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MULTIVARIATE APPROXIMATION WITH RATES BY PERTURBED KANTOROVICH-SHILKRET NEURAL NETWORK OPERATORS

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ABSTRACT. This paper deals with the determination of the rate of convergence to the unit of Perturbed Kantorovich-Shilkret multivariate normalized neural network operators of one hidden layer. These are given through the multivariate modulus of continuity of the engaged multivariate function or its high order partial derivatives and that appears in the associated multivariate Jackson type inequalities. The activation function is very general and it can derive from any multivariate sigmoid or bell-shaped functions. The right hand sides of our Jackson type inequalities do not depend on the activation function. The sample functionals are Kantorovich-Shilkret type. We provide an application for the first partial derivatives of the involved function.

1. Introduction

Our work is motivated by [2], [3]. The basic ingredient of our perturbed multivariate neural network operators here is the Shilkret ([5]) integral.

Our operators fall into the category of feed-forward neural networks (FNNs) with one hidden layer, that are mathematicaly expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x\rangle + b_j\right), \quad x \in \mathbb{R}^s,\ s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x\rangle$ is the inner product of a_j and x, and σ is the activation function of the network.

In many fundamental network models, the activation function is the sigmoid function of logistic type or other sigmoid function or bell-shaped function. It is well known that FNN's are universal approximators. Theoretically any continuous function defined on a compact set can be approximated to

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any desired degree of accuracy by increasing the number of hidden neurons. Similar approximations are possible for measurable functions in the L_p norm. P. Cardaliaguet and G. Euvrad were the first ([4]) to describe precisely and study neural network approximation operators to the unit operator. The author in [1], [2], [3] was the first to put neural network approximation in a quantitative form and produce Jackson type inequalities with moduli of continuity.

The author here continues these studies by employing the non-additive Shilkret integral ([5]) and going very generally by using general activation functions of compact support under perturbation. Because Shilkret integral is so general and simple and non-additive, it has great applications to economic theory.

2. Background

Here we follow [5].

Let $\mathcal{F}$ be a σ-field of subsets of an arbitrary set Ω. An extended non-negative real valued function μ on $\mathcal{F}$ is called maxitive if $\mu(\varnothing) = 0$ and

$$\mu\left(\cup_{i \in I} E_i\right) = \sup_{i \in I} \mu\left(E_i\right),$$

where the set I is of cardinality at most countable. We also call μ a maxitive measure. Here f stands for a non-negative measurable function on Ω. In [5], Niel Shilkret developed his non-additive integral defined as follows:

$$\left(N^*\right) \int_D f d\mu := \sup_{y \in Y} \left\{ y \cdot \mu\left(D \cap \{f \geq y\}\right) \right\},$$

where $Y = [0, m]$ or $Y = [0, m)$ with $0 < m \leq \infty$, and $D \in \mathcal{F}$. Here we take $Y = [0, \infty)$.

It is easily proved that

$$\left(N^*\right) \int_D f d\mu = \sup_{y > 0} \left\{ y \cdot \mu\left(D \cap \{f > y\}\right) \right\}.$$

The Shilkret integral takes values in $[0, \infty]$.

The Shilkret integral ([5]) has the following properties:

$$\left(N^*\right) \int_\Omega \chi_E d\mu = \mu(E),$$

where χ_E is the indicator function on $E \in \mathcal{F}$,

$$\left(N^*\right) \int_D c f d\mu = c \left(N^*\right) \int_D f d\mu, \quad c \geq 0,$$

$$\left(N^*\right) \int_D \sup_{n \in \mathbb{N}} f_n d\mu = \sup_{n \in \mathbb{N}} \left(N^*\right) \int_D f_n d\mu,$$

where f_n, $n \in \mathbb{N}$, is an increasing sequence of elementary (countably valued) functions converging uniformly to f. Furthermore we have

$$(N^*) \int_D f d\mu \geq 0,$$

$$f \geq g \text{ implies } (N^*) \int_D f d\mu \geq (N^*) \int_D g d\mu,$$

where $f, g : \Omega \to [0, \infty]$ are measurable.

Let $a \leq f(\omega) \leq b$ for almost every $\omega \in E$, then

$$a\mu(E) \leq (N^*) \int_E f d\mu \leq b\mu(E);$$

$$(N^*) \int_E 1 d\mu = \mu(E);$$

$f > 0$ almost everywhere and $(N^*) \int_E f d\mu = 0$ imply $\mu(E) = 0$; $(N^*) \int_\Omega f d\mu = 0$ if and only $f = 0$ almost everywhere; $(N^*) \int_\Omega f d\mu < \infty$ implies that

$$\overline{N}(f) := \{\omega \in \Omega | f(\omega) \neq 0\} \text{ has } \sigma\text{-finite measure;}$$

$$(N^*) \int_D (f+g) d\mu \leq (N^*) \int_D f d\mu + (N^*) \int_D g d\mu;$$

and

$$\left| (N^*) \int_D f d\mu - (N^*) \int_D g d\mu \right| \leq (N^*) \int_D |f - g| d\mu.$$

From now on in this article we assume that $\mu : \mathcal{F} \to [0, +\infty)$.

We need

Proposition 2.1 ([1]). *Let $a \leq b$, $a, b \in \mathbb{R}$. Let $card(K)$ (≥ 0) be the maximum number of integers contained in $[a, b]$. Then*

$$\max(0, (b-a) - 1) \leq card(K) \leq (b-a) + 1.$$

3. Main Results

Here the activation function $b : \mathbb{R}^d \to \mathbb{R}_+$, $d \in \mathbb{N}$, is of compact support $B := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$, $i = 1, ..., d$. That is $b(x) > 0$ for any $x \in B$, and clearly b may have jump discontinuities. Also the shape of the graph of b is immaterial.

Typically in neural networks approximation we take b to be a d-dimensional bell-shaped function (i.e. per coordinate is a centered bell-shaped function), or a product of univariate centered bell-shaped functions, or a product of sigmoid functions, in our case all of them are of compact support B.

Example Take $b(x) = \beta(x_1)\beta(x_2)...\beta(x_d)$, where β is any of the following functions, $i = 1, ..., d$:

(i) $\beta(x_j)$ is the characteristic function on $[-1, 1]$,

(ii) $\beta(x_j)$ is the hat function over $[-1+\varepsilon, 1-\varepsilon]$, where $\varepsilon > 0$ is small, that is,

$$\beta(x_j) = \begin{cases} 1 + x_j, & -1 \leq x_j \leq 0, \\ 1 - x_j, & 0 < x_j \leq 1, \\ 0, & \text{elsewhere}, \end{cases}$$

(iii) the truncated sigmoids

$$\beta(x_j) = \begin{cases} \frac{1}{1+e^{-x_j}} \text{ or } \tanh x_j \text{ or } erf(x_j), \text{ for } x_j \in [-T_j, T_j], \text{ with large } T_j > 0, \\ 0, \quad x_j \in \mathbb{R} \setminus [-T_j, T_j], \end{cases}$$

(iv) the truncated Gompertz function

$$\beta(x_j) = \begin{cases} e^{-\alpha e^{-\beta x j}}, & x_j \in [-T_j, T_j]; \ \alpha, \beta > 0; \text{ large } T_j > 0, \\ 0, & x_j \in \mathbb{R} \setminus [-T_j, T_j], \end{cases}$$

Thus the general activation function b we will be using here includes all kinds of activation functions in neural network approximations.

Here we consider functions $f : \mathbb{R}^d \to \mathbb{R}_+$ that either continuous and bounded, or uniformly continuous.

Let here the parameters: $0 < \alpha < 1$, $x = (x_1, ..., x_d) \in \mathbb{R}^d$, $n \in \mathbb{N}$; $r = (r_1, ..., r_d) \in \mathbb{N}^d$, $i = (i_1, ..., i_d) \in \mathbb{N}^d$, with $i_j = 1, 2, ..., r_j$, $j = 1, ..., d$; also let $w_i = w_{i_1,...,i_d} \geq 0$, such that $\sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} ... \sum_{i_d=1}^{r_d} w_{i_1,...,i_d} = 1$, in brief written as $\sum_{i=1}^{r} w_i = 1$. We further consider the parameters $k = (k_1, ..., k_d) \in \mathbb{Z}^d$; $\mu_i = (\mu_{i_1}, ..., \mu_{i_d}) \in \mathbb{R}_+^d$, $\nu_i = (\nu_{i_1}, ..., \nu_{i_d}) \in \mathbb{R}_+^d$; and $\lambda_i = \lambda_{i_1,...,i_d}$, $\rho_i = \rho_{i_1,...,i_d} \geq 0$.

We use here the first multivariate modulus of continuity, with $\delta > 0$,

$$\omega_1(f, \delta) := \sup_{\substack{x, y \in \mathbb{R}^d \\ \|x - y\|_\infty \leq \delta}} |f(x) - f(y)|,$$

where $\|x\|_\infty = \max(|x_1|, ..., |x_d|)$.

Given that f is uniformly continuous we get $\lim_{\delta \to 0} \omega_1(f, \delta) = 0$.

So in this section mainly we study the pointwise convergence with rates over $\mathbb{R}^d$, to the unit operator, of the following one hidden layer multivariate normalized neural network perturbed operator:

Definition 3.1. Let $\mathcal{L}$ be the Lebesgue σ-algebra on $\mathbb{R}^N$, $N \in \mathbb{N}$, and the maxitive measure $\mu : \mathcal{L} \to [0, +\infty)$, such that for any $A \in \mathcal{L}$ with $A \neq \varnothing$, we get $\mu(A) > 0$. We define the multivariate Kantorovich-Shilkret type neural network operators for any $x \in \mathbb{R}^N$, $n \in \mathbb{N}$:

$$(M_n^\mu(f))(x) =$$

$$\frac{\left[\sum_{k=-n^2}^{n^2}\left(\sum_{i=1}^{r} w_i \frac{1}{\mu\left(\left[0,\frac{1}{n+\rho_i}\right]^d\right)}(N^*)\int_0^{\frac{1}{n+\rho_i}} f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right)d\mu(t)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right]}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)} \tag{1}$$

$$= \sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_d=-n^2}^{n^2} \left(\sum_{i_1=1}^{r_1} \cdots \sum_{i_d=1}^{r_d} w_{i_1,\dots,i_d} \frac{1}{\mu\left(\left[0, \frac{1}{n+\rho_{i_1,\dots,i_d}}\right]^d\right)} \right.$$

$$\left. (N^*) \underset{\left[0,\frac{1}{n+\rho_{i_1,\dots,i_d}}\right]^d}{\int \cdots \int \cdots \int} f\left(t_1 + \frac{k_1+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}, \dots, t_d + \frac{k_d+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}\right) d\mu(t_1,\dots,t_d) \right)$$

$$\frac{b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\dots,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}{\sum_{k_1=-n^2}^{n^2}\cdots\sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\dots,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}.$$

Remark 3.2. For f continuous and bounded we get:

$$\|M_n^{\mu}(f)\|_{\infty} \leq \|f\|_{\infty}, \text{ and } M_n^{\mu}(1) = 1.$$

Remark 3.3. The terms in the ratio of sums in (1) can be nonzero, iff simultaneously

$$\left| n^{1-\alpha}\left(x_j - \frac{k_j}{n}\right)\right| \leq T_j, \text{ all } j = 1,\dots,d,$$

i.e. $\left|x_j - \frac{k_j}{n}\right| \leq \frac{T_j}{n^{1-\alpha}}$, all $j = 1,\dots,d$, iff

$$nx_j - T_j n^{\alpha} \leq k_j \leq nx_j + T_j n^{\alpha}, \quad \text{all } j = 1,\dots,d.$$

To have the order

$$-n^2 \leq nx_j - T_j n^{\alpha} \leq k_j \leq nx_j + T_j n^{\alpha} \leq n^2, \tag{2}$$

we need $n \geq T_j + |x_j|$, all $j = 1,\dots,d$. So (2) is true when we take

$$n \geq \max_{j\in\{1,\dots,d\}} (T_j + |x_j|).$$

When $x \in B$ in order to have (2) it is enough to assume that $n \geq 2T^*$, where $T^* := \max\{T_1,\dots,T_d\} > 0$. Consider

$$\widetilde{I}_j := [nx_j - T_j n^{\alpha}, nx_j + T_j n^{\alpha}], \ \ j = 1,\dots,d, \ n \in \mathbb{N}.$$

The length of $\widetilde{I}_j$ is $2T_j n^{\alpha}$. By Proposition 2.1, we get that the cardinality of $k_j \in \mathbb{Z}$ that belong to $\widetilde{I}_j$ is $card(k_j) \geq \max(2T_j n^{\alpha} - 1, 0)$, any $j \in \{1,\dots,d\}$.

In order to have $card\,(k_j) \geq 1$, we need $2T_j n^{\alpha} - 1 \geq 1$ if and only if $n \geq T_j^{-\frac{1}{\alpha}}$, any $j \in \{1, ..., d\}$.

Therefore, a sufficient condition in order to obtain the order (2) along with the interval $\widetilde{I}_j$ to contain at least one integer for all $j = 1, ..., d$ is that

$$n \geq \max_{j \in \{1,...,d\}} \left\{ T_j + |x_j|, T_j^{-\frac{1}{\alpha}} \right\}. \tag{3}$$

Clearly as $n \to +\infty$ we get that $card\,(k_j) \to +\infty$, all $j = 1, ..., d$. Also notice that $card\,(k_j)$ equals to the cardinality of integers in $[\lceil nx_j - T_j n^{\alpha} \rceil, [nx_j + T_j n^{\alpha}]]$ for all $j = 1, ..., d$. Here $[\cdot]$ denotes the integral part of the number while $\lceil \cdot \rceil$ denotes its ceiling.

From now on, in this section we assume (3).

We shall denote by $T = (T_1, ..., T_d)$, $[nx + Tn^{\alpha}] = ([nx_1 + T_1 n^{\alpha}], ..., [nx_d + T_d n^{\alpha}])$, and $\lceil nx - Tn^{\alpha} \rceil = (\lceil nx_1 - T_1 n^{\alpha} \rceil, ..., \lceil nx_d - T_d n^{\alpha} \rceil)$. Furthermore it holds

$$\left(M_n^{\mu}(f)\right)(x) =$$

$$\frac{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i \frac{1}{\mu\left(\left[0, \frac{1}{n+\rho_i}\right]^d\right)} (N^*) \int_0^{\frac{1}{n+\rho_i}} f\left(t + \frac{k+\lambda_i}{n+\rho_i}\right) d\mu(t) \right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)} \tag{4}$$

$$= \sum_{k_1 = \lceil nx_1 - T_1 n^{\alpha} \rceil}^{[nx_1 + T_1 n^{\alpha}]} \cdots \sum_{k_d = \lceil nx_d - T_d n^{\alpha} \rceil}^{[nx_d + T_d n^{\alpha}]} \left(\sum_{i_1=1}^{r_1} \cdots \sum_{i_d=1}^{r_d} w_{i_1,...,i_d} \frac{1}{\mu\left(\left[0, \frac{1}{n+\rho_{i_1,...,i_d}}\right]^d\right)} \right.$$

$$\left. (N^*) \int \cdots \int_{\left[0, \frac{1}{n+\rho_{i_1,...,i_d}}\right]^d} \cdots \int f\left(t_1 + \frac{k_1 + \lambda_{i_1,...,i_d}}{n + \rho_{i_1,...,i_d}}, ..., t_d + \frac{k_d + \lambda_{i_1,...,i_d}}{n + \rho_{i_1,...,i_d}}\right) d\mu(t_1, ..., t_d) \right)$$

$$\frac{b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{\sum_{k_1 = \lceil nx_1 - T_1 n^{\alpha} \rceil}^{[nx_1 + T_1 n^{\alpha}]} \cdots \sum_{k_d = \lceil nx_d - T_d n^{\alpha} \rceil}^{[nx_d + T_d n^{\alpha}]} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}.$$

So if $\left| n^{1-\alpha}\left(x_j - \frac{k_j}{n}\right) \right| \leq T_j$, all $j = 1, ..., d$, we get that

$$\left\| x - \frac{k}{n} \right\|_{\infty} \leq \frac{T^*}{n^{1-\alpha}}.$$

For convinience we call

$$V(x) := \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) =$$

$$\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]} \dots \sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\dots,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right).$$

We make

Remark 3.4. Here always k is as in (2).

We have for

$$0 \le t_j \le \frac{1}{n+\rho_{i_1,\dots,i_d}}, \quad j=1,\dots,d,$$

that

$$\left\|t+\frac{k+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}-x\right\|_{\infty} \le \frac{1}{n+\rho_{i_1,\dots,i_d}}+\left\|\frac{k+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}-x\right\|_{\infty} \le$$

$$\frac{1+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}+\left\|\frac{k}{n+\rho_{i_1,\dots,i_d}}-x\right\|_{\infty} \le$$

$$\frac{1+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}+\left\|\frac{k}{n+\rho_{i_1,\dots,i_d}}-\frac{k}{n}\right\|_{\infty}+\left\|\frac{k}{n}-x\right\|_{\infty} \le$$

$$\frac{1+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}+\frac{T^*}{n^{1-\alpha}}+\frac{\rho_{i_1,\dots,i_d}\|k\|_{\infty}}{(n+\rho_{i_1,\dots,i_d})\,n} \le$$

$$\frac{1+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}+\frac{T^*}{n^{1-\alpha}}+\frac{\rho_{i_1,\dots,i_d}}{n\,(n+\rho_{i_1,\dots,i_d})}\left(n\|x\|_{\infty}+T^*n^{\alpha}\right) =$$

$$\frac{1+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}+\frac{T^*}{n^{1-\alpha}}+\left(\frac{\rho_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}\right)\left(\|x\|_{\infty}+\frac{T^*}{n^{1-\alpha}}\right).$$

We have found that

$$\left\|t+\frac{k+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}-x\right\|_{\infty} \le$$

$$\left(\frac{\rho_{i_1,\dots,i_d}\|x\|_{\infty}+\lambda_{i_1,\dots,i_d}+1}{n+\rho_{i_1,\dots,i_d}}\right)+\left(1+\frac{\rho_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}\right)\frac{T^*}{n^{1-\alpha}}. \tag{5}$$

So when $0 \le t_j \le \frac{1}{n+\rho_{i_1,\dots,i_d}}$, $j=1,\dots,d$, we get that

$$\omega_1\left(f,\left\|t+\frac{k+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}-x\right\|_{\infty}\right) \le$$

$$\leq \omega_1\left(f, \left(\frac{\rho_{i_1,\dots,i_d}\|x\|_\infty + \lambda_{i_1,\dots,i_d} + 1}{n + \rho_{i_1,\dots,i_d}}\right) + \left(1 + \frac{\rho_{i_1,\dots,i_d}}{n + \rho_{i_1,\dots,i_d}}\right)\frac{T^*}{n^{1-\alpha}}\right), \quad (6)$$

with dominant speed $\frac{1}{n^{1-\alpha}}$.

We present

Theorem 3.5. *Let $f : \mathbb{R}^d \to \mathbb{R}_+$ be continuous and bounded or uniformly continuous. Let $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$ such that $n \geq \max_{j\in\{1,\dots,d\}}\left(T_j + |x_j|, T_j^{-\frac{1}{\alpha}}\right)$, $T_j > 0$, $0 < \alpha < 1$. Then*

$$\sup_\mu |(M_n^\mu(f))(x) - f(x)| \leq$$

$$\sum_{i=1}^r w_i\omega_1\left(f, \left(\frac{\rho_i\|x\|_\infty + \lambda_i + 1}{n + \rho_i}\right) + \left(1 + \frac{\rho_i}{n + \rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right) = \quad (7)$$

$$\sum_{i_1=1}^{r_1}\dots\sum_{i_d=1}^{r_d} w_{i_1,\dots,i_d}\omega_1\left(f, \left(\frac{\rho_{i_1,\dots,i_d}\|x\|_\infty + \lambda_{i_1,\dots,i_d} + 1}{n + \rho_{i_1,\dots,i_d}}\right) + \left(1 + \frac{\rho_{i_1,\dots,i_d}}{n + \rho_{i_1,\dots,i_d}}\right)\frac{T^*}{n^{1-\alpha}}\right).$$

Proof. We notice the following:

$$f\left(t + \frac{k + \lambda_i}{n + \rho_i}\right) = f\left(t + \frac{k + \lambda_i}{n + \rho_i}\right) - f(x) + f(x) \leq \left|f\left(t + \frac{k + \lambda_i}{n + \rho_i}\right) - f(x)\right| + f(x).$$

Hence

$$\frac{1}{\mu\left(\left[0, \frac{1}{n+\rho_i}\right]^d\right)}(N^*)\int_0^{\frac{1}{n+\rho_i}} f\left(t + \frac{k + \lambda_i}{n + \rho_i}\right)d\mu(t) \leq$$

$$\frac{1}{\mu\left(\left[0, \frac{1}{n+\rho_i}\right]^d\right)}(N^*)\int_0^{\frac{1}{n+\rho_i}} \left|f\left(t + \frac{k + \lambda_i}{n + \rho_i}\right) - f(x)\right|d\mu(t) + f(x).$$

That is

$$\frac{1}{\mu\left(\left[0, \frac{1}{n+\rho_i}\right]^d\right)}(N^*)\int_0^{\frac{1}{n+\rho_i}} f\left(t + \frac{k + \lambda_i}{n + \rho_i}\right)d\mu(t) - f(x) \leq$$

$$\frac{1}{\mu\left(\left[0, \frac{1}{n+\rho_i}\right]^d\right)}(N^*)\int_0^{\frac{1}{n+\rho_i}} \left|f\left(t + \frac{k + \lambda_i}{n + \rho_i}\right) - f(x)\right|d\mu(t).$$

Similarly, we have

$$f(x) = f(x) - f\left(t + \frac{k+\lambda_i}{n+\rho_i}\right) + f\left(t + \frac{k+\lambda_i}{n+\rho_i}\right) \leq$$

$$\left| f\left(t + \frac{k+\lambda_i}{n+\rho_i}\right) - f(x) \right| + f\left(t + \frac{k+\lambda_i}{n+\rho_i}\right).$$

Hence

$$f(x) \leq \frac{1}{\mu\left(\left[0, \frac{1}{n+\rho_i}\right]^d\right)} (N^*) \int_0^{\frac{1}{n+\rho_i}} \left| f\left(t + \frac{k+\lambda_i}{n+\rho_i}\right) - f(x) \right| d\mu(t) +$$

$$\frac{1}{\mu\left(\left[0, \frac{1}{n+\rho_i}\right]^d\right)} (N^*) \int_0^{\frac{1}{n+\rho_i}} f\left(t + \frac{k+\lambda_i}{n+\rho_i}\right) d\mu(t).$$

That is,

$$f(x) - \frac{1}{\mu\left(\left[0, \frac{1}{n+\rho_i}\right]^d\right)} (N^*) \int_0^{\frac{1}{n+\rho_i}} f\left(t + \frac{k+\lambda_i}{n+\rho_i}\right) d\mu(t) \leq$$

$$\frac{1}{\mu\left(\left[0, \frac{1}{n+\rho_i}\right]^d\right)} (N^*) \int_0^{\frac{1}{n+\rho_i}} \left| f\left(t + \frac{k+\lambda_i}{n+\rho_i}\right) - f(x) \right| d\mu(t).$$

We have proved that

$$\left| \frac{1}{\mu\left(\left[0, \frac{1}{n+\rho_i}\right]^d\right)} (N^*) \int_0^{\frac{1}{n+\rho_i}} f\left(t + \frac{k+\lambda_i}{n+\rho_i}\right) d\mu(t) - f(x) \right| \leq$$

$$\frac{1}{\mu\left(\left[0, \frac{1}{n+\rho_i}\right]^d\right)} (N^*) \int_0^{\frac{1}{n+\rho_i}} \left| f\left(t + \frac{k+\lambda_i}{n+\rho_i}\right) - f(x) \right| d\mu(t).$$

Using the last inequality we derive

$$\left| (M_n^{\mu}(f))(x) - f(x) \right| \overset{(4)}{=}$$

$$\left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i \frac{1}{\mu\left(\left[0, \frac{1}{n+\rho_i}\right]^d\right)} (N^*) \int_0^{\frac{1}{n+\rho_i}} f\left(t + \frac{k+\lambda_i}{n+\rho_i}\right) d\mu(t) \right) \cdot \right.$$

$$\left. \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{(x)} - f(x) \right| =$$

$$\left| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i \frac{1}{\mu\left(\left[0,\frac{1}{n+\rho_i}\right]^d\right)} (N^*) \int_0^{\frac{1}{n+\rho_i}} f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right) d\mu(t) \right) \cdot \right.$$

$$\left. \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} - \frac{f(x)V(x)}{V(x)} \right| =$$

$$\left| \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left[\left(\sum_{i=1}^{r} w_i \frac{1}{\mu\left(\left[0,\frac{1}{n+\rho_i}\right]^d\right)} (N^*) \int_0^{\frac{1}{n+\rho_i}} f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right) d\mu(t) \right) - f(x) \right]}{V(x)} \right.$$

$$\left. \cdot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) \right| \leq$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left[\sum_{i=1}^{r} w_i \left| \frac{1}{\mu\left(\left[0,\frac{1}{n+\rho_i}\right]^d\right)} (N^*) \int_0^{\frac{1}{n+\rho_i}} f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right) d\mu(t) - f(x) \right| \right]}{V(x)}$$

$$\cdot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) \leq$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left[\sum_{i=1}^{r} w_i \left[\frac{1}{\mu\left(\left[0,\frac{1}{n+\rho_i}\right]^d\right)} (N^*) \int_0^{\frac{1}{n+\rho_i}} \left| f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right) - f(x) \right| d\mu(t) \right] \right]}{V(x)}$$

$$\cdot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) \leq$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left[\sum_{i=1}^{r} w_i \left[\frac{1}{\mu\left(\left[0,\frac{1}{n+\rho_i}\right]^d\right)} (N^*) \int_0^{\frac{1}{n+\rho_i}} \omega_1\left(f, \left\| t+\frac{k+\lambda_i}{n+\rho_i} - x \right\|_{\infty}\right) d\mu(t) \right] \right]}{V(x)}$$

$$\cdot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) \overset{\text{(by (6))}}{\leq}$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left[\sum_{i=1}^{r} w_i \left[\frac{1}{\mu\left(\left[0,\frac{1}{n+\rho_i}\right]^d\right)} (N^*) \int_0^{\frac{1}{n+\rho_i}} \omega_1\left(f, \left(\frac{\rho_i \|x\|_{\infty} + \lambda_i + 1}{n+\rho_i}\right) + \right.\right.\right.}{V(x)}$$

$$\left.\left.\left. \left(1+\frac{\rho_i}{n+\rho_i}\right) \frac{T^*}{n^{1-\alpha}}\right) d\mu(t) \right] \right] \cdot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) =$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left[\sum_{i=1}^{r} w_i\omega_1\left(f,\left(\frac{\rho_i\|x\|_{\infty}+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right)\right]}{V(x)}\cdot$$

$$\cdot\, b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)=$$

$$\sum_{i=1}^{r} w_i\omega_1\left(f,\left(\frac{\rho_i\|x\|_{\infty}+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right),$$

proving the claim. □

Corollary 3.6. *Let $f:\mathbb{R}^d\to\mathbb{R}_+$ continuous and bounded or uniformly continuous function. Let $x\in\prod_{j=1}^{d}[-\gamma_j,\gamma_j]\subset\mathbb{R}^d$, $\gamma_j>0$, $\gamma^*=\max\{\gamma_1,...,\gamma_d\}$ and $n\in\mathbb{N}:n\geq\max_{j\in\{1,...,d\}}\left\{T_j+\gamma_j,T_j^{-\frac{1}{\alpha}}\right\}$. Then*

$$\sup_{\mu}\|M_n^{\mu}(f)-f\|_{\infty,\prod_{j=1}^{d}[-\gamma_j,\gamma_j]}\leq$$

$$\sum_{i=1}^{r} w_i\omega_1\left(f,\left(\frac{\rho_i\gamma^*+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right)= \quad (8)$$

$$\sum_{i_1=1}^{r_1}...\sum_{i_d=1}^{r_d} w_{i_1,...,i_d}\omega_1\left(f,\left(\frac{\rho_{i_1,...,i_d}\gamma^*+\lambda_{i_1,...,i_d}+1}{n+\rho_{i_1,...,i_d}}\right)+\left(1+\frac{\rho_{i_1,...,i_d}}{n+\rho_{i_1,...,i_d}}\right)\frac{T^*}{n^{1-\alpha}}\right).$$

Proof. By (7). □

Next here that the right hand sides of (7) and (8) do not depend on μ and b.

We continue with the higher order of approximation results based on the high order differentiability of the approximated function.

Theorem 3.7. *Let $x\in\mathbb{R}^d$ and $n\in\mathbb{N}$ such that $n\geq\max_{j\in\{1,...,d\}}\left(T_j+|x_j|,T_j^{-\frac{1}{\alpha}}\right)$, $T_j>0$, $0<\alpha<1$. Let also $f\in C^N\left(\mathbb{R}^d,\mathbb{R}_+\right)$, $N\in\mathbb{N}$, such that all of its partial derivatives $f_{\widetilde{\alpha}}$ of order N, $\widetilde{\alpha}:|\widetilde{\alpha}|=\sum_{j=1}^{d}\alpha_j=N$, are uniformly continuous or continuous and bounded. Then*

$$\sup_{\mu}|(M_n^{\mu}(f))(x)-f(x)|\leq$$

$$\sum_{l=1}^{N}\frac{1}{l!}\left(\sum_{i=1}^{r} w_i\left[\left(\frac{\rho_i\|x\|_{\infty}+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^l\right)\cdot$$

$$\left(\left(\sum_{j=1}^{d}\left|\frac{\partial}{\partial x_j}\right|\right)^{l} f(x)\right)+ \tag{9}$$

$$\frac{d^N}{N!}\sum_{i=1}^{r} w_i\left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^N \cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]\right).$$

Inequality (9) implies the pointwise convergence with rates of $(M_n^\mu(f))(x)$ *to* $f(x)$*, as* $n\to\infty$*, at speed* $\frac{1}{n^{1-\alpha}}$*.*

Proof. As in the proof of Theorem 3.5 we get:

$$|(M_n^\mu(f))(x)-f(x)|\leq$$

$$\frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{\lfloor nx+Tn^\alpha\rfloor}\left[\sum_{i=1}^{r} w_i\left[\frac{1}{\mu\left(\left[0,\frac{1}{n+\rho_i}\right]^d\right)}(N^*)\int_0^{\frac{1}{n+\rho_i}}\left|f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right)-f(x)\right|d\mu(t)\right]\right]}{V(x)} \tag{10}$$

$$\cdot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right).$$

Here we consider $0\leq t_j\leq\frac{1}{n+\rho_{i_1,\dots,i_d}}$, $j=1,\dots,d$.

Set

$$g_{t+\frac{k+\lambda_i}{n+\rho_i}}(\lambda^*)=f\left(x+\lambda^*\left(t+\frac{k+\lambda_i}{n+\rho_i}-x\right)\right),\quad 0\leq\lambda^*\leq 1.$$

Then we have

$$g^{(l)}_{t+\frac{k+\lambda_i}{n+\rho_i}}(\lambda^*)=$$

$$\left[\left(\sum_{j=1}^{d}\left(t_j\frac{k_j+\lambda_i}{n+\rho_i}-x_j\right)\frac{\partial}{\partial x_j}\right)^{l} f\right]\left(x+\lambda^*\left(t+\frac{k+\lambda_i}{n+\rho_i}-x\right)\right),$$

and

$$g_{t+\frac{k+\lambda_i}{n+\rho_i}}(0)=f(x).$$

By multivariate Taylor's formula, we get

$$f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right)-f(x)=g_{t+\frac{k+\lambda_i}{n+\rho_i}}(1)-f(x)=$$

$$\sum_{l=1}^{N}\frac{g^{(l)}_{t+\frac{k+\lambda_i}{n+\rho_i}}(0)}{l!}+R_N\left(t+\frac{k+\lambda_i}{n+\rho_i},0\right), \tag{11}$$

where

$$R_N\left(t+\frac{k+\lambda_i}{n+\rho_i},0\right)=$$

$$\int_0^1\left(\int_0^{\lambda_1^*}...\left(\int_0^{\lambda_{N-1}^*}\left(g^{(N)}_{t+\frac{k+\lambda_i}{n+\rho_i}}\left(\lambda_N^*\right)-g^{(N)}_{t+\frac{k+\lambda_i}{n+\rho_i}}(0)\right)d\lambda_N^*\right)...\right)d\lambda_1^*.$$

Here we denote by

$$f_{\widetilde{\alpha}}:=\frac{\partial^{\widetilde{\alpha}}f}{\partial x^{\widetilde{\alpha}}},\quad \widetilde{\alpha}:=(\alpha_1,...,\alpha_d),\ \ \alpha_j\in\mathbb{Z}^+,\ j=1,...,d,$$

such that $|\widetilde{\alpha}|=\sum_{j=1}^{d}\alpha_j=N$.

Notice that

$$\left|g^{(l)}_{t+\frac{k+\lambda_i}{n+\rho_i}}(0)\right|\overset{(5)}{\leq}\left(\left(\sum_{j=1}^{d}\left|\frac{\partial}{\partial x_j}\right|\right)^l f(x)\right)\cdot$$

$$\left[\left(\frac{\rho_i\left\|x\right\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^l, \tag{12}$$

We observe that $(0\leq\lambda_N^*\leq 1)$

$$\left|g^{(N)}_{t+\frac{k+\lambda_i}{n+\rho_i}}\left(\lambda_N^*\right)-g^{(N)}_{t+\frac{k+\lambda_i}{n+\rho_i}}(0)\right|=$$

$$\left|\left[\left(\sum_{j=1}^{d}\left(t_j+\frac{k_j+\lambda_i}{n+\rho_i}-x_j\right)\frac{\partial}{\partial x_j}\right)^N f\right]\left(x+\lambda_N^*\left(t+\frac{k+\lambda_i}{n+\rho_i}-x\right)\right)-\right.$$

$$\left.\left[\left(\sum_{j=1}^{d}\left(t_j+\frac{k_j+\lambda_i}{n+\rho_i}-x_j\right)\frac{\partial}{\partial x_j}\right)^N f\right](x)\right|\overset{\text{(by (5), (6))}}{\leq}$$

$$d^N\left[\left(\frac{\rho_i\left\|x\right\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^N\cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\left[\left(\frac{\rho_i\left\|x\right\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]\right).$$

Thus we find

$$\left|R_N\left(t+\frac{k+\lambda_i}{n+\rho_i},0\right)\right|\leq$$

$$\int_0^1\left(\int_0^{\lambda_1^*}...\left(\int_0^{\lambda_{N-1}^*}\left|g^{(N)}_{t+\frac{k+\lambda_i}{n+\rho_i}}\left(\lambda_N^*\right)-g^{(N)}_{t+\frac{k+\lambda_i}{n+\rho_i}}(0)\right|d\lambda_N^*\right)...\right)d\lambda_1^*\leq \tag{13}$$

$$\frac{d^N}{N!}\left[\left(\frac{\rho_i\left\|x\right\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^N\cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n+\rho_i} \right) + \left(1 + \frac{\rho_i}{n+\rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right] \right).$$

Therefore it holds

$$\left| f\left(t + \frac{k+\lambda_i}{n+\rho_i} \right) - f(x) \right| \overset{(11)}{\leq}$$

$$\sum_{l=1}^{N} \frac{\left| g^{(l)}_{t+\frac{k+\lambda_i}{n+\rho_i}}(0) \right|}{l!} + \left| R_N \left(t + \frac{k+\lambda_i}{n+\rho_i}, 0 \right) \right| \overset{(\text{by } (12), (13))}{\leq}$$

$$\sum_{l=1}^{N} \frac{1}{l!} \left(\left(\sum_{j=1}^{d} \left| \frac{\partial}{\partial x_j} \right| \right)^l f(x) \right) \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n+\rho_i} \right) + \left(1 + \frac{\rho_i}{n+\rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right]^l \quad (14)$$

$$+ \frac{d^N}{N!} \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n+\rho_i} \right) + \left(1 + \frac{\rho_i}{n+\rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right]^N \cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n+\rho_i} \right) + \left(1 + \frac{\rho_i}{n+\rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right] \right).$$

Hence we have that

$$\sum_{i=1}^{r} w_i \left[\frac{1}{\mu\left(\left[0, \frac{1}{n+\rho_i} \right]^d \right)} (N^*) \int_0^{\frac{1}{n+\rho_i}} \left| f\left(t + \frac{k+\lambda_i}{n+\rho_i} \right) - f(x) \right| d\mu(t) \right] \overset{(14)}{\leq}$$

$$\sum_{i=1}^{r} w_i \left\{ \sum_{l=1}^{N} \frac{1}{l!} \left(\left(\sum_{j=1}^{d} \left| \frac{\partial}{\partial x_j} \right| \right)^l f(x) \right) \cdot \quad (15)$$

$$\left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n+\rho_i} \right) + \left(1 + \frac{\rho_i}{n+\rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right]^l$$

$$+ \frac{d^N}{N!} \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n+\rho_i} \right) + \left(1 + \frac{\rho_i}{n+\rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right]^N \cdot$$

$$\left. \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n+\rho_i} \right) + \left(1 + \frac{\rho_i}{n+\rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right] \right) \right\}.$$

Consequently we derive

$$\left| (M_n^\mu(f))(x) - f(x) \right| \overset{(\text{by } (10), (15))}{\leq}$$

$$\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \left[\sum_{l=1}^{N} \frac{1}{l!} \sum_{i=1}^{r} w_i \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n+\rho_i} \right) + \left(1 + \frac{\rho_i}{n+\rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right]^l \cdot \right.$$

$$\left(\left(\sum_{j=1}^{d} \left| \frac{\partial}{\partial x_j} \right| \right)^l f(x) \right) + \frac{d^N}{N!} \sum_{i=1}^{r} w_i \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n+\rho_i} \right) + \left(1 + \frac{\rho_i}{n+\rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right]^N \cdot$$

$$\frac{\max\limits_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]\right)\right]b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}=$$

$$\sum_{l=1}^{N}\frac{1}{l!}\sum_{i=1}^{r}w_i\left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^l\cdot$$

$$\left(\left(\sum_{j=1}^{d}\left|\frac{\partial}{\partial x_j}\right|\right)^l f(x)\right)+\frac{d^N}{N!}\sum_{i=1}^{r}w_i\left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^N\cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]\right),$$

proving the claim. $\square$

We have

Corollary 3.8. *Let all as in Theorem 3.7. Additionally, assume that* $f_{\widetilde{\alpha}}(x)=0$, $\widetilde{\alpha}:|\widetilde{\alpha}|=\rho$, $1\le\rho\le N$. *Then*

$$\sup_\mu\left|\left(M_n^\mu(f)\right)(x)-f(x)\right|\le$$

$$\frac{d^N}{N!}\sum_{i=1}^{r}w_i\left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^N\cdot \tag{16}$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]\right).$$

Inequality (16) implies the pointwise convergence with rates of $\left(M_n^\mu(f)\right)(x)$ *to* $f(x)$, *as* $n\to\infty$, *at speed* $\frac{1}{n^{(1-\alpha)(N+1)}}$.

Proof. By (9) we get the desired result. $\square$

The uniform convergence with rates follows from the following

Corollary 3.9. *All as in Theorem 3.7. Let* $x\in G:=\prod_{j=1}^{d}[-\gamma_j,\gamma_j]\subset\mathbb{R}^d$, $\gamma_j>0$, $\gamma^*:=\max\{\gamma_1,...,\gamma_d\}$ *and* $n\in\mathbb{N}:n\ge\max\limits_{j\in\{1,...,d\}}\left(T_j+\gamma_j,T_j^{-\frac{1}{\alpha}}\right)$. *Then*

$$\sup_\mu\left\|M_n^\mu(f)-f\right\|_{\infty,\prod_{j=1}^{d}[-\gamma_j,\gamma_j]}\le\sum_{l=1}^{N}\frac{1}{l!}\left(\left(\sum_{j=1}^{d}\left|\frac{\partial}{\partial x_j}\right|\right)^l\|f\|_{\infty,\prod_{j=1}^{d}[-\gamma_j,\gamma_j]}\right)$$

$$\left[\sum_{i=1}^{r}w_i\left[\left(\frac{\rho_i\gamma^*+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^l\right]+ \tag{17}$$

$$\frac{d^N}{N!}\sum_{i=1}^{r} w_i \left[\left(\frac{\rho_i \gamma^* + \lambda_i + 1}{n+\rho_i}\right) + \left(1 + \frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^N \cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \left[\left(\frac{\rho_i \gamma^* + \lambda_i + 1}{n+\rho_i}\right) + \left(1 + \frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]\right).$$

Inequality (17) implies the pointwise convergence with rates of $M_n^{\mu}(f)$ *to* f *on* G*, as* $n \to \infty$*, at speed* $\frac{1}{n^{1-\alpha}}$*.*

Proof. By (9) we get the desired result. □

Corollary 3.10. *All as in Theorem 3.7 with* $N = 1$ *hold. Then*

$$\sup_{\mu} |(M_n^{\mu}(f))(x) - f(x)| \leq \left(\sum_{j=1}^{d} \left|\frac{\partial f(x)}{\partial x_j}\right|\right) \cdot$$

$$\left[\sum_{i=1}^{r} w_i \left[\left(\frac{\rho_i \|x\|_{\infty} + \lambda_i + 1}{n+\rho_i}\right) + \left(1 + \frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]\right] + \tag{18}$$

$$d \sum_{i=1}^{r} w_i \left[\left(\frac{\rho_i \|x\|_{\infty} + \lambda_i + 1}{n+\rho_i}\right) + \left(1 + \frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right] \cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \left[\left(\frac{\rho_i \|x\|_{\infty} + \lambda_i + 1}{n+\rho_i}\right) + \left(1 + \frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]\right).$$

Inequality (18) implies the pointwise convergence with rates of $(M_n^{\mu}(f))(x)$ *to* $f(x)$*, as* $n \to \infty$*, at speed* $\frac{1}{n^{1-\alpha}}$*.*

Proof. By (9) we get the desired result. □

Notice here that the right hand sides of (9), (16), (17) and (18), do not depend on μ and b.

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