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ON SELF COMPLEMENTARITY OF THE INDUCED COMPLEMENT OF A GRAPH

KIRAN KIGGAL UDAYASHANKAR AND PRAMEELA KOLAKE

ABSTRACT. Let $G=(V,E)$ be a graph and $S \subseteq V$. The *induced complement of the graph G with respect to the set S*, denoted by G_S, is the graph obtained from the graph G by removing the edges of $\langle S \rangle$ of G and adding the edges which are not in $\langle S \rangle$ of G. Given a set $S \subseteq V$, the graph G is said to be *S-induced self complementary* if $G_S \cong G$. The graph G is said to be *S-induced co-complementary* if $G_S \cong \overline{G}$. This paper presents the study of the different properties of the S-i.s.c. and S-i.c.c. graphs.

1. INTRODUCTION

We consider here only the finite undirected graphs with no loops or multiple edges. Further, we denote any graph G by a pair (V,E), where V is the set of all vertices of G and E, the set of all edges of G. By an (n,m)-graph G, we mean, a graph G on n vertices (also referred to as *order* of G) and m edges (also referred to as *size* of G). Let $G=(V,E)$ be an (n,m) graph. For any set $S \subseteq V$, the *induced subgraph* $\langle S \rangle$ is the maximal subgraph of G with the vertex set S. For any vertex $v \in V$, the *neighborhood of v* is $N(v)=\{u \in V \mid uv \in E\}$. The *degree* of a vertex $v \in V$ is $deg(v)=|N(v)|$. The *degree sequence* of a graph is the list of vertex degrees, usually written in decreasing order, such as $d_1 \geq d_2 \geq ..... \geq d_n$. For the notations and terminologies used here, the reader is referred to [5, 12] unless specified otherwise.

The *complement* $\overline{G}$ of a graph $G=(V,E)$ also has V as its vertex set, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Two graphs G and H are *isomorphic (written $G \cong H$)* if there exists a one-to-one correspondence between their vertex sets $V(G)$ and $V(H)$, which preserves the adjacency. A graph $G=(V,E)$ is self-complementary if $G \cong \overline{G}$. Prameela Kolake [7] in her Ph.D thesis defined the concept of Induced complementation of a Graph as follows.

Definition 1.1. *Let $G=(V,E)$ be a graph and $S \subseteq V$. The induced complement of G with respect to S, denoted by G_S, is a graph (V,E_S), where for any two vertices $u,v \in V$, $uv \in E_S$ if and only if one of the following conditions holds.*

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(1) *($u \notin S$ or $v \notin S$) and $uv \in E$.*
(2) *$u, v \in S$ and $uv \notin E$.*

The graph G is said to be
(1) S-induced self complementary (S-i.s.c.) if $G_S \cong G$.
(2) S-induced co-complementary (S-i.c.c.) if $G_S \cong \overline{G}$.

Remark 1.1. Let $G = (V, E)$ be a (n, m) graph. Then we observe the following.
(1) Every graph is S-induced self complementary for $|S| = 1$.
(2) If $S = V$, then $G_S \cong \overline{G}$.
(3) A graph is always n-induced co-complementary, i.e., $G_S \cong \overline{G}$ for $|S| = n$.

Some basic observations and results on the Induced complement of a graph G are listed below.

Note 1.1. (1) Let G be a (n, m) graph and $S \subseteq V$ with $|S| = k$ where $\langle S \rangle$ contains m_S edges. Then G_S contains $m - 2m_S + \binom{k}{2}$ edges.
(2) Let G be a graph and $S \subseteq V$. Then $\overline{(G_S)} \cong (\overline{G})_S$.
(3) For any tree T of order n,
$$T_{\{v_i,v_j\}} = \begin{cases} T_1 \cup T_2 & if\ v_i\ and\ v_j\ are\ adjacent. \\ unicyclic\ graph, & otherwise. \end{cases}$$
where, T_1 and T_2 are two trees of order less than n.
(4) In particular for any path P_n,
$$(P_n)_{\{v_i,v_j\}} = \begin{cases} P_i \cup P_{n-i} & if\ j = i+1 \\ unicyclic\ graph, & otherwise. \end{cases}$$
(5) Let C_n be a cycle on n vertices. Then $(C_n)_{\{i,j\}} \cong P_n$ for any pair of adjacent vertices i and j of the cycle C_n.
(6) $(C_n)_{\{i,j,k\}} \cong C_{n-1} \cup K_1$ for any three consecutively adjacent vertices i, j and k of C_n.
(7) Thus in general, if $S = \{v_1, v_2, ...v_k\}$ is the set of k consecutively adjacent vertices of a cycle C_n, then $(C_n)_S$ is the graph with a cycle C_{n-k+2} and $\overline{P}_k$ with an edge in common.

2. Results on S-i.s.c. and S-i.c.c. graphs

The graph G is said to be S-induced self complementary (S-i.s.c.) if $G_S \cong G$ and S-induced co-complementary (S-i.c.c.) if $G_S \cong \overline{G}$. In this section a few results are discussed with respect to S-i.s.c. and S-i.c.c. graphs.

Consider a graph $G = (V, E)$. The induced complement of G with respect to every set $S \subseteq V$ with $|S| = k$, is called a k-induced complement of the graph G and is denoted by G_k. We have the following result.

Theorem 2.1. *For a positive integer k, if G_k is self complementary then G must be a connected graph.*

Proof. Let G_k be a self complementary graph for every set S with cardinality k. If G is not connected then we show that it is possible to construct a set $S_1 \subseteq V$ with $|S_1| = k$ such that, G_{S_1} is not self complementary.
Assume that the graph G is a disconnected graph and let $G^1, G^2, \ldots.., G^r$ be the components of G with order $n_1 \geq n_2 \geq \ldots\ldots.. \geq n_r$ respectively. If $k = n$, then clearly $G_k \cong \overline{G}$. Then $\overline{(G_k)} \cong \overline{(\overline{G})} \cong G$. Thus $G \cong \overline{(G_k)} \cong G_k \cong \overline{G}$ for $k = n$. Thus if G_k is self complementary then G must be self complementary. Hence G must be a connected graph.
Let us assume that $k < n$. Suppose $k \leq \sum_{i=1}^{r-1} n_i$. Then we can pick any k vertices from the components $G^1, G^2, \ldots.., G^{r-1}$ to form S_1. So, there exists at least one component G^r such that no vertex of G^r lies in S_1. Then G_{S_1} will be a disconnected graph and hence not self complementary. Thus $\overline{(G_k)} \ncong G_k$, a contradiction.
Suppose $n > k > \sum_{i=1}^{r-1} n_i$. Then choose S_1 to be the set of all vertices of $G^1, G^2, \ldots, G^{r-1}$ together with m vertices of the last component G^r of G such that $\sum_{i=1}^{r-1} n_i + m = k$. If $n_r = 1$, then $k = n$ and so, $n_r \geq 2$ and there exists at least one vertex v in G^r such that $v \notin S_1$. Clearly, $n_i \geq 2$ for all i. We are choosing m vertices from the component G^r to form the set S_1.

Case i: $m \neq \frac{n_r}{2}$. Then, since $n_i \geq n_r$ for all i, the degree sequence of the vertices of G_{S_1} and $\overline{(G_{S_1})}$ will be different. Hence $G_{S_1} \ncong \overline{(G_{S_1})}$.

Case ii: $m = \frac{n_r}{2}$ and the degree sequence of the vertices in the equal partition of $V(G^r)$ is the same in G. Then the m vertices of G^r are adjacent to every vertex of $\overline{(\cup_{i=1}^{r-1} G^i)}$ in G_{S_1} and the remaining m vertices of G^r are adjacent to every vertex of $\cup_{i=1}^{r-1} G^i$ in $\overline{(G_{S_1})}$. Then G_{S_1} is self complementary only if $\cup_{i=1}^{r-1} G^i$ is self complementary. Since $\cup_{i=1}^{r-1} G^i$ is disconnected, it can not be a self complementary graph. Hence, G_{S_1} is not self complementary.

Thus in either case, G_k is not self complementary, a contradiction. Hence G must be a connected graph. □

Theorem 2.2. *Let G be a self complementary graph. Then G_S is a self complementary graph if and only if $\langle S \rangle$ is a self complementary graph.*

Proof. Let G be a self complementary graph and suppose G_S is a self complementary graph with respect to a set $S \subseteq V$. Let $u_i, u_j \in S$ be such that u_i and u_j are adjacent in G. Then u_i and u_j are non-adjacent in G_S. Since $G_S \cong \overline{(G_S)} \cong (\overline{G})_S$, there exists a map $f : V(G) \longrightarrow V(\overline{G})$ such that, we can find two non-adjacent vertices w_i and w_j in $(\overline{G})_S$ corresponding to u_i and u_j of G_S. By the definition of the induced complements, either

(i) $w_i, w_j \in S$ and are adjacent in $\overline{G}$ or

(ii) at least one of w_i, w_j lies in $V-S$ and are non-adjacent in $\overline{G}$.

If (ii) is true, then f which maps u_i to w_i will not preserve the adjacency, a contradiction. Hence (i) must be true. So, for each vertex $u_i \in S$, we can find a vertex w_i in $\overline{\langle S\rangle}$ which preserves the adjacency. Thus $\langle S\rangle$ is a self complementary graph.
Conversely, let G be a self complementary graph and $S \subseteq V$ such that $\langle S\rangle$ is a self complementary graph. Suppose u_i and u_j are two adjacent vertices in G_S. Then either u_i and u_j lie in S and are not adjacent in G or at least one of u_i and u_j lies in $V-S$ and they are adjacent in G. Since G and $\langle S\rangle$ are self complementary graphs, there exist two vertices v_i and v_j in $\overline{G}$ such that v_i and v_j are non-adjacent in $\overline{G}$ if $v_i, v_j \in S$ or v_i and v_j are adjacent in $\overline{G}$ if at least one of v_i and v_j lies in $V-S$. Then v_i and v_j are adjacent in $(\overline{G})_S$. Since $(\overline{G})_S \cong \overline{(G_S)}$ (by Note 2.1), v_i and v_j are adjacent in $\overline{(G_S)}$. Thus $\overline{(G_S)} \cong G_S$. □

Theorem 2.3. *A graph $G=(V,E)$ is a S-i.s.c. graph for a given set $S \subseteq V$ if and only if $\langle S\rangle \cong \overline{\langle S\rangle}$ and one of the following conditions holds.*
(1) $\langle S\rangle$ is a component of G or
(2) at least $|S|-1$ vertices of S are adjacent to every vertex of a set $A \subseteq V-S$ and $N(v)\cap(V-S)=A$ for every vertex $v \in S$ with $deg_{\langle S\rangle}(v) \neq deg_{\overline{\langle S\rangle}}(v)$.

Proof. Let $G_S \cong G$ for some set $S \subseteq V$. By the definition of G_S, two adjacent vertices u_i, u_j of G are adjacent in G_S if and only if at least one of them lies in $V-S$. Thus the adjacency within the vertices of $V-S$ and between the vertices of S and $V-S$ remains unaltered in G_S. Also, two adjacent vertices of G are non-adjacent in G_S if and only if they lie in S. Thus if $G_S \cong G$, then $\langle S\rangle \cong \overline{\langle S\rangle}$.
Suppose, $S \subseteq V$ is such that, $G_S \cong G$ and $\langle S\rangle \cong \overline{\langle S\rangle}$. If no vertex of S is adjacent to any vertex of $V-S$, then $\langle S\rangle$ is a component of G.
Suppose there exist at least two vertices $v_i, v_j \in S$ such that, $A_i = N(v_i)\cap(V-S)$ and $A_j = N(v_j)\cap(V-S)$ and v_i and v_j are adjacent in G (See figure 1).

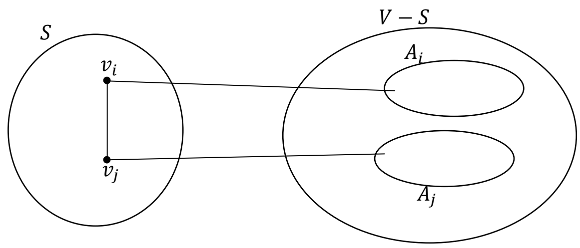

FIGURE 1. Adjacencies in graph G

Since $\langle S\rangle$ is self complementary, there exist two non-adjacent vertices u_i, u_j in S such that, $A_i = N(u_i)\cap(V-S)$ and $A_j = N(u_j)\cap(V-S)$ in G. If $\langle\{v_i, v_j, u_i, u_j\}\rangle$ is not self complementary, then $G_S \ncong G$. Suppose $u_i v_i v_j u_j$ form a path P_4 in G such that u_i, v_i, v_j, u_j are adjacent to the sets A_i, A_i, A_j, A_j respectively in G (See figure 2).

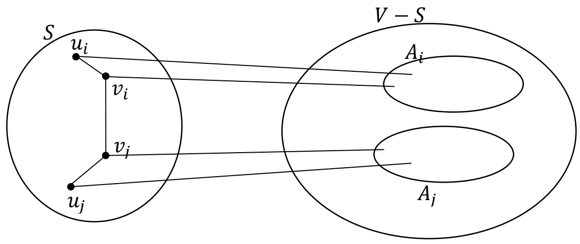

FIGURE 2. Adjacencies in graph G

Then in G_S, $v_iu_ju_iv_j$ forms a path P_4 with v_i,u_j,u_i,v_j adjacent to the sets A_i,A_j,A_i,A_j respectively (See figure 3).

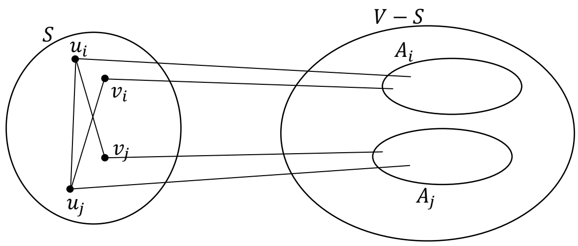

FIGURE 3. Adjacencies in graph G_S

Thus, the adjacency is not preserved in G_S. So, at least $|S|-1$ vertices of S are adjacent to every vertex of a set $A \subseteq V-S$ in G.
Suppose $v \in S$ and $deg_{\langle S\rangle}(v) \neq deg_{\overline{\langle S\rangle}}(v)$ and let $N(v)\cap(V-S) \neq A$. Let $N(v)\cap(V-S)=B$. Since $\langle S\rangle$ is self complementary, there exists a vertex $w \in S$ such that $N(w)\cap(V-S)=B$ in G. Then whenever B is adjacent to two adjacent vertices v,w in G, it is adjacent to the same v,w in G_S wherein they are non-adjacent. Similar is the case when B is adjacent to two non-adjacent vertices in G. Hence $G_S \ncong G$, a contradiction.
Conversely, let $S \subseteq V$ where $\langle S\rangle \cong \overline{\langle S\rangle}$ be such that, $\langle S\rangle$ is a component of G. Then clearly, $G_S \cong G$.
Suppose that at least $|S|-1$ vertices of S are adjacent to every vertex of a set $A \subseteq V-S$. Also, $N(v)\cap(V-S)=A$ for every vertex $v \in S$ with $deg_{\langle S\rangle}(v) \neq deg_{\overline{\langle S\rangle}}(v)$. Then at most one vertex in S, having the same degree in $\langle S\rangle$ and $\overline{\langle S\rangle}$, can be adjacent to vertices outside the set A. Since the degree of this vertex can not change in G_S and $\langle S\rangle \cong \overline{\langle S\rangle}$, we can find a one to one correspondence between the vertices of G and G_S, which preserves the adjacency. Thus $G_S \cong G$. □

Theorem 2.4. *For any graph $G=(V,E)$ of order n, if $G_S \cong G$ for some set $S \subseteq V$ with $|S|=k$, then there is always a set $S_1 \subseteq V$ with $|S_1|=k$ such that $G_{S_1} \ncong G$, except for $k=1,n$.*

Proof. Let $G = (V,E)$ be a disconnected graph such that, $G_S \cong G$ for some set $S \subseteq V$ with $|S| = k$. Then pick any vertex from a component of the graph and $k-1$ vertices from the remaining components of the graph to form the set S_1. Then $G_{S_1} \ncong G$.
Let us assume that G is a connected graph. Let $S \subseteq V$ with $|S| = k$ such that, $G_S \cong G$. Then from Theorem 2.3, $\langle S \rangle$ is self complementary and $V - S$ contains a set A such that at least $k-1$ vertices of S are adjacent to every vertex of A. Now pick $k-1$ vertices from the set S and one vertex from the set A to form the set S_1. Then $\langle S_1 \rangle$ is not self complementary. Hence by the Theorem 2.3, $G_{S_1} \ncong G$. □

Theorem 2.5. *A graph $G = (V,E)$ is S-i.c.c. for a given set $S \subseteq V$ if and only if $S = V$ or $S = V - \{v\}$ for some vertex $v \in V$ such that S is partitioned into two equal partitions S_1 and S_2 such that, $S_1 = N(v) \cap S$ and the vertices of S_1 and S_2 have the same degree sequence in $\langle S \rangle$ of G.*

Proof. Let $G_S \cong \overline{G}$ for a given set $S \subseteq V$. Suppose $V - S \neq \emptyset$ and no vertex of S is adjacent to any vertex of $V - S$ in G. Then G is disconnected and so is G_S. But then $\overline{G}$ is a connected graph, a contradiction.
Suppose $V - S$ contains more than one vertex, say, $u_i, u_j \in V - S$ are the two adjacent vertices in G and hence in G_S. Let $S_i = N(u_i) \cap S$ and $S_j = N(u_j) \cap S$ in G_S. Suppose $S_i \cap S_j \neq \emptyset$ or $S_i \cup S_j \neq S$ or $|S_i| \neq |S_j|$. Then in any case, $deg_{\overline{G}}(u_i) \neq deg_{G_S}(u_i)$. Hence the set S is partitioned into two equal sets S_i and S_j.
Suppose $V - S = \{u_i, u_j\}$. Then the sets S_i and S_j are adjacent to two adjacent vertices u_i and u_j in G_S which are non-adjacent vertices in $\overline{G}$. Thus $G_S \ncong \overline{G}$, a contradiction.
Suppose $S_i \cup S_j = S$ and there exist $u_m, u_n \in V - S$ which are non-adjacent in G_S, such that, u_m is adjacent to $S - S_i$ and u_n is adjacent to $S - S_j$ in G_S. If $\langle \{u_i, u_j, u_m, u_n\} \rangle$ is not self complementary, then $|E(\overline{G})| \neq |E(G_S)|$, a contradiction.
Suppose $\langle \{u_i, u_j, u_m, u_n\} \rangle$ is a self complementary graph. Let us assume that $u_m u_i u_j u_n$ forms a path P_4 of $\langle V - S \rangle$ in G_S such that the vertices u_m, u_i, u_j, u_n are adjacent to the sets $S - S_i, S_i, S_j, S - S_j$ respectively in G_S. Then $u_j u_m u_n u_i$ forms a path P_4 in $\langle V - S \rangle$ of $\overline{G}$ such that the vertices u_j, u_m, u_n, u_i are adjacent to the sets $S - S_j, S_i, S_j, S - S_i$ respectively in $\overline{G}$. This shows that the adjacency between the vertices of G_S and $\overline{G}$ is not preserved and hence $G_S \ncong \overline{G}$, a contradiction.
Thus $V - S$ can have only one vertex, say v. Let $S_1 = N(v) \cap S$ and $S_2 = S - S_1$. If $|S_1| \neq |S_2|$, then $deg_{G_S}(v) \neq deg_{\overline{G}}(v)$, a contradiction. Thus S is partitioned into two equal partitions S_1 and S_2 such that only S_1 is adjacent to v in G.
Suppose the degree sequences of the vertices of S_1 and S_2 are not the same in $\langle S \rangle$ of G. Then note that the vertex v is adjacent to S_1 in G_S but adjacent to S_2 in $\overline{G}$. Hence the degree sequence of the vertices of G_S and $\overline{G}$ will be different. This implies that $G_S \ncong \overline{G}$, a contradiction.
Conversely, let $S = S_1 \cup S_2$ with $S_1 \cap S_2 = \phi$ and $|S_1| = |S_2| = \frac{|S|}{2}$ such that, exactly S_1 is adjacent to $\{v\} = V - S$ in G and hence in G_S. Then in $\overline{G}$, v is adjacent to the

same number of vertices as in G_S. Since the vertices of S_1 and S_2 have the same degree sequence in $\langle S\rangle$ of G and $\langle S\rangle_{G_S} \cong \langle S\rangle_{\overline{G}}$, the adjacency between the vertices of G_S and $\overline{G}$ is preserved. Hence $G_S \cong \overline{G}$. □

Theorem 2.6. *Let $G = (V,E)$ be a graph on n vertices such that, $G_S \cong \overline{G}$ for some set $S \subseteq V$ with $|S| = k$ and $k \neq n$. Then it is always possible to find a set $S' \subseteq V$ with $|S'| = k$ such that, $G_{S'} \ncong \overline{G}$.*

Proof. Let G be a graph on n vertices such that, $G_S \cong \overline{G}$ for some set $S \subseteq V$. Then by Theorem 2.5, $S = V - \{v\}$ and S is partitioned into two equal partitions S_1 and S_2 such that, $S_1 = N(v) \cap S$ and the degree sequences of the vertices of S_1 and S_2 are the same in $\langle S\rangle$ of G. Then there exists at least one vertex $u \in S$ such that $|N(u)| \neq \frac{n-1}{2}$. For otherwise, let every vertex in S be adjacent to $\frac{n-1}{2}$ vertices of G. Then G is a regular graph of degree $\frac{n-1}{2}$. Thus $\overline{G}$ is also a regular graph of degree $\frac{n-1}{2}$. But then $\langle S\rangle$ is not a regular graph which in turn implies that $\overline{\langle S\rangle}$ is not a regular graph. Thus G_S is not a regular graph and hence $G_S \ncong \overline{G}$, a contradiction. Thus there exists at least one vertex $u \in S$ such that, $|N(u)| \neq \frac{n-1}{2}$. Choose $S' = V - \{u\}$ and by Theorem 2.5, $G_{S'} \ncong \overline{G}$. □

Theorem 2.7. *Let $S \subseteq V$ be a set of vertices of a graph $G = (V,E)$. Then $G_S \cong G_{(V-S)}$ if and only if one of the following is true:*

(1) *$\langle S\rangle \cong \langle V-S\rangle$ and if $v \in S$ and $u \in V-S$ are adjacent in G, then $deg_{\langle S\rangle}(v) = deg_{\langle V-S\rangle}(u)$ or*
(2) *$\langle S\rangle$ and $\langle V-S\rangle$ are self complementary graphs such that,*
   i. *there exists no edge between S and $V-S$ in G or*
   ii. *the edges between S and $V-S$ form a complete bipartite graph with partite sets V_1, V_2 such that, V_1 is one among $S, S-\{v\}$ and $\{v\}$ and V_2 is one among $V-S, (V-S)-\{u\}$ and $\{u\}$ where $deg_{\langle S\rangle}(v) = deg_{\overline{\langle S\rangle}}(v)$ and $deg_{\langle V-S\rangle}(u) = deg_{\overline{\langle V-S\rangle}}(u)$.*

Proof. Suppose $\langle S\rangle \cong \langle V-S\rangle$ and let $v \in S$ and $u \in V-S$ be adjacent in G but $deg_{\langle S\rangle}(v) \neq deg_{\langle V-S\rangle}(u)$. Then the degree sequence of G_S and $G_{(V-S)}$ are not equal, a contradiction. Thus the condition 1 holds.

By definition of the induced complement of G with respect to the set S,

$$\langle S\rangle_{G_S} \cong \langle S\rangle_{\overline{G}} \text{ and } \langle V-S\rangle_{G_S} \cong \langle V-S\rangle_G. \tag{2.1}$$

Similarly, with respect to the set $V-S$,

$$\langle V-S\rangle_{G_{(V-S)}} \cong \langle V-S\rangle_{\overline{G}} \text{ and } \langle S\rangle_{G_{(V-S)}} \cong \langle S\rangle_G. \tag{2.2}$$

Since $G_S \cong G_{(V-S)}$, from equations (2.1) and (2.2)

$$\langle S\rangle_{G_S} \cong \langle S\rangle_{\overline{G}} \cong \langle S\rangle_G \text{ and } \langle V-S\rangle_{G_S} \cong \langle V-S\rangle_G \cong \langle V-S\rangle_{\overline{G}}. \tag{2.3}$$

Thus $\langle S\rangle$ and $\langle V-S\rangle$ are self complementary graphs.

Suppose $\langle S\rangle$ and $\langle V-S\rangle$ are two components of G, then there is nothing to prove.

Suppose the edges between S and $V-S$ do not form a complete bipartite graph. Then we can find at least two vertices $v_1, v_2 \in S$ and at least two vertices $u_1, u_2 \in V-S$ such that v_i is not adjacent to both u_1, u_2 for $i = 1,2$ in G. Then in G_S, two adjacent/ non-adjacent vertices $v_1, v_2 \in S$ are adjacent to two adjacent/ non-adjacent vertices $u_1, u_2 \in V-S$. But $v_1, v_2 \in S$ are non-adjacent/ adjacent in $G_{(V-S)}$ and are adjacent to two non-adjacent/ adjacent vertices $u_1, u_2 \in V-S$. Hence the adjacency is not preserved between the vertices of G_S and $G_{(V-S)}$, a contradiction. Thus the edges between S and $V-S$ form a complete bipartite graph with partite sets V_1 and V_2.
Without loss of generality, let us assume that $V_1 \subseteq S$ and $V_2 \subseteq V-S$. Let $v_i, v_j \in S$ such that, v_i, v_j are non-adjacent to any vertex of V_2 and v_i, v_j are adjacent in G_S. Then v_i, v_j are non-adjacent in $G_{(V-S)}$ and are not adjacent to any vertex of V_2. Thus the adjacency is not preserved and hence $G_{(V-S)} \ncong G_S$, a contradiction. Thus $V_1 = S$ or $S - \{v\}$ or $\{v\}$ and $V_2 = V - S$ or $(V-S) - \{u\}$ or $\{u\}$.
Suppose $deg_{\langle S\rangle}(v) \neq deg_{\overline{\langle S\rangle}}(v)$, then we can find a vertex with different degrees in G_S and $G_{(V-S)}$, which is non-adjacent to any vertex of V_2, a contradiction. Thus $deg_{\langle S\rangle}(v) = deg_{\overline{\langle S\rangle}}(v)$. Similarly, $deg_{\langle V-S\rangle}(u) = deg_{\overline{\langle V-S\rangle}}(u)$.
Conversely, if the second condition holds, then $G_S \cong G$ and $G_{(V-S)} \cong G$, and hence $G_S \cong G_{(V-S)}$.
Let v_i, v_j be two adjacent vertices of G_S. Then

Case 1: $v_i, v_j \in S$. Then v_i, v_j are non-adjacent in G and hence in $\langle S\rangle_G$. Since $\langle S\rangle_G \cong \langle V-S\rangle_G$, there exists two adjacency preserving non-adjacent vertices $u_i, u_j \in V-S$. Hence u_i, u_j are adjacent in $G_{(V-S)}$.

Case 2: One of v_i, v_j lies in $V-S$. Then v_i, v_j are adjacent in G and hence in $G_{(V-S)}$.

Case 3: $v_i, v_j \in V-S$. Then one can argue in the lines of case 1 and prove the result.

Thus in all the possibilities, $G_S \cong G_{(V-S)}$. □

Proposition 2.1. *Let G be a graph of order n and let $S \subseteq V$. If $G_S \cong G_{(V-S)}$, then*

(1) $|S| < n-4$ *for* $n > 8$,

(2) $|S| = \frac{n}{2}$ *for n even.*

Proof. Suppose $G_S \cong G_{(V-S)}$. As in Theorem 2.7, let us consider two possibilities.

Case 1: $\langle S\rangle$ and $\langle V-S\rangle$ are self complementary graphs satisfying some conditions. Then $|S|$ and $|V-S|$ must be at least 4. Thus if $|V-S| \geq 4$, then $|S| < n-4$ for $n > 8$.

Case 2: $\langle S\rangle \cong \langle V-S\rangle$ satisfying some conditions. Then note that $|S| = |V-S|$ and hence n must be even. □

Remark 2.1. From the proof of the Proposition 2.1, it may be observed that, the graphs with orders $3,5$ or 7 do not contain a set S of vertices, such that, $G_S \cong G_{(V-S)}$.

Theorem 2.8. *Suppose G is a disconnected graph. Then G_S is connected if and only if S contains at least one vertex from each component of G.*

Proof. Let $G_i = (V_i, E_i)$, $i = 1, 2, ..., k$ be the k components of $G = (V, E)$. Suppose no vertex of a component G_j $(1 \leq j \leq k)$ lies in S. Then $V_j \subseteq V - S$ and hence no vertex of S is adjacent to any vertex of G_j in G and hence in G_S. Thus G_j is a component in G_S, too and hence G_S is disconnected. Thus S contains at least one vertex from each component of G.
Conversely, suppose S contains at least one vertex from each component of G. Then $\langle S \rangle$ is disconnected and $\overline{\langle S \rangle}$ is connected. Also, note that there exists at least one vertex u_i in the component G_i such that u_i is adjacent to some vertex of S. Thus there exists at least one path between any two vertices in G_S and hence G_S is connected. □

Lemma 2.1. *The complement $\overline{G}$ of a connected graph $G = (V, E)$ is disconnected if and only if there exists a set $A \subset V$ such that every vertex of A is adjacent to every vertex of $V - A$.*

Proof. Suppose $\overline{G}$ is disconnected graph, where G is connected. Then there exist at least two components G_1 and G_2 of $\overline{G}$. Since $\overline{(\overline{G})} \cong G$, every vertex of $V(G_1)$ is adjacent to every vertex of $V(G_2)$ in G. Thus $V(G_1)$ is the set A mentioned in the theorem.
Conversely, let every vertex of A be adjacent to every vertex of $V - A$ in G. Then $\langle A \rangle$ is a component of $\overline{G}$. Thus $\overline{G}$ is disconnected. □

Theorem 2.9. *Let $G = (V, E)$ be a connected graph. Then G_S is disconnected if and only if $\langle S \rangle$ is connected and there exists a set $S_1 \subset S$ such that every vertex of S_1 is adjacent to every vertex of $S - S_1$ and $N(S_1) \cap (V - S)$ is not adjacent to $S - S_1$ in G.*

Proof. Suppose G_S is a disconnected graph and assume that $\langle S \rangle$ is disconnected. Then $\overline{\langle S \rangle}$ is connected and hence in G_S, there exists at least one path between every pair of vertices through the vertices of S, a contradiction. So, $\langle S \rangle$ must be connected. Since G is connected, there exists at least one path between the vertices of S and $V - S$ in G. Thus G_S is disconnected if $\overline{\langle S \rangle}$ is disconnected. The complement of a connected graph is disconnected if and only if there exists a set A of vertices which are adjacent to all vertices of $V - A$ (by the Lemma 2.1). So, there exists a set $S_1 \subset S$ such that every vertex of S_1 is adjacent to every vertex of $S - S_1$.
Suppose some vertex $u_i \in N(S_1) \cap (V - S)$ is adjacent to a vertex $v_i \in S - S_1$ in G (and hence in G_S). Then every path between the vertices of S_1 and $S - S_1$ must have u_i and v_i on it in G_S, a contradiction. So, no vertex of $N(S_1) \cap (V - S)$ is adjacent to any vertex of $S - S_1$.
Conversely, suppose $\langle S \rangle$ is connected and there exists a set $S_1 \subset S$ such that every vertex of S_1 is adjacent to every vertex of $S - S_1$. Then S_1 and $S - S_1$ are two components of $\overline{\langle S \rangle}$. Also, note that no vertex of $N(S_1) \cap (V - S)$ is adjacent to any vertex of $S - S_1$. Thus $[N(S_1) \cap (V - S)] \cup S_1$ forms a component of G_S and hence G_S is disconnected. □

3. CONCLUSION

Induced complement of a graph is all about complementing a set of vertices instead of complementing the entire graph. This paper attempts to present some results on S-i.s.c. and S-i.c.c. graphs which are the self complementary graphs with respect to the induced complement of a graph.

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Kiran Kiggal Udayashankar
Maharani's Science College for Women (Autonomous), Mysuru
Department: Electronics
Maharani's Science College for Women (Autonomous)
Mysuru - 570 005, Karnataka, INDIA
e-mail: *kirangsch@gmail.com*
and
Prameela Kolake
Alva's Institute of Engineering and Technology, Moodbidri
Department: Mathematics
Alva's Institute of Engineering and Technology
Moodbidri, Shobhavana Campus
Mijar - 574 225, Karnataka, INDIA
e-mail: *prameela.kolake@gmail.com*