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# ON ORDERS OF APPROXIMATION FUNCTIONS OF GENERALIZED MIXED SMOOTHNESS IN LORENTZ SPACES

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ABSTRACT. We consider the Lorentz space with mixed norm of periodic functions of many variables and Nikol'skii-Besov type classes of the generalized mixed smoothness. We have obtained estimates of the best approximation by trigonometric polynomials with the harmonics from the hyperbolic crosses of functions from Nikol'skii-Besov's type classes of the generalized mixed smoothness in the Lorentz space with the mixed norm.

## 1. Introduction

Let $\overline{x} = (x_1, ..., x_m) \in I^m = [0, 2\pi]^m$ and let $\theta_j, p_j \in [1, \infty)$, $j = 1, ..., m$, $\mathbf{N}$ be the set of natural numbers.

We shall denote by $L_{\overline{p},\overline{\theta}}(I^m)$ the Lorentz spaces with mixed norm of Lebesgue measurable functions $f(\overline{x})$ defined on $\mathbf{R}^m$ with of period $2\pi$ for each variable such that $\|f\|_{\overline{p},\overline{\theta}} = \|...\|f\|_{p_1,\theta_1}...\|_{p_m,\theta_m} < +\infty$, where

$$\|g\|_{p,\theta} = \left\{ \int\limits_0^{2\pi} (g^*(t))^\theta t^{\frac{\theta}{p}-1} dt \right\}^{\frac{1}{\theta}},$$

where $g^*$ is a non-increasing rearrangement of the function $|g|$ (see [13]).

As we know, that in case when $p_j = \theta_j$, $j = 1, ..., m$, the space $L_{\overline{p},\overline{\theta}}(I^m)$ coincides with the Lebesgue space $L_{\overline{p}}(I^m)$ with mixed norm (for the definition see [23], p. 128):

$$\|f\|_{\bar{p}} = \left[\int_0^{2\pi} \left[\cdots \left[\int_0^{2\pi} |f(\bar{x})|^{p_1} dx_1\right]^{\frac{p_2}{p_1}} \cdots\right]^{\frac{p_m}{p_{m-1}}} dx_m\right]^{\frac{1}{p_m}}.$$

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We will consider a set of functions $f \in L_{\overline{q},\overline{\theta}}(I^m)$ such that

$$\int\limits_0^{2\pi} f(\overline{x})dx_j = 0, \quad j = 1, ..., m,$$

and let $a_{\overline{n}}(f)$ be the Fourier coefficients of the function $f \in L_1(I^m)$ with respect to the multiple trigonometric system $\{e^{i\langle\overline{n},\overline{x}\rangle}\}_{\bar{n}\in\mathbf{Z}^m}$, where $\langle\bar{y},\bar{x}\rangle = \sum\limits_{j=1}^{m} y_j x_j$. Then, we set $\delta_{\overline{s}}(f,\overline{x}) = \sum\limits_{\overline{n}\in\rho(\overline{s})} a_{\overline{n}}(f)\, e^{i\langle\overline{n},\overline{x}\rangle}$,

$$\rho(\bar{s}) = \left\{\overline{k} = (k_1, ..., k_m) \in \mathbf{Z}^m : \quad 2^{s_j-1} \le |k_j| < 2^{s_j}, j = 1, ..., m\right\},$$

where $\bar{s} = (s_1, ..., s_m)$, $s_j \in \mathbf{Z}_+$. A function $\Omega(\bar{t}) = \Omega(t_1, ..., t_m)$ is a function of the type of mixed modulus smoothness of an order $l \in \mathbf{N}$ if it satisfies the following conditions:

1) $\Omega(\bar{t}) > 0$, $t_j > 0$, $j = 1, ..., m$, $\Omega(\bar{t}) = 0$, if $\prod_{j=1}^m t_j = 0$;
2) $\Omega(\bar{t})$ increases in each variable;
3) $\Omega(k_1t_1, ..., k_mt_m) \le \left(\prod_{j=1}^m k_j\right)^l \Omega(t_1, ..., t_m)$, $k_j \in \mathbf{N}$, $j = 1, ..., m$;
4) $\Omega(\bar{t})$ is continuous for $t_j > 0$, $j = 1, ..., m$.

Let us consider the following sets

$$\Gamma(\Omega, N) = \left\{\bar{s} = (s_1, ..., s_m) \in \mathbf{Z}_+^m : \Omega(2^{-s_1}, ..., 2^{-s_m}) \ge \frac{1}{N}\right\},$$

$$Q(\Omega, N) = \cup_{\bar{s}\in\Gamma(\Omega,N)}\rho(\bar{s}),$$

$$\Gamma^{\perp}(\Omega, N) = \mathbf{Z}_+^m \setminus \Gamma(\Omega, N), \tag{1.1}$$

$$\Lambda(\Omega, N) = \Gamma^{\perp}(\Omega, N) \setminus \Gamma^{\perp}(\Omega, 2^l N). \tag{1.2}$$

It follows from (1.1) and (1.2) that $\Lambda(\Omega, N) \subset \Gamma^{\perp}(\Omega, N)$ and

$$\frac{1}{2^l N} \le \Omega(2^{-\bar{s}}) < \frac{1}{N} \tag{1.3}$$

for $\bar{s} \in \Lambda(\Omega, N)$. In [25], N.N. Pustovoitov proved that $\Lambda(\Omega, N) \neq \emptyset$ and

$$|\Lambda(\Omega, N)| \asymp (\log_2 N)^{m-1}, \tag{1.4}$$

where $|F|$ is the number of elements of the set $F$.

We will use the notation $S_{Q(\Omega,N)}(f, \bar{x}) = \sum_{\bar{k}\in Q(\Omega,N)} a_{\bar{k}}(f)e^{i\langle\bar{k},\bar{x}\rangle}$ for a partial sum of the Fourier series of a function $f$. $E_{Q(\Omega,N)}(f)_{\overline{p},\overline{\theta}}$ is the best approximation of function $L_{\overline{p},\overline{\theta}}(I^m)$ by trigonometric polynomials with the harmonics from hyperbolic cross $Q(\Omega, N)$.

The idea of using of trigonometric polynomials with the harmonics from generalized step hyperbolic crosses similar to $Q(\Omega, N)$ belongs to A.S. Romanyuk [28].

For a sequence of numbers we write $\{a_{\overline{n}}\}_{\overline{n}\in\mathbf{Z}^m} \in l_{\overline{p}}$ if

$$\left\|\left\{a_{\overline{n}}\right\}_{\bar{n}\in\mathbf{Z}^m}\right\|_{l_{\overline{p}}} = \left\{\sum_{n_m=-\infty}^{\infty}\left[...\left[\sum_{n_1=-\infty}^{\infty}\left|a_{\overline{n}}\right|^{p_1}\right]^{\frac{p_2}{p_1}}...\right]^{\frac{p_m}{p_{m-1}}}\right\}^{\frac{1}{p_m}} < +\infty,$$

where $\overline{p} = (p_1, ..., p_m)$, $1 \le p_j < +\infty$, $j = 1, 2, ..., m$.

If $p_j = \infty$, $j = 1, ..., m$, then $\left\|\{a_{\bar{n}}\}\right\|_{l_\infty} = \sup\limits_{\bar{n}\in\mathbf{Z}^m} |a_{\bar{n}}|$.

The notation $A(y) \asymp B(y)$ means that there exists positive constants $C_1$, $C_2$ such that $C_1 A(y) \le B(y) \le C_2 A(y)$. If $B \le C_2 A$ or $A \ge C_1 B$, then we write $B << A$ or $A >> B$.

For a given function of the type of mixed modulus smoothness $\Omega(\bar{t})$ consider Nikol'skii – Besov type classes of the generalized mixed smoothness

$$S^{\Omega}_{\overline{p},\overline{\theta},\overline{\tau}}B = \left\{f \in L_{\overline{p},\overline{\theta}}(I^m) : \left\|\left\{\Omega^{-1}(2^{-\bar{s}})\left\|\delta_{\overline{s}}(f)\right\|_{\overline{p},\overline{\theta}}\right\}_{\bar{s}\in\mathbf{Z}^m_+}\right\|_{l_{\overline{\tau}}} \le 1\right\},$$

where $\overline{p} = (p_1, ..., p_m)$, $\overline{\theta} = (\theta_1, ..., \theta_m)$, $\overline{\tau} = (\tau_1, ..., \tau_m)$, $1 < p_j < +\infty$, $1 < \theta_j < \infty$, $1 \le \tau_j \le +\infty$, $j = 1, ..., m$, and $\Omega(2^{-\bar{s}}) = \Omega(2^{-s_1}, ..., 2^{-s_m})$.

If $\Omega(\bar{t}) = \prod_{j=1}^m t_j^{r_j}$, $r_j > 0$, $j = 1, ..., m$, then this class is denoted by $S^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}}B$.

In the case $p_j = \theta_j = p$ and $\Omega(\bar{t}) = \prod_{j=1}^m t_j^{r_j}, r_j < l, \tau_j = +\infty, j = 1, ..., m$, $S^{\Omega}_{\overline{p},\overline{\theta},\overline{\tau}}B$ was defined by S.M. Nikol'skii [22], and for $1 \le \tau_j < +\infty, j = 1, ..., m$, by T.I. Amanov [6] and P.I. Lizorkin, S.M. Nikol'skii [20].

As pointed out in [?], [17] one of the difficulties in the theory of approximation of functions of several variables of mixed smoothness is the choice of harmonics of the approximating polynomials. The first author, who suggested to approximate functions of several variables of mixed smoothness by polynomials with harmonics in hyperbolic crosses, was K.I. Babenko [7]. After that approximations of various classes of smooth functions by this method were considered by S.A. Telyakovskii [38], B.S. Mityagin [21], Ya. S. Bugrov [14], N.S. Nikol'skaya [24], E.M. Galeev [18], [19], V.N. Temlyakov [39], [40], Dinh Dung [16], A.S. Romanyuk [27], [28], [29], R.A. DeVore, S.V. Konyagin and V.N. Temlyakov [15], H. - J. Schmeisser and W. Sickel [33], W. Sickel and T. Ullrich [31].

For Nikol'skii – Besov type classes of the generalized mixed smoothness this problem was considered by N.N. Pustovoitov [25], [26], Sun Yongsheng and Wang Heping [37], M. Sikhov [32], D.B. Bazakhanov [10], S.A. Stasyuk [34], [35], S.A. Stasyuk and S.Ya. Yanchenko [36], Sh.A. Balgimbaeva and T.I. Smirnov [8].

Exact orders of the approximation of the Nikol'skii–Besov classes in the metric of the Lorentz space were found by the author [1], [2] and K.A. Bekmaganbetov [11], [12].

An order of approximation of the class $S^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}}B$ by partial Fourier sums $S_n^{\bar{\gamma}}(f,\bar{x}) = \sum_{\langle \bar{s},\bar{\gamma}\rangle<n} \delta_{\overline{s}}(f,\bar{x})$ was found in [1].

The exact estimate of the quantity $\sup\limits_{f\in S^{\Omega}_{\overline{p},\bar{\theta}^{(1)}\overline{\tau}}B} \|f - S_{Q(\Omega,N)}(f)\|_{\overline{q},\overline{\theta}^{(2)}}$ was proved in [4].

The main aim of the present paper is to estimate the order of the quantity

$$E_{Q(\Omega,N)}(S^{\Omega}_{\overline{p},\overline{\theta},\overline{\tau}}B)_{\overline{q},\overline{\theta}} = \sup_{f\in S^{\Omega}_{\overline{p},\overline{\theta},\overline{\tau}}B} E_{Q(\Omega,N)}(f)_{\overline{q},\overline{\theta}}.$$

This paper is organized as follows. In the second section some auxiliary results are given. The third section establishes the estimate of the order approximation of the Nikol'skii–Besov classes in the Lorentz space with mixed norm.

## 2. Auxiliary results

In what follows, we denote by $\chi_{\kappa(n)}(\bar{s})$ the characteristic function of the set $\kappa(n) = \{\bar{s} = (s_1,...,s_m) \in \mathbf{Z}_+^m : \quad \langle \bar{s},\bar{\gamma}\rangle = n\}$, $n \in \mathbf{N}$, where $\bar{\gamma} = (\gamma_1,...,\gamma_m)$, $\gamma_j \in (0,\infty)$, $j = 1,...,m$.

**Lemma 2.1.** *Let $\bar{\tau} = (\tau_1,...,\tau_m)$, $1 \le \tau_j < +\infty$, $j = 1,...,m$. Then the following relation holds:*

$$\left\| \left\{ \chi_{\kappa(n)}(\bar{s}) \right\}_{\bar{s}\in\kappa(n)} \right\|_{l_{\bar{\tau}}} \asymp n^{\sum\limits_{j=2}^{m} \frac{1}{\tau_j}}, \ \ n \in \mathbf{N}.$$

Lemma 2.1, is proved in [2].

Let us recall definitions of the conditions $(S)$, $(S_l)$ given by S.B.Stechkin and N.K. Bary [9].

**Definition 2.2.** A function $g(t)$ satisfies the condition $(S)$, if for some $\alpha \in (0,1)$ the function $t^{-\alpha}g(t)$ almost increases on $(0,1]$.

We say that a function $\Omega(\bar{t})$ satisfies the condition $(S)$ on $(0,1]^m$, if it satisfies this condition on each variable.

**Definition 2.3.** A function $g(t)$ satisfies the condition $(S_l)$, if for some $\alpha \in (0,l)$ the function $t^{-\alpha}g(t)$ almost decreases on $(0,1]$.

We say that a function $\Omega(\bar{t})$ satisfies the condition $(S_l)$ on $(0,1]^m$, if it satisfies this condition on each variable.

**Lemma 2.4.** ([4]) *Let $1 \le \theta_j < +\infty$, $j = 1,...,m$, and $\Omega(\bar{t})$ be a function of the type of mixed modulus smoothness of an order $l$ which satisfies the $(S)$-condition for $\bar{\alpha} = (\alpha_1,...,\alpha_m)$, $\alpha_j > \beta_j \ge 0$, $j = 1,...,m$. Then for $1 \le \theta_j < +\infty$, $j = 1,...,m$, the following relation holds*

$$\left\|\left\{\Omega(2^{-\bar{s}})\prod_{j=1}^{m}2^{s_j\beta_j}\right\}_{\bar{s}\in\Gamma^{\perp}(\Omega,N)}\right\|_{l_{\bar{\theta}}}\asymp\left\|\left\{\Omega(2^{-\bar{s}})\prod_{j=1}^{m}2^{s_j\beta_j}\right\}_{\bar{s}\in\Lambda(N)}\right\|_{l_{\bar{\theta}}}.$$

**Lemma 2.5.** ([4]) *Let $\Omega(\bar{t})$ be a function of the type of mixed modulus smoothness of an order $l$, which satisfies the conditions $(S)$ and $(S_l)$, $1\leq\tau_j<+\infty, j=1,...,m,$ and $\Lambda(\Omega,N)=\Gamma^{\perp}(\Omega,N)\setminus\Gamma^{\perp}(\Omega,2^lN)$. Then*

$$\left\|\left\{\chi_{\Lambda(\Omega,N)}(\bar{s})\right\}_{\bar{s}\in\Lambda(\Omega,N)}\right\|_{l_{\bar{\tau}}}\asymp(\log_2(N+1))^{\sum\limits_{j=2}^{m}\frac{1}{\tau_j}},\ N\in\mathbf{N}.$$

*Remark* 2.6. Note that for the case $\tau_1=...=\tau_m=1$ Lemma 2.5 was proved by N.N. Pustovoitov [25].

**Theorem 2.7.** *Let $\bar{q}=(q_1,...,q_m)$, $1<q_j<\infty$, $j=1,...,m$, $\beta=min\{q_1,...,q_m,2\}$. Then, for any function $f\in L_{\bar{q}}(I^m)$, the following inequality holds*

$$\|f\|_{\bar{q}}<<\left\{\sum_{\bar{s}\in\mathbf{Z}_+^m}\|\delta_{\bar{s}}(f)\|_{\bar{q}}^{\beta}\right\}^{\frac{1}{\beta}}.$$

The proof of the Theorem is given in [3].

**Theorem 2.8.** ([1]) *Let $\bar{p}=(p_1,...,p_m)$, $\bar{q}=(q_1,...,q_m)$, $\bar{\theta}^{(1)}=(\theta_1^{(1)},...,\theta_m^{(1)})$, $\bar{\theta}^{(2)}=(\theta_1^{(2)},...,\theta_m^{(2)})$. Assume that $1\leq p_j<q_j<+\infty$, $1\leq\theta_j^{(1)},\theta_j^{(2)}<+\infty$, $j=1,...,m$. If $f\in L_{\bar{p},\bar{\theta}^{(1)}}(I^m)$, $\max_{j=1,...,m-1}\theta_j^{(2)}<\min_{j=2,...,m}q_j$ and the quantity*

$$\sigma(f)\equiv\left\|\left\{\prod_{j=1}^{m}2^{s_j(\frac{1}{p_j}-\frac{1}{q_j})}\|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{\theta}^{(1)}}\right\}_{\bar{s}\in\mathbb{Z}_+^m)}\right\|_{l_{\bar{\theta}^{(2)}}}$$

*is finite, then $f\in\tilde{L}_{\bar{q},\bar{\theta}^{(2)}}(I^m)$ and $\|f\|_{\bar{q},\bar{\theta}^{(2)}}<<\sigma(f)$.*

**Theorem 2.9.** ([1]) *Let $\bar{q}=(q_1,...,q_m)$, $\bar{\theta}=(\theta_1,...,\theta_m)$, $\bar{\lambda}=(\lambda_1,...,\lambda_m)$. Assume that $1<q_j<\tau_j<+\infty$, $1<\theta_j<+\infty$, $j=1,...,m$. If $f\in L_{\bar{q},\bar{\theta}}(I^m)$ and*

$$f(\bar{x})\sim\sum_{\bar{s}\in\mathbf{Z}_+^m}b_{\bar{s}}\sum_{\bar{k}\in\rho(\bar{s})}e^{i\langle\bar{k},\bar{x}\rangle},$$

*then*

$$\|f\|_{\bar{q},\bar{\theta}}>>\left\|\left\{\prod_{j=1}^{m}2^{s_j(\frac{1}{\lambda_j}-\frac{1}{q_j})}\|\delta_{\bar{s}}(f)\|_{\bar{\lambda},\bar{\theta}}\right\}_{\bar{s}\in\mathbb{Z}_+^m)}\right\|_{l_{\bar{\theta}}}.$$

## 3. Main results

Let us prove the main results of the present paper. Consider the function $\Omega_1(\bar{t})=\Omega(\bar{t})\prod\limits_{j=1}^{m}t_j^{-(\frac{1}{p_j}-\frac{1}{q_j})}$, where $t_j\in(0,1], j=1,...,m$ and respectively set $Q(\Omega_1,N),\Gamma^{\perp}(\Omega_1,N),\Lambda(\Omega_1,N)$.

**Theorem 3.1.** *Let* $1 \le \theta_j^{(1)}, \theta_j^{(2)}, \tau_j < +\infty$, $1 < p_j < q_j < \infty$, $j = 1, ..., m$, *and* $\Omega(\bar{t})$ *be a function of the type of mixed modulus smoothness of an order* $l$, *which satisfies the conditions* $(S)$ *and* $(S_l)$, $\alpha_j > \frac{1}{p_j} - \frac{1}{q_j}$, $j = 1, ..., m$ *i* $\Omega_1(\bar{t}) = \Omega(\bar{t}) \prod_{j=1}^{m} t_j^{-(\frac{1}{p_j} - \frac{1}{q_j})}$.

*1) If* $1 \le \theta_j^{(2)} < \tau_j < +\infty$, $j = 1, ..., m$, *then*

$$E_{Q(\Omega_1,N)}(S^{\Omega}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B)_{\overline{q},\overline{\theta}^{(2)}} \asymp \frac{1}{N}(\log_2 N)^{\sum_{j=2}^{m}(\frac{1}{\theta_j^{(2)}} - \frac{1}{\tau_j})}$$

*for* $N = 2, 3, ....$

*2) If* $\tau_j \le \theta_j^{(2)}, j = 1, ..., m$, *then* $E_{Q(\Omega_1,N)}(S^{\Omega}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B)_{\overline{q},\overline{\theta}^{(2)}} \asymp \frac{1}{N}$, $N \in \mathbf{N}$.

*Proof.* Let $f \in S^{\Omega}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B$. For the proof of the first part it remains to apply Holder's inequality with exponents $\beta_j = \frac{\tau_j}{\theta_j^{(2)}}$, $\frac{1}{\beta_j} + \frac{1}{\beta_j'} = 1$, $j = 1, ..., m$ taking into account $\Omega(\bar{t})$ satisfies $(S)$ - condition for $\alpha_j > \frac{1}{p_j} - \frac{1}{q_j}$, $j = 1, ..., m$ we obtain

$$\Big\|\Big\{\prod_{j=1}^{m} 2^{s_j(\frac{1}{p_j} - \frac{1}{q_j})}\|\delta_{\bar{s}}(f)\Big\|_{\overline{p},\bar{\theta}^{(1)}}\Big\}_{\bar{s}\in\mathbf{Z}_+^m}\Big\|_{l_{\bar{\theta}^{(2)}}} \le$$

$$\le \Big\|\Big\{\Omega^{-1}(2^{-\bar{s}})\Big\|\delta_{\bar{s}}(f)\Big\|_{\overline{p},\bar{\theta}^{(1)}}\Big\}_{\bar{s}\in\mathbf{Z}_+^m}\Big\|_{l_{\bar{\tau}}}\Big\|\Big\{\prod_{j=1}^{m} 2^{s_j(\frac{1}{p_j} - \frac{1}{q_j})}\Omega(2^{-\bar{s}})\Big\}_{\bar{s}\in\mathbf{Z}_+^m}\Big\|_{l_{\bar{\varepsilon}}}$$

is finite, where $\bar{\varepsilon} = (\varepsilon_1, ..., \varepsilon_m)$, $\varepsilon_j = \tau_j\beta_j'$, $j = 1, ..., m$. Therefore, by Theorem 2.8 we have $S^{\Omega}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B \subset L_{\bar{q},\bar{\theta}^{(2)}}(I^m)$. Taking into account $\delta_{\bar{s}}(f - S_{Q(\Omega_1,N)}(f)) = 0$ , if $\bar{s} \in Q(\Omega_1, N)$ and $\delta_{\bar{s}}(f - S_{Q(\Omega_1,N)}(f)) = \delta_{\bar{s}}(f)$ , if $\bar{s} \notin Q(\Omega_1, N)$ by Theorem 2.8, we have

$$E_{Q(\Omega_1,N)}(f)_{\overline{q},\overline{\theta}^{(2)}} <<$$

$$<< \Big\|\Big\{\prod_{j=1}^{m} 2^{s_j\left(\frac{1}{p_j} - \frac{1}{q_j}\right)}\Big\|\delta_{\bar{s}}(f - S_{Q(\Omega_1,N)}(f))\Big\|_{\overline{p},\bar{\theta}^{(1)}}\Big\}_{\bar{s}\in\Gamma^{\perp}(\Omega_1,N)}\Big\|_{l_{\bar{\theta}^{(2)}}}.$$

for any function $f \in S^{\Omega}_{\overline{p},\bar{\theta}^{(1)},\overline{\tau}}B$. Since $\beta_j = \frac{\tau_j}{\theta_j^{(2)}} > 1, j = 1, ..., m$, and by applying the Holder inequality we obtain the following

$$E_{Q(\Omega_1,N)}(f)_{\overline{q},\overline{\theta}^{(2)}} << \Big\|\Big\{\Omega^{-1}(2^{-\bar{s}})\,\|\delta_{\bar{s}}(f)\|_{\overline{p},\bar{\theta}^{(1)}}\Big\}_{\bar{s}\in\mathbf{Z}_+^m}\Big\|_{l_{\bar{\tau}}} \times \Big\|\Big\{\Omega(2^{-\bar{s}})\prod_{j=1}^{m} 2^{s_j(\frac{1}{p_j} - \frac{1}{q_j})}\Big\}_{\bar{s}\in\Gamma^{\perp}(\Omega_1,N)}\Big\|_{l_{\bar{\epsilon}}}, \tag{3.1}$$

where $\bar{\epsilon} = (\epsilon_1, ..., \epsilon_m)$, $\epsilon_j = \frac{\tau_j\theta_j^{(2)}}{\tau_j - \theta_j^{(2)}}$, $j = 1, ..., m$.

Since, by the assumption of the Theorem, the function $\Omega(\bar{t})$ satisfies $S$ and $S_l$ conditions and $\alpha_j > \frac{1}{p_j} - \frac{1}{q_j}$, $j = 1, ..., m$, then the function $\Omega_1(\bar{t})$ satisfies the conditions $S$ and $S_l$. Therefore, by Lemma 2.4, Lemma 2.5 and the definition of the set $\Gamma^{\perp}(\Omega_1, N)$ in (3.1), we have

$$E_{Q(\Omega_1,N)}(S^{\Omega}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B)_{\overline{q},\overline{\theta}^{(2)}} << \left\|\left\{\Omega_1(2^{\bar{s}})\right\}_{\bar{s}\in\Lambda(\Omega_1,N)}\right\|_{l_{\bar{\epsilon}}} <<$$

$$<< \frac{1}{N}\left\|\left\{\chi_{\Lambda(\Omega_1,N)}(\bar{s})\right\}_{\bar{s}\in\Lambda(\Omega_1,N)}\right\|_{l_{\bar{\epsilon}}} << \frac{1}{N}(\log_2 N)^{\sum\limits_{j=2}^{m}\left(\frac{1}{\theta_j^{(2)}}-\frac{1}{\tau_j}\right)}.$$

In item 1) of the Theorem, the upper bound has been proved.

Let us prove the lower bound. Consider the function

$$f_0(\bar{x}) = (\log_2 N)^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}} \sum_{\bar{s}\in\Lambda(\Omega_1,N)} \prod_{j=1}^{m}\Omega(2^{-\bar{s}})2^{-s_j(1-\frac{1}{p_j})} \sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle}.$$

In one-dimensional case for the Dirichlet kernel $D_n(x) = \frac{1}{2} + \sum\limits_{k=1}^{n} e^{ikx}$ the following statement holds $\|D_n\|_{p,\theta} \asymp n^{1-\frac{1}{p}}$, $1 < p < +\infty$, $1 < \theta < +\infty$. Then, by the property of the norm, we have

$$\left\|\sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle}\right\|_{\overline{p},\bar{\theta}^{(1)}} = \prod_{j=1}^{m}\left\|\sum_{k_j=2^{s_j-1}}^{2^{s_j}-1} e^{ik_jx_j}\right\|_{p_j,\theta_j^{(1)}} << \prod_{j=1}^{m} 2^{s_j(1-\frac{1}{p_j})}.$$

for $1 < p_j < +\infty$, $1 < \theta_j^{(1)} < +\infty$, $j = 1, ..., m$.

Let us prove the rest of the equality. By Lemma B in [1], the following inequality holds

$$\max_{\bar{x}\in I^m}|\sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle}| << \prod_{j=1}^{m} 2^{s_j\left(1-\frac{1}{p_j}\right)}\left\|\sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle}\right\|_{\overline{p},\bar{\theta}^{(1)}}. \tag{3.2}$$

It is known that $\max_{\bar{x}\in I^m}|\sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle}| \geq 2^{-m}\prod_{j=1}^{m} 2^{s_j}$. Therefore, it follows from (3.2) that $\prod_{j=1}^{m} 2^{s_j(1-\frac{1}{p_j})} << \left\|\sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle}\right\|_{\overline{p},\bar{\theta}^{(1)}}$. Thus, we have proved the relation

$$\left\|\sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle}\right\|_{\overline{p},\bar{\theta}^{(1)}} \asymp \prod_{j=1}^{m} 2^{s_j(1-\frac{1}{p_j})}. \tag{3.3}$$

Therefore, by Lemma 2.5 and by estimation (3.3), we have

$$\left\|\left\{\Omega^{-1}(2^{-\bar{s}})\left\|\delta_{\bar{s}}(f_0)\right\|_{\overline{p},\bar{\theta}^{(1)}}\right\}_{\bar{s}\in\mathbf{Z}_+^m}\right\|_{l_{\bar{\tau}}} \asymp$$

$$\asymp (\log_2 N)^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}}\left\|\left\{\chi_{\Lambda(\Omega_1,N)}(\bar{s})\right\}_{\bar{s}\in\Lambda(\Omega_1,N)}\right\|_{l_{\bar{\tau}}} \leq C_0.$$

Hence $C_0^{-1}f_0 \in S^{\Omega}_{\overline{p},\overline{\tau}}B$. Now taking into account that $S^{\bar{\gamma}}_{Q(\Omega_1,N)}(f_0,\bar{x}) = 0, \bar{x} \in I^m$ and using Theorem 2.9 and (3.3), Lemma 2.5, we obtain

$$E_{Q(\Omega_1,N)}(f_0)_{\overline{q},\overline{\theta}^{(2)}} = \|f_0\|_{\overline{q},\overline{\theta}^{(2)}} >>$$

$$>> \left\| \left\{ \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{\lambda_j}-\frac{1}{q_j}\right)} \|\delta_{\bar{s}}(f_0)\|_{\overline{\lambda},\bar{\theta}^{(1)}} \right\}_{\bar{s}\in\mathbf{Z}^m_+} \right\|_{l_{\bar{\theta}^{(2)}}} >>$$

$$>> (\log_2 N)^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}} \left\| \left\{ \Omega_1(2^{-\bar{s}}) \right\}_{\bar{s}\in\Lambda(\Omega_1,N)} \right\|_{l_{\bar{\theta}^{(2)}}} >>$$

$$>> \frac{1}{N}(\log_2 N)^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}} \left\| \left\{ \chi_{\Lambda(\Omega_1,N)} \right\}_{\bar{s}\in\Lambda(\Omega_1,N)} \right\|_{l_{\bar{\theta}^{(2)}}} >>$$

$$>> \frac{1}{N}(\log_2 N)^{\sum\limits_{j=2}^{m}(\frac{1}{\theta_j^{(2)}}-\frac{1}{\tau_j})}.$$

Thus, $E_{Q(\Omega_1,N)}(S^{\Omega}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B)_{\overline{q},\overline{\theta}^{(2)}} >> \frac{1}{N}(\log_2 N)^{\sum\limits_{j=2}^{m}(\frac{1}{\theta_j^{(2)}}-\frac{1}{\tau_j})}$. Item 1) of the theorem has been proved.

Let us prove item 2) of the theorem. Since $\tau_j \leq \theta_j^{(2)}$, $j = 1, ..., m$, then by applying Theorem 2.8 and the Jensen inequality (see [23], p. 125), we obtain $S^{\Omega}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B \subset \tilde{L}_{\bar{q},\bar{\theta}^{(2)}}(I^m)$ and

$$\|f - S_{Q(\Omega_1,N)}(f)\|_{\overline{q},\overline{\theta}^{(2)}} << \left\| \left\{ \prod_{j=1}^{m} 2^{s_j(\frac{1}{p_j}-\frac{1}{q_j})} \left\|\delta_{\bar{s}}(f)\right\|_{\overline{p},\overline{\theta}^{(1)}} \right\}_{\bar{s}\in\Gamma^{\perp}(\Omega_1,N)} \right\|_{l_{\bar{\tau}}} <<$$

$$\left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) \left\|\delta_{\bar{s}}(f)\right\|_{\overline{p},\overline{\theta}^{(1)}} \right\}_{\bar{s}\in\mathbf{Z}^m_+} \right\|_{l_{\bar{\tau}}} \sup_{\bar{s}\in\Gamma^{\perp}(\Omega_1,N)} \Omega(2^{-\bar{s}}) \prod_{j=1}^{m} 2^{s_j(\frac{1}{p_j}-\frac{1}{q_j})} << \frac{1}{N}$$

for any function $f \in S^{\Omega}_{\overline{p},\bar{\theta}^{(1)},\overline{\tau}}B$, which proves the upper bound in item 2). For the lower bound, consider the function

$$f_1(\bar{x}) = \Omega(2^{-\bar{\tilde{s}}}) 2^{-\sum\limits_{j=1}^{m}\tilde{s}_j(1-\frac{1}{p_j})} \sum_{\bar{k}\in\rho(\bar{\tilde{s}})} e^{i\langle\bar{k},\bar{x}\rangle},$$

where $\bar{\tilde{s}} = (\tilde{s}_1, ..., \tilde{s}_m) \in \Lambda(\Omega_1, N)$. Then $f_1 \in S^{\Omega}_{\overline{p},\bar{\theta}^{(1)},\overline{\tau}}B$. Next, by (3.3), we have

$$E_{Q(\Omega_1,N)}(f_1)_{\overline{q},\overline{\theta}^{(2)}} = \|f_1\|_{\overline{q},\overline{\theta}^{(2)}} >>$$

$$>> \Omega(2^{-\bar{\tilde{s}}}) 2^{-\sum\limits_{j=1}^{m}\tilde{s}_j(1-\frac{1}{p_j})} \prod_{j=1}^{m} 2^{\tilde{s}_j(1-\frac{1}{q_j})} = C\Omega(2^{-\bar{\tilde{s}}}) \prod_{j=1}^{m} 2^{\tilde{s}_j(\frac{1}{p_j}-\frac{1}{q_j})}$$

for $\bar{\tilde{s}} \in \Lambda(\Omega_1, N)$. Hence , by (1.3) we obtain $E_{Q(\Omega_1,N)}(S^{\Omega}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B)_{\overline{q},\overline{\theta}^{(2)}} >> N^{-1}$. This proves the lower bound in item 2). $\square$

**Theorem 3.2.** *Let $\Omega(\bar{t})$ be a function of the type of mixed modulus smoothness of an order $l$ which satisfies the conditions $(S)$ and $(S_l)$, $1 < q_j < p_j < \infty$, $p_j \geq 2$, $1 < \theta_j < \infty$, $1 \leq \tau_j \leq +\infty$, $j = 1, ..., m$.*

*1) If $2 < \tau_j < +\infty$, $j = 1, ..., m$, then*

$$E_{Q(\Omega,N)}(S^{\Omega}_{\overline{p},\overline{\tau}}B)_{\overline{q},\overline{\theta}} \asymp \frac{1}{N} (\log_2 N)^{\sum\limits_{j=2}^{m}(\frac{1}{2}-\frac{1}{\tau_j})}.$$

*2) If $\tau_j \leq 2$, $j = 1, ..., m$, then $E_{Q(\Omega,N)}(S^{\Omega}_{\overline{p},\overline{\tau}}B)_{\overline{q},\overline{\theta}} \asymp N^{-1}$.*

*3) If $1 < q_j < p_j \leq 2$, $p_0 = \min\{p_1, ..., p_m\} < \tau_j, j = 1, ..., m$, then*

$$N^{-1}(\log_2 N)^{\sum\limits_{j=2}^{m}(\frac{1}{p_j}-\frac{1}{\tau_j})} << E_{Q(\Omega,N)}(S^{\Omega}_{\overline{p},\overline{\tau}}B)_{\overline{q},\overline{\theta}} << N^{-1}(\log_2 N)^{\sum\limits_{j=2}^{m}(\frac{1}{p_0}-\frac{1}{\tau_j})}.$$

*Proof.* Item 1) proved in [4]. Let us prove item 2). Since $q_j < p_j$, $j = 1, ..., m$, then $L_{\bar{p}}(I^m) \subset L_{\bar{q},\bar{\theta}}(I^m)$ and we have $\|f\|_{\overline{q},\overline{\theta}} << \|f\|_{\overline{p}}, \quad f \in L_{\bar{p}}(I^m)$. Therefore $S^{\Omega}_{\overline{p},\overline{\tau}}B \subset L_{\bar{q},\bar{\theta}}(I^m)$ and

$$\|f - S_{Q(\Omega,N)}(f)\|_{\overline{q},\overline{\theta}} << \left\| \sum_{\bar{s}\in\Gamma^{\perp}(\Omega,N)} \delta_{\bar{s}}(f) \right\|_{\overline{p}} \tag{3.4}$$

for any function $f \in S^{\Omega}_{\overline{p},\overline{\tau}}B$.

Now, since $2 \leq p_j < +\infty$, $j = 1, ..., m$, using Theorem 2.7 from (3.4) we obtain

$$\|f - S_{Q(\Omega,N)}(f)\|_{\overline{q},\overline{\theta}} << \left\{ \sum_{\bar{s}\in\Gamma^{\perp}(\Omega,N)} \left\|\delta_{\bar{s}}(f)\right\|^2_{\overline{p}} \right\}^{\frac{1}{2}} =$$

$$= C\left\{ \sum_{\bar{s}\in\Gamma^{\perp}(\Omega,N)} \Omega^2(2^{-\bar{s}})\left(\Omega^{-1}(2^{-\bar{s}})\left\|\delta_{\bar{s}}(f)\right\|_{\overline{p}}\right)^2 \right\}^{\frac{1}{2}} \tag{3.5}$$

for any function $f \in S^{\Omega}_{\overline{p},\overline{\tau}}B$.

If $\tau_j \leq 2$, $j = 1, ..., m$, then using the Jensen inequality (see [23], p. 125) we have

$$\left\{ \sum_{\bar{s}\in\Gamma^{\perp}(\Omega,N)} \left\|\delta_{\bar{s}}(f)\right\|^2_{\overline{p}} \right\}^{\frac{1}{2}} <<$$

$$<< \left\| \left\{\Omega^{-1}(2^{-\bar{s}})\left\|\delta_{\bar{s}}(f)\right\|_{\overline{p}}\right\}_{\bar{s}\in\mathbf{Z}^m_+} \right\|_{l_{\bar{\tau}}} \sup_{\bar{s}\in\Gamma^{\perp}(\Omega,N)} \Omega(2^{-\bar{s}}).$$

Therefore, from the inequality (3.5) we obtain $E_{Q(\Omega,N)}(S^{\Omega}_{\overline{p},\overline{\tau}}B)_{\overline{q},\overline{\theta}} << \frac{1}{N}$, in case $2 < p_j < +\infty$, $\tau_j \leq 2$, $j = 1, ..., m$. This proves the upper bound. The lower bound in item 2) proved in [4].

Let us prove item 3). Since $1 < p_j \leq 2$, $j = 1, ..., m$, using Theorem 2.7 from (3.4) we obtain

$$\|f - S_{Q(\Omega,N)}(f)\|_{\overline{q},\overline{\theta}} << \Big\{ \sum_{\bar{s}\in\Gamma^{\perp}(\Omega,N)} \|\delta_{\bar{s}}(f)\|_{\overline{p}}^{p_0} \Big\}^{\frac{1}{p_0}} =$$

$$= C\Big\{ \sum_{\bar{s}\in\Gamma^{\perp}(\Omega,N)} \Omega^{p_0}(2^{-\bar{s}})\Big(\Omega^{-1}(2^{-\bar{s}})\|\delta_{\bar{s}}(f)\|_{\overline{p}}\Big)^{p_0}\Big\}^{\frac{1}{p_0}} \quad (3.6)$$

for any function $f \in S^{\Omega}_{\overline{p},\overline{\tau}}B$. If $p_0 < \tau_j < +\infty$, $j = 1,...,m$, then by the Holder inequality from (3.6), we get

$$\|f - S_{Q(\Omega,N)}(f)\|_{\overline{q},\overline{\theta}} << \Big\|\Big\{\Omega^{-1}(2^{-\bar{s}})\|\delta_{\bar{s}}(f)\|_{\overline{p}}\Big\}_{\bar{s}\in\mathbf{Z}^m_+}\Big\|_{l_{\bar{\tau}}} \times \Big\|\Big\{\Omega(2^{-\bar{s}})\Big\}_{\bar{s}\in\Gamma^{\perp}(\Omega,N)}\Big\|_{l_{\bar{\epsilon}}}, \quad (3.7)$$

where $\bar{\epsilon} = (\epsilon_1, ..., \epsilon_m)$, $\epsilon_j = 2\beta'_j$, $\frac{1}{\beta_j} + \frac{1}{\beta'_j} = 1$, $\beta_j = \frac{\tau_j}{p_0}$, $j = 1,...,m$.

Now by Lemma 2.4 and Lemma 2.5 from (3.7) we obtain

$$\|f - S_{Q(\Omega,N)}(f)\|_{\overline{q},\overline{\theta}} << N^{-1}(\log_2 N)^{\sum\limits_{j=2}^{m}(\frac{1}{p_0}-\frac{1}{\tau_j})}$$

for any function $f \in S^{\Omega}_{\overline{p},\overline{\tau}}B$. This proves the upper bound.

Let us prove the lower bound. Consider the set similarly in [26]

$$\Lambda'(\Omega,N) = \Big\{\ \bar{s} \in \Lambda(\Omega,N) : s_j > \frac{1}{2ml}\log_2(C_3N),\ j = 1,...,m\Big\}.$$

N.N. Pustovoitov [25] has been proved that, number of point is equal to $|\Lambda'(\Omega,N)| \asymp (\log_2 N)^{m-1}$.

After this we choose set $\Lambda(\bar{\Omega},N)$. Lets take a number $v = [|\Lambda'(\Omega,N)|^{\frac{1}{m}}]$ - which is whole part of a number $|\Lambda'(\Omega,N)|^{\frac{1}{m}}$. Divide set $\mathbf{I}^m = [-\pi,\pi]^m$ to $v^m$ cubes with side equal to $\frac{2\pi}{v}$. Then choose set $\Lambda(\bar{\Omega},N) \subset \Lambda(\Omega,N)$, such that $|\Lambda(\bar{\Omega},N)| = v^m$, and define bijection between this set $\Lambda(\bar{\Omega},N)$ and the set of cubes .

Let for $\bar{s} \in \Lambda(\bar{\Omega},N)$ point $\bar{x}^{\bar{s}}$ denote the centre of the cube. Further we set the notation $u = \Big[2^{\frac{1}{m-1}\sum\limits_{j=2}^{m}(1-\frac{1}{p_j})(\sum\limits_{j=1}^{m}(1-\frac{1}{p_j}))^{-1}\log_2|\Lambda(\Omega,N)|}\Big]$. Also, consider the function $f_3(\bar{x}) = \frac{1}{N}(\log_2 N)^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}} u^{-\sum\limits_{j=1}^{m}(1-\frac{1}{p_j})}\Psi(\bar{x})$, where (see [25]) $K_u(\bar{x}) = 2^m\prod_{j=1}^{m}K_u(x_j)$,

$$\Psi(\bar{x}) = \sum_{\bar{s}\in\Lambda(\bar{\Omega},N)} e^{i\langle\bar{k}^{\bar{s}}}K_u(\bar{x}-\bar{x}^{\bar{s}}),\ \ \bar{k}^{\bar{s}} = (k_1^{\bar{s}},...,k_m^{\bar{s}}), k_j^{\bar{s}} = 2^{s_j} + 2^{s_j-1},$$

$j = 1,...,m$, $K_u(x_j)$ – is the Fejer kernel of order $u$ by variable $x_j, j = 1,...,m$. Note that,

$$u \asymp (\log_2 N)^{\sum_{j=2}^{m}(1-\frac{1}{p_j})(\sum_{j=1}^{m}(1-\frac{1}{p_j}))^{-1}}. \tag{3.8}$$

In [25] we proved that

$$E_{Q(\Omega,N)}(\Psi)_1 >> |\Lambda(\Omega,N)|. \tag{3.9}$$

Lets show that $C_3 f_3 \in S^{\Omega}_{\overline{p},\overline{\tau}}B$ for some constant $C_3 > 0$. Since for Fejer core with one variable we have got the estimation $\|K_u(y)\|_p \asymp u^{1-\frac{1}{p}}$, $1 \leq p \leq \infty$, then $\|K_u(\bar{x})\|_{\bar{p}} \asymp u^{\sum_{j=1}^{m}(1-\frac{1}{p_j})}$. Using this relation and $|\Lambda'(\Omega,N)| \asymp |\Lambda(\Omega,N)| \asymp (\log_2 N)^{m-1}$ we get

$$\left\|\left\{\Omega^{-1}(2^{-\bar{s}})\|\delta_{\bar{s}}(f_3)\|_{\overline{p}}\right\}_{\bar{s}\in\mathbf{Z}^m_+}\right\|_{l_{\bar{\tau}}} <<$$

$$<< N^{-1}(\log_2 N)^{-\sum_{j=2}^{m}\frac{1}{\tau_j}} u^{-\sum_{j=1}^{m}(1-\frac{1}{p_j})}\left\|\left\{\Omega^{-1}(2^{-\bar{s}})u^{\sum_{j=1}^{m}(1-\frac{1}{p_j})}\right\}_{\bar{s}\in\Lambda(\bar{\Omega},N)}\right\|_{l_{\bar{\tau}}} <<$$

$$<< (\log_2 N)^{-\sum_{j=2}^{m}\frac{1}{\tau_j}}\|\{1\}_{\bar{s}\in\Lambda(\bar{\Omega},N)}\|_{l_{\bar{\tau}}}.$$

Since by Lemma 2.5 the estimation $\|\{1\}_{\bar{s}\in\Lambda(\bar{\Omega},N)}\|_{l_{\bar{\tau}}} << (\log_2 N)^{\sum_{j=2}^{m}\frac{1}{\tau_j}}$ is true, then $\left\|\left\{\Omega^{-1}(2^{-\bar{s}})\|\delta_{\bar{s}}(f)\|_{\overline{p}}\right\}_{\bar{s}\in\mathbf{Z}^m_+}\right\|_{l_{\bar{\tau}}} << C$. Because $C_3 f_3 \in S^{\Omega}_{\overline{p},\overline{\tau}}B$. Since $L_{\bar{q},\bar{\theta}}(I^m) \subset L_1(I^m)$ and $\|f\|_1 << \|f\|_{\bar{q},\bar{\theta}}$, then

$$E_{Q(\Omega,N)}(f_3)_{\bar{q},\bar{\theta}} >> E_{Q(\Omega,N)}(f_3)_1 = \frac{C}{N}(\log_2 N)^{-\sum_{j=2}^{m}\frac{1}{\tau_j}} u^{-\sum_{j=1}^{m}(1-\frac{1}{p_j})} E_{Q(\Omega,N)}(\Psi)_1.$$

Therefore, by the estimates (3.9), (3.8) we have got

$$E_{Q(\Omega,N)}(f_3)_{\bar{q},\bar{\theta}} >> \frac{1}{N}(\log_2 N)^{-\sum_{j=2}^{m}\frac{1}{\tau_j}} u^{-\sum_{j=1}^{m}(1-\frac{1}{p_j})}|\Lambda(\Omega,N)|$$

$$>> C\frac{1}{N}(\log_2 N)^{\sum_{j=2}^{m}(\frac{1}{p_j}-\frac{1}{\tau_j})}.$$

The Theorem 3.2 is proved. □

Now consider the case $q_j = p_j$, $j = 1,...,m$ and $\Omega(\bar{t}) = \prod_{j=1}^{m} t_j^{r_j}$, $r_j > 0$, $t_j \in [0,1]$, $j = 1,...m$. Next $E_{Q(\Omega,N)}(f)_{\overline{p},\overline{\theta}} = E_N^{\bar{\gamma}}(f)_{\overline{p},\overline{\theta}}$, where $\bar{\gamma} = (\gamma_1,...,\gamma_m)$, $\gamma_j = \frac{r_j}{r_1}$, $j = 1,...,m$.

**Theorem 3.3.** *Let* $\bar{r} = (r_1,...,r_m)$, $0 < r_1 = ... = r_\nu < r_{\nu+1} \leq ... \leq r_m$ *and* $2 \leq p_j < \infty$, $1 < \theta_j < \infty$, $1 \leq \tau_j \leq +\infty$, $j = 1,...,m$. *If* $2 \leq p_j < \theta_j < \infty$, $2 \leq \tau_j < +\infty$, $j = 1,...,m$, *then*

$$E_N^{\bar{\gamma}}(S^{\Omega}_{\overline{p},\overline{\theta},\overline{\tau}}B)_{\overline{p}} << N^{-r_1}(\log_2 N)^{\sum_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{\theta_j}\right)}(\log_2 N)^{\sum_{j=2}^{m}\left(\frac{1}{2}-\frac{1}{\tau_j}\right)}$$

*and if* $p_1 = ... = p_m = p$, *then*

$$N^{-r_1} (\log_2 N)^{\sum\limits_{j=1}^{m}\left(\frac{1}{p}-\frac{1}{\theta_j}\right)} (\log_2 N)^{\sum\limits_{j=2}^{m}\left(\frac{1}{p}-\frac{1}{\tau_j}\right)} << E_N^{\bar\gamma}(S^{\Omega}_{\overline{p},\overline{\theta},\overline{\tau}}B)_{\overline{p}}.$$

*Proof.* Let $f \in S^{\bar r}_{\overline{p},\overline{\theta},\overline{\tau}}B$. Now, since $2 \leq p_j < +\infty$, $2 \leq \tau_j < +\infty$, $j = 1, ..., m$, using Theorem 2.7 and the inequality of different metric for trigonometric polynomials (see [5]), the Holder inequality we obtain

$$\|f\|_{\overline{p}} << \Big\{\sum_{\bar s\in \mathbf{Z}^m} \|\delta_{\bar s}(f)\|^2_{\overline{p}}\Big\}^{\frac12} << \Big\{\sum_{\bar s\in \mathbf{Z}^m} \|\delta_{\bar s}(f)\|^2_{\overline{p},\bar\theta} \prod_{j=1}^{m}(s_j+1)^{\frac{1}{p_j}-\frac{1}{\theta_j}}\Big\}^{\frac12}$$

$$<< \left\|\Big\{2^{\langle \bar s,\bar r\rangle}\|\delta_{\bar s}(f)\|_{\overline{p}}\Big\}_{\bar s\in\mathbf{Z}^m_+}\right\|_{l_{\bar\tau}} \left\|\Big\{2^{-\langle \bar s,\bar r\rangle}\prod_{j=1}^{m}(s_j+1)^{\frac{1}{p_j}-\frac{1}{\theta_j}}\Big\}_{\bar s\in\mathbf{Z}^m_+}\right\|_{l_{\bar\epsilon}}, \tag{3.10}$$

where $\bar\epsilon = (\epsilon_1, ..., \epsilon_m)$, $\epsilon_j = \frac{2\tau_j}{\tau_j-2}$, $j = 1, ..., m$. Taking into account that $r_j > 0$, $j = 1, ..., m$ we get

$$\left\|\Big\{2^{-\langle \bar s,\bar r\rangle}\prod_{j=1}^{m}(s_j+1)^{\frac{1}{p_j}-\frac{1}{\theta_j}}\Big\}_{\bar s\in\mathbf{Z}^m_+}\right\|_{l_{\bar\epsilon}} < \infty.$$

Hence, it follows from (3.10) that $S^{\bar r}_{\overline{p},\overline{\theta},\overline{\tau}}B \subset L_{\bar p}(I^m)$ and

$$\|f - S^{\bar\gamma}_N(f)\|_{\overline{p}} << \left\|\Big\{2^{\langle \bar s,\bar r\rangle}\|\delta_{\bar s}(f)\|_{\overline{p}}\Big\}_{\bar s\in\mathbf{Z}^m_+}\right\|_{l_{\bar\tau}} \times \left\|\Big\{2^{-\langle \bar s,\bar r\rangle}\prod_{j=1}^{m}(s_j+1)^{\frac{1}{p_j}-\frac{1}{\theta_j}}\Big\}_{\bar s\in\Gamma^{\perp}(N)}\right\|_{l_{\bar\epsilon}}, \tag{3.11}$$

where $\Gamma^{\perp}(N) = \{\bar s \in \mathbf{Z}^m_+ :\ \langle \bar s, \bar\gamma\rangle \geq \log_2 N^{\frac{1}{r_1}}\}$. Next applying inequality

$$I_N = \left\|\Big\{2^{-\langle \bar s,\bar\gamma\rangle\beta}\prod_{j=1}^{m}s_j^{d_j}\Big\}_{\bar s\in\Gamma^{\perp}(N)}\right\|_{l_{\bar\theta}} << 2^{-n\beta} n^{\sum_{j=1}^m d_j+\sum_{j=2}^m \frac{1}{\theta_j}}$$

for $\beta > 0$, $d_j \geq 0, j = 1, ..., m$, then

$$\left\|\Big\{2^{-\langle \bar s,\bar\gamma\rangle r_1}\prod_{j=1}^{m}(s_j+1)^{\frac{1}{p_j}-\frac{1}{\theta_j}}\Big\}_{\bar s\in\Gamma^{\perp}(N)}\right\|_{l_{\bar\epsilon}} <<$$

$$<< \frac{1}{N^{r_1}}(\log_2 N)^{\sum_{j=1}^m(\frac{1}{p_j}-\frac{1}{\theta_j})+\sum_{j=2}^m\frac{1}{\varepsilon_j}}.$$

Therefore from (3.11) we obtain

$$E^{\bar\gamma}_N(f)_{\overline{p}} \leq \|f - S^{\bar\gamma}_N(f)\|_{\overline{p}} << N^{-r_1}(\log_2 N)^{\sum_{j=1}^m(\frac{1}{p_j}-\frac{1}{\theta_j})+\sum_{j=2}^m(\frac12-\frac{1}{\tau_j})}$$

for any function $f \in S^{\bar r}_{\overline{p},\bar\theta,\overline{\tau}}B$. This proves the upper bound.

Let us prove the lower bound. Consider the function

$$f_4(\bar{x}) = (\log_2 N)^{-\sum\limits_{j=2}^{m} \frac{1}{\tau_j}} \times$$

$$\times \sum_{\langle \bar{s}, \bar{\gamma} \rangle = \log_2 N^{\frac{1}{r_1}}} \prod_{j=1}^{m} 2^{-s_j r_j} s_j^{-\frac{1}{\theta_j}} \sum_{\bar{k} \in \rho(\bar{s})} \prod_{j=1}^{m} (k_j - 2^{s_j - 1} + 1)^{\frac{1}{p} - 1} e^{i\langle \bar{k}, \bar{x} \rangle}.$$

Then $f_4 \in L_{\bar{p},\bar{\theta}}(I^m)$. In one-dimensional case the following statement holds (see [30])

$$\Big\| \sum_{k=2^{s-1}}^{2^s - 1} (k - 2^{s-1} + 1)^{\frac{1}{p} - 1} e^{ikx} \Big\|_{p,\theta} \asymp (s+1)^{\frac{1}{\theta}}, \ 1 < p, \theta < \infty,$$

for $s \in \mathbf{N}$. Therefore

$$\Big\| \sum_{\bar{k} \in \rho(\bar{s})} \prod_{j=1}^{m} (k_j - 2^{s_j - 1} + 1)^{\frac{1}{p} - 1} e^{i\langle \bar{k}, \bar{x} \rangle} \Big\|_{\bar{p},\bar{\theta}} \asymp \prod_{j=1}^{m} (s_j + 1)^{\frac{1}{\theta_j}} \tag{3.12}$$

for $1 < p_j < \infty$, $1 < \theta_j < \infty, j = 1, ..., m$ Now by relation (3.12) and by Lemma 1 [4] we get

$$\Big\| \Big\{ 2^{\langle \bar{s}, \bar{r} \rangle} \|\delta_{\bar{s}}(f_4)\|_{\bar{p},\bar{\theta}} \Big\}_{\bar{s} \in \mathbf{Z}_+^m} \Big\|_{l_{\bar{\tau}}} << (\log_2 N)^{-\sum\limits_{j=2}^{m} \frac{1}{\tau_j}} \Big\| \{1\}_{\bar{s} \in \kappa(N)} \Big\|_{l_{\bar{\tau}}} \leq C,$$

where $\kappa(N) = \{\bar{s} \in \mathbf{Z}_+^m : \ \langle \bar{s}, \bar{\gamma} \rangle = \log_2 N^{\frac{1}{r_1}}\}$. Hence the function $C_4 f_4 \in S^{\bar{r}}_{\bar{p},\bar{\theta},\bar{\tau}} B$, for some constant $C_4 > 0$.

Since $2 \leq p = p_1 = ... = p_m < \infty$, then by Littlewood-Paley theorem [23] we obtain

$$E_N^{\bar{\gamma}}(C_4 f_4)_p = C_4 \|f_4\|_p >> \Big\| \Big\{ \sum_{\bar{s} \in \kappa(N)} |\delta_{\bar{s}}(f_4)|^2 \Big\}^{\frac{1}{2}} \Big\|_p >> \Big\{ \sum_{\bar{s} \in \kappa(N)} \|\delta_{\bar{s}}(f_4)\|_p^p \Big\}^{\frac{1}{p}}.$$

By relation (3.12), for $\theta_j = p_j = p, j = 1, ..., m$ , it follows that

$$E_N^{\bar{\gamma}}(C_4 f_4)_p >> (\log_2 N)^{-\sum\limits_{j=2}^{m} \frac{1}{\tau_j}} \Big( \sum_{\bar{s} \in \kappa(N)} 2^{-\langle \bar{s}, \bar{r} \rangle p} \prod_{j=1}^{m} (s_j + 1)^{(\frac{1}{p} - \frac{1}{\theta_j})p} \Big)^{\frac{1}{p}} >>$$

$$>> N^{-r_1} (\log_2 N)^{\sum\limits_{j=1}^{m} (\frac{1}{p} - \frac{1}{\theta_j})} (\log_2 N)^{\sum\limits_{j=2}^{m} (\frac{1}{p} - \frac{1}{\tau_j})}$$

for function $C_4 f_4 \in S^{\bar{r}}_{p,\bar{\theta},\bar{\tau}} B$. So Theorem 3.3 has been proved. □

*Remark* 3.4. Note that for the case $q_j = \theta_j = q$, $p_j = p$, $\tau_j = \tau$, $j = 1, ..., m$, Theorem 3.2 was proved by S.A. Stasyuk [34]. For the case $p_j = \theta_j^{(1)} = p$, $q_j = \theta_j^{(2)} = q$, $\tau_j = \tau < +\infty$, $j = 1, ..., m$, Theorem 3.1 was proved by S.A. Stasyuk [35].

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