is given by

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} = h_{ij} + l_i l_j.$$

Using equations (2.1) and (2.3) the fundamental metric tensor g_{ij} is given by

$$g_{ij} = p_{-1}a_{ij} + p_0b_i b_j + p_{-2}(a_i b_j + a_j b_i) + p_{-4}a_i a_j, \quad (2.5)$$

where,

$$p_0 = \frac{3\gamma^4}{(\gamma - \beta)^4}, \quad p_{-2} = \frac{\gamma^2 - 4\gamma\beta}{(\gamma - \beta)^4}, \quad p_{-4} = \frac{2\beta^2 - \gamma^2 + 2\gamma\beta}{\gamma^2(\gamma - \beta)^4}.$$

Now using the above values of p_0, p_{-2}, p_{-4} in equation (2.4) and using the Euler theorem of homogeneous function, we have the following two identities:

$$\begin{cases} p_0\beta + p_{-2}\gamma^3 = FF_\beta, \\ p_{-2}\beta + p_{-4}\gamma^3 = 0. \end{cases} \quad (2.6)$$

Proposition 2.1. *The line supporting element l_i , the angular metric tensor h_{ij} and the fundamental metric tensor g_{ij} for the Finsler space endowed with the cubic changed Matsumoto metric are described by equations (2.1), (2.3) and (2.5) respectively.*

Proposition 2.2. *The coefficients q_0, q_{-2}, q_{-4} of the angular metric tensor h_{ij} in the Finsler space with a cubic changed Matsumoto metric adhere to the relation given by equation (2.4).*

Proposition 2.3. *The coefficients p_0, p_{-2}, p_{-4} of the fundamental metric tensor g_{ij} in the Finsler space with a cubic changed infinite series metric adhere to the relation given by equation (2.6).*

From [2] we have:

Proposition 2.4. *Let (A_{ij}) be a non-singular symmetric $n \times n$ matrix and let c_i be n quantities. Define $B_{ij} = A_{ij} + c_i c_j$. The inverse matrix (B^{ij}) of (B_{ij}) and the determinant $\det(B_{ij})$ are given by:*

$$B^{ij} = A^{ij} - \frac{1}{1 + c^2} c^i c^j, \quad \det(B_{ij}) = A(1 + c^2),$$

where, (A^{ij}) is the inverse matrix of (A_{ij}) , $A = \det(A_{ij})$, $c^i = A^{ij}c_j$ and $c^2 = c^i c_i$.

Now, equation (2.5) can be rewritten as

$$g_{ij} = p_{-1}a_{ij} + c_i c_j + d_i d_j,$$

where,

$$c_i = \pi b_i, \quad d_i = \pi_0 b_i + \pi_{-2} a_i, \\ \pi^2 + \pi_0^2 = p_0, \quad \pi_0 \pi_{-2} = p_{-2}, \quad \pi_{-2}^2 = p_{-4}.$$

Again, the above equation can be rewritten as

$$g_{ij} = B_{ij} + d_i d_j, \quad (2.7)$$

where,

$$B_{ij} = p_{-1}a_{ij} + c_i c_j. \quad (2.8)$$

Using the properties of the Kronecker-delta and tensor algebra, we have $B_{ij}B^{jk} = \delta_i^k$ where δ_i^k is the Kronecker delta, and it confirms that B^{jk} is indeed the inverse of B_{ij} . Then

$$B^{ij} = \frac{1}{p_{-1}} \left(a^{ij} - \frac{c^i c^j}{p_{-1} + c^2} \right), \quad (2.9)$$

where, a^{ij} is the reciprocal of a_{ij} , $c^i = a^{ij}c_j$.

Now, by using proposition 2.4,

$$g^{ij} = B^{ij} - \frac{d^i d^j}{1 + d^2}, \quad (2.10)$$

where, $d^i = B^{ij}d_j$, $d^i d_i = d^2$.

Using equation (2.9) in the above equation, we have

$$g^{ij} = \frac{1}{p_{-1}} a^{ij} - \frac{c^i c^j}{p_{-1}(p_{-1} + c^2)} - \frac{d^i d^j}{1 + d^2}. \quad (2.11)$$

Now,

$$d^i = B^{ij}d_j = \frac{1}{p_{-1}} \left[\frac{(\pi_0 p_{-1} - \pi^2 \pi_{-2} \bar{a}) B^i}{p_{-1} + c^2} + \pi_{-2} a^i \right],$$

where, $B^i b_i = b^2 = a^{im} b_m b_i$, $a^i B^i = a^{im} a_i b_m = a^i b_i = \bar{a}$, $\pi^2 b^2 = c^2$.

Again,

$$d^i d^j = \frac{1}{p_{-1}^2} \left[\frac{(\pi_0 p_{-1} - \pi^2 \pi_{-2} \bar{a})^2 B^i B^j}{(p_{-1} + c^2)^2} + \frac{(p_{-1} p_{-2} - \pi^2 p_{-4} \bar{a})}{p_{-1} + c^2} (B^i a^j + a^i B^j) + p_{-2} a^i a^j \right].$$

Now,

$$d^2 = d_i d^i = \frac{1}{p_{-1}(p_{-1} + c^2)} \left[\pi_0^2 p_{-1} b^2 + 2p_{-1} p_{-2} \bar{a} + p_{-1} p_{-4} a^2 + p_{-4} c^2 a^2 - p_{-4} \pi^2 \bar{a}^2 \right].$$

The determinant of the nonsingular matrix g_{ij} is given by

$$|g_{ij}| = p_{-1}^{n-1} a(p_{-1} + c^2)(1 + d^2) = p_{-1}^{n-1} a \tau_{-2}$$

where, a is the determinant of a_{ij} , $\tau_{-2} = p_{-1}(p_{-1} + p_0 b^2 + p_{-2} \bar{a}) + (p_{-1} p_{-2} \bar{a} - p_{-2}^2 \bar{a}^2) + p_{-1} p_{-4} a^2 + p_{-4} c^2 a^2$, $B^i b_i = b^2 = a^{im} b_m b_i$, $a^i B^i = a^{im} a_i b_m = a^i b_i = \bar{a}$, $\pi^2 b^2 = c^2 = c^i c_i$, $a^i a_i = a^2$.

Thus the inverse matrix g^{ij} of g_{ij} is given by

$$g^{ij} = \frac{1}{p_{-1}} a^{ij} - s_2 B^i B^j - s_0 (a^i B^j + a^j B^i) - s_{-2} a^i a^j, \quad (2.12)$$

$$\begin{aligned} \text{where, } s_2 &= \frac{\pi_0^2 p_{-1}^2 + \pi^2 (\tau_{-2} - \pi^2 p_{-4} \bar{a}^2 - 2 p_{-1} p_{-2} \bar{a})}{\tau_{-2} p_{-1} (p_{-1} + c^2)} \\ &= \frac{1}{\tau_{-2} (\gamma^2 - 2\gamma\beta) \{ \gamma^2 - 2\gamma\beta + (\gamma - \beta)^3 c^2 \}} \left[\pi_0^2 (\gamma^2 - 2\gamma\beta)^2 + \pi^2 \tau_{-2} (\gamma - \beta)^6 + \right. \\ &\quad \left. \pi^4 \bar{a}^2 \frac{(\gamma - \beta)^2 (2\beta^2 - \gamma^2 + 2\gamma\beta)}{\gamma^2} - 2 \bar{a} \pi^2 \frac{(\gamma^2 - 2\gamma\beta)(\gamma^2 - 4\gamma\beta)}{(\gamma - \beta)} \right], \\ s_0 &= \frac{p_{-1} p_{-2} - \pi^2 p_{-4} \bar{a}}{\tau_{-2} p_{-1}} \\ &= \frac{1}{\tau_{-2} (\gamma^2 - 2\gamma\beta)} \left[\frac{\gamma^4 - 6\gamma^3 \beta + 8\gamma^2 \beta^2}{(\gamma - \beta)^4} - \frac{\pi^2 \bar{a} (2\beta^2 - \gamma^2 + 2\gamma\beta)}{\gamma^2 (\gamma - \beta)} \right], \\ s_{-2} &= \frac{p_{-1} p_{-4} + c^2 p_{-4}}{\tau_{-2} p_{-1}} \\ &= \frac{1}{\tau_{-2} (\gamma^2 - 2\gamma\beta)} \left[\frac{(2\beta^2 - \gamma^2 + 2\gamma\beta)}{\gamma^2 (\gamma - \beta)} \left\{ \frac{(\gamma^2 - 2\gamma\beta)}{(\gamma - \beta)^3} + c^2 \right\} \right]. \end{aligned}$$

Proposition 2.5. *The reciprocal g^{ij} of the fundamental metric tensor g_{ij} for a Finsler space equipped with the cubic-changed Matsumoto metric is given by equation (2.12).*

Since by definition, the Cartan torsion tensor C_{ijk} is

$$C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}.$$

So, differentiating (2.5) partially with respect to y^k , we get

$$2p_{-1} C_{ijk} = 2p_{-1}^2 a_{ijk} + p_0 \beta r_{-2} b_i b_j b_k + r_{-8} a_i a_j a_k \quad (2.13)$$

$$\Pi_{(ijk)} (P_i h_{jk} + r_{-4} a_i b_j b_k + r_{-6} a_i a_j b_k)$$

where, $\Pi_{(ijk)}$ represents the sum of cyclic permutations of i, j, k ;

$$P_i = p_{-4} a_i + p_{-2} b_i, \quad (2.14)$$

and, $r_{-2} = p_{-1} p_0 \beta - 3p_{-2} q_0 = \frac{6\gamma^6}{(\gamma - \beta)^8}$,

$$\begin{aligned} r_{-4} &= p_{-1} p_{-2} \beta - q_0 p_{-4} - 2p_{-2} q_{-2} = \frac{2\gamma^4 + 12\gamma^2 \beta^2 - 12\gamma^3 \beta}{(\gamma - \beta)^8}, \\ r_{-6} &= p_{-1} p_{-4} \beta - 2p_{-4} q_{-2} - p_{-2} q_{-4} = \frac{38\gamma^2 \beta^2 - 4\gamma^3 \beta - 22\gamma \beta^3}{\gamma^2 (\gamma - \beta)^8}, \\ r_{-8} &= p_{-1} \frac{p_{-4} \beta}{\gamma^2} - 3p_{-4} q_{-4} = \frac{4\beta^4 - 2\gamma^4 - 20\gamma^2 \beta^2 + 12\gamma^3 \beta}{\gamma^4 (\gamma - \beta)^8}. \end{aligned}$$

From (2.4) and (2.6), we obtain

$$r_{-\mu}\beta + r_{-\mu-2}\gamma^3 = 0, \quad \mu = 2, 4, 6. \quad (2.15)$$

From (2.6) and (2.15), we have

$$p_{-4} = \phi p_{-2}, \quad r_{-\mu-2} = \phi^{\frac{\mu}{2}} r_{-2}, \quad \mu = 2, 4, 6 \quad (2.16)$$

where, $\phi = -\frac{\beta}{\gamma^3}$.

The Cartan torsion tensor C_{ijk} can be rewritten as

$$2p_{-1}C_{ijk} = 2p_{-1}^2 a_{ijk} + \Pi_{(ijk)}(H_{jk}P_i) \quad (2.17)$$

where, $H_{ij} = h_{ij} + \frac{r_{-2}}{3p_{-2}^3} P_i P_j$.

Now, contracting C_{ijk} by g^{ij} , we have,

$$C_i = p_{-1}A_i + Aa_i + Bb_i, \quad (2.18)$$

where, C_i is the mean Cartan tensor of order one and $A_i = a_{ijk}g^{jk}$ is also a tensor of order one but different from the mean Cartan tensor, while

$$\begin{aligned} A = \frac{1}{2p_{-1}} & \left[p_{-4} + \frac{p_{-4}}{p_{-1}} q_{-2}\bar{a} + \frac{p_{-4}}{p_{-1}} q_{-4} + \frac{p_{-2}}{p_{-1}} q_{-4}\bar{a} - s_2(q_{-2}p_{-4}\bar{a}b^2 + p_{-2}q_{-2}b^4 \right. \\ & + q_{-4}p_{-4}\bar{a}^2 + q_{-4}p_{-2}\bar{a}b^2) - s_0(p_{-1}p_{-4}\bar{a} + p_{-1}p_{-2}b^2 + q_{-2}p_{-4}a^2b^2 + q_{-2}p_{-4}\bar{a}^2 \\ & + 2q_{-2}p_{-2}\bar{a}b^2 + 2q_{-4}p_{-4}\bar{a}a^2 + q_{-4}p_{-2}\bar{a}^2 + q_{-4}p_{-2}a^2b^2) - s_{-2}(p_{-1}p_{-4}a^2 \\ & + p_{-1}p_{-2}\bar{a} + q_{-2}p_{-4}\bar{a}a^2 + q_{-2}p_{-2}\bar{a}^2 + q_{-4}p_{-4}a^4 + q_{-4}p_{-4}\bar{a}a^2) + \frac{r_{-2}}{3p_{-1}}(\phi^3a^2 \\ & + 2\phi^2\bar{a} + \phi b^2) - \frac{r_{-2}}{3}s_2(\phi^3\bar{a}^2 + 2\phi^2\bar{a}b^2 + \phi b^4) - \frac{r_{-2}}{3}s_0(2\phi^3a^2\bar{a} + 2\phi^2\bar{a} \\ & \left. + 2\phi^2a^2b^2 + 2\phi\bar{a}b^2) - \frac{r_{-2}}{3}s_{-2}(\phi^3a^4 + 2\phi^2a^2\bar{a} + \phi a^2) \right] \end{aligned}$$

and

$$\begin{aligned} B = \frac{1}{2p_{-1}} & \left[p_{-2} + \frac{p_{-2}}{p_{-1}} q_0b^2 + \frac{p_{-4}}{p_{-1}} q_0\bar{a} + \frac{p_{-4}}{p_{-1}} q_{-2}\bar{a} - s_2(p_{-1}p_{-4}\bar{a} + p_{-1}p_{-2}b^2 \right. \\ & + p_{-4}q_0\bar{a}b^2 + p_{-2}q_0b^4 + p_{-4}q_{-2}\bar{a}^2 + q_{-2}p_{-2}\bar{a}b^2) - s_0(p_{-1}p_{-4}a^2 + p_{-1}p_{-2}\bar{a} \\ & + q_0p_{-4}a^2b^2 + q_0p_{-4}\bar{a}^2 + 2q_0p_{-2}\bar{a}b^2 + 2q_{-2}p_{-4}\bar{a}a^2 + q_{-2}p_{-2}\bar{a}^2 + q_{-2}p_{-2}a^2b^2) \\ & - s_{-2}(q_0p_{-4}a^2\bar{a}^2 + q_0p_{-2}\bar{a}^2 + q_{-2}p_{-4}a^4 + q_{-2}p_{-2}\bar{a}a^2) + \frac{r_{-2}}{3p_{-1}}(\phi^2a^2 + 2\phi\bar{a} \\ & + b^2) - \frac{r_{-2}}{3}s_2(\phi^2a^2 + 2\phi\bar{a}b^2 + b^4) - \frac{r_{-2}}{3}s_0(2\phi^2\bar{a}a^2 + 2\phi a^2b^2 + 2\phi\bar{a}^2 + 2\bar{a}b^2) \\ & \left. - \frac{r_{-2}}{3}s_{-2}(\phi^2a^4 + 2\phi a^2\bar{a} + a^2) \right] \end{aligned}$$

are scalars.

Proposition 2.6. *The Cartan torsion tensor C_{ijk} of a Finsler space endowed with the cubic changed Matsumoto metric is described by equation (2.13).*

Proposition 2.7. *The Mean tensor C_i of the Cartan torsion tensor C_{ijk} for a Finsler space endowed with the cubic changed Matsumoto metric is described by equation (2.18).*

3. REDUCIBILITY OF THE CARTAN TENSOR FOR THE FINSLER SPACE WITH THE CUBIC CHANGED MATSUMOTO METRIC

In this section, we explore the reducibility of the Cartan tensor for the Finsler space equipped with the cubic-changed Matsumoto metric. The Cartan tensor is a crucial component in determining how a Finsler metric approximates or deviates from being Riemannian. For the Finsler space with the cubic-changed Matsumoto metric, the Cartan tensor's reducibility provides insights into the specific classification of the Finsler space into quasi C-reducible Finsler space, semi C-reducible Finsler space, C-reducible Finsler space and C2-like Finsler space.

Definition 3.1. *A Finsler space is said to be quasi-C-reducible Finsler space if its Cartan torsion tensor C_{ijk} can be expressed in the form:*

$$C_{ijk} = Q_{ij}C_k + Q_{jk}C_i + Q_{ki}C_j \quad (3.1)$$

where, Q_{ij} is any symmetric tensor of order two and C_i is the mean Cartan tensor for the Finsler space.

Now, from equations (2.14) and (2.17), we have

$$2p_{-1}C_{ijk} = 2p_{-2}^2a_{ijk} + \Pi_{(ijk)} \{H_{jk}(p_{-4}a_i + p_{-2}b_i)\}.$$

Now, substituting the value of a_i from (2.18) in the above equation, we have

$$\begin{aligned} 2p_{-1}C_{ijk} &= 2p_{-2}^2a_{ijk} + \Pi_{(ijk)} \left\{ H_{jk} \left(\frac{p_{-4}}{A} (C_i - p_{-1}a_i - Bb_i) + p_{-2}b_i \right) \right\}, \\ C_{ijk} &= p_{-1}a_{ijk} + \frac{p_{-4}}{2p_{-1}A} \Pi_{(ijk)} (H_{jk}C_i) - \\ &\quad \Pi_{(ijk)} \left\{ \left(\frac{p_{-4}A_i}{2A} + \left(\frac{p_{-4}B}{2p_{-1}A} - \frac{p_{-2}}{2p_{-1}} \right) b_i \right) H_{jk} \right\}. \end{aligned}$$

Thus, for a Finsler space with a cubic changed Matsumoto metric, the Cartan tensor can be written in the form

$$C_{ijk} = V_{ijk} + \Pi_{(ijk)}(Q_{jk}C_i), \quad (3.2)$$

where, $Q_{jk} = \frac{p_{-4}}{2p_{-1}A}H_{jk}$ is a symmetric tensor of order two and

$$V_{ijk} = p_{-1}a_{ijk} - \Pi_{(ijk)} \left\{ \left(\frac{p_{-4}A_i}{2A} + \left(\frac{p_{-4}B}{2p_{-1}A} - \frac{p_{-2}}{2p_{-1}} \right) b_i \right) H_{jk} \right\}.$$

Thus, comparing equations (3.1) and (3.2), we can conclude the following proposition:

Theorem 3.1. *A Finsler space equipped with the cubic changed Matsumoto metric will exhibit a quasi C-reducible Finsler space if and only if the tensor V_{ijk} in equation (3.2) vanishes identically and $\frac{p-4}{2p-1A} \rightarrow 1$.*

Definition 3.2. *A Finsler space is said to be semi C-reducible Finsler space if its Cartan torsion tensor C_{ijk} can be written as*

$$C_{ijk} = \frac{r}{n+1}(C_i h_{jk} + C_j h_{ki} + C_k h_{ij}) + \frac{t}{c^2} C_i C_j C_k \quad (3.3)$$

where, r and t are scalar functions satisfying $r + t = 1$, while h_{ij} and C_i denote the angular metric tensor and mean Cartan tensor of the Finsler space under consideration.

Now, using equations (2.14) and (2.17), the Cartan torsion tensor for the Finsler space with a cubic changed Matsumoto metric can be rewritten as

$$C_{ijk} = p_{-1} a_{ijk} + \frac{1}{2p_{-1}} \Pi_{ijk} (h_{jk} P_i) + \frac{r_{-2}}{p_{-1} p_{-2}^3} P_i P_j P_k.$$

By using equation (2.18), we have

$$C_{ijk} = \frac{r}{n+1} \Pi_{ijk} (h_{jk} C_i) + \frac{t}{C^2} C_i C_j C_k + U_{ijk} \quad (3.4)$$

where, $U_{ijk} = p_{-1} a_{ijk} - \Pi_{(ijk)} \left[\frac{p_{-4}}{2p_{-1}A} \{ p_{-1}A_i + (B - \frac{p_{-2}A}{p_{-4}}) b_i \} h_{jk} + \frac{r_{-2}}{2p_{-1}p_{-2}^3} \left\{ -\frac{p_{-1}p_{-4}^3}{A} C_i C_j A_k - \frac{p_{-4}^2}{A^2} (\frac{p_{-4}B}{A} - p_{-2}) C_i C_j b_k + \frac{p_{-1}p_{-4}^2}{A^3} C_i A_j A_k + \frac{p_{-1}p_{-4}^2}{A^2} (\frac{p_{-4}B}{A} - p_{-2}) C_i A_j b_k + \frac{p_{-4}}{A} (\frac{p_{-4}B}{A} - p_{-2})^2 C_i b_j b_k - \frac{p_{-1}p_{-4}^2}{A^2} (\frac{p_{-4}B}{A} - p_{-2}) A_i A_j b_k - \frac{p_{-1}p_{-4}}{A} (\frac{p_{-4}B}{A} - p_{-2})^2 A_i b_j b_k \right\} \right] - \frac{r_{-2}}{2p_{-1}p_{-2}^3} \left\{ \frac{p_{-1}p_{-4}^3}{A^3} A_i A_j A_k + (\frac{p_{-4}B}{A} - p_{-2})^3 b_i b_j b_k \right\}$, is a tensor of order three, whereas, $r = \frac{(n+1)p_{-4}}{2Ap_{-1}}$ and $t = \frac{C^2 r_{-2} p_{-4}^3}{2p_{-1} p_{-2}^3 A^3}$ are some scalars.

Now, from equations (3.3) and (3.4) we can conclude:

Theorem 3.2. *A Finsler space equipped with the cubic changed Matsumoto metric will exhibit a semi-C-reducible Finsler space if and only if the tensor U_{ijk} in equation (3.4) vanishes identically and $r + t = 1$.*

Definition 3.3. *A Finsler space is said to be C-reducible Finsler space if its Cartan torsion tensor C_{ijk} can be written as*

$$C_{ijk} = \frac{1}{n+1} (C_i h_{jk} + C_j h_{ki} + C_k h_{ij}) \quad (3.5)$$

where, C_i is the mean Cartan tensor and h_{ij} is the symmetric angular metric tensor for the Finsler space.

From (2.17) we have

$$2p_{-1}C_{ijk} = 2p_{-1}^2a_{ijk} + \Pi_{(ijk)}(h_{jk}P_i) + \frac{r_{-2}}{p_{-2}^3}P_iP_jP_k. \quad (3.6)$$

Now, using equation (3.5) in the above equation we have

$$\begin{aligned} 2p_{-1}\left(\frac{1}{n+1}\right)\Pi_{(ijk)}(h_{jk}C_k) &= 2p_{-1}^2a_{ijk} + \Pi_{(ijk)}(h_{jk}P_i) + \frac{r_{-2}}{p_{-2}^3}P_iP_jP_k, \\ 2p_{-1}^2a_{ijk} + \frac{r_{-2}}{p_{-2}^3}P_iP_jP_k &= 2p_{-1}\left(\frac{1}{n+1}\right)\Pi_{(ijk)}(h_{jk}C_k) - \Pi_{(ijk)}(h_{jk}P_i), \\ 2p_{-1}^2a_{ijk} + \frac{r_{-2}}{p_{-2}^3}P_iP_jP_k &= \Pi_{(ijk)}\left(\frac{2p_{-1}}{(n+1)}C_k - P_k\right)h_{ij}, \\ 2p_{-1}^2a_{ijk} + \frac{r_{-2}}{p_{-2}^3}P_iP_jP_k &= \Pi_{(ijk)}(h_{ij}N_k), \end{aligned} \quad (3.7)$$

where, $N_k = \frac{2p_{-1}}{(n+1)}C_k - P_k$. Conversely, if (3.7) is satisfied for a certain covariant vector N_k , then from (2.17) we have

$$\begin{aligned} 2p_{-1}C_{ijk} &= \Pi_{(ijk)}(h_{ij}N_k) + \Pi_{(ijk)}(h_{ij}P_k), \\ 2p_{-1}C_{ijk} &= \Pi_{(ijk)}\{h_{ij}(N_k + P_k)\}, \end{aligned} \quad (3.8)$$

which gives (3.5).

Theorem 3.3. *A Finsler space with a cubic changed Matsumoto metric is C-reducible Finsler space if and only if (3.7) holds.*

Definition 3.4. *A Finsler space is said to be a C2-like Finsler space if its Cartan torsion tensor C_{ijk} can be written as*

$$C_{ijk} = \frac{1}{C^2}C_iC_jC_k \quad (3.9)$$

where, $C^2 = C_iC^i$ and C_i is the mean Cartan tensor.

Now, comparing equation (3.4) and (3.9), we can conclude that:

Theorem 3.4. *A Finsler space equipped with the cubic changed Matsumoto metric will exhibit C2-like Finsler space if and only if the tensor U_{ijk} in equation (3.4) and scalar r vanishes identically and $t \rightarrow 1$.*

4. CONCLUSION

In the present paper, we investigated the Finsler space with the cubic changed Matsumoto metric and derived its fundamental tensors: the supporting line element, angular metric tensor, fundamental metric tensor and its reciprocal, Cartan torsion tensor and mean Cartan tensor in Propositions 2.1, 2.5, 2.6 and 2.7 respectively. Furthermore, we identified conditions under which the Finsler space with a cubic changed infinite series metric can be classified as quasi C-reducible, semi C-reducible, C-reducible, and C2-like Finsler spaces in Theorems 3.2, 3.4, 3.6 and 3.8 respectively. In future research, we plan to extend our investigation to explore the geometric properties associated with various curvature characteristics of the Finsler spaces with a cubic changed Matsumoto metric.

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