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EXTENDED ALGEBRAIC STRUCTURE OF QUASIMODULES

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ABSTRACT. A Module is one of the common and significant algebraic structures of modern algebra. We have introduced in our paper "An Associated Structure of a Module" published in *Revista de la Academia Canaria de Ciencias*, Volume XXV, 9–22 (2013), the concept of a *quasimodule* which is a generalisation of a module that speaks of a topological hyperspace structure as well as a module structure in some sense. A quasimodule is a conglomeration of semigroup structure, ring multiplication and partial order; this structure always contains a module. In the present paper we shall introduce the concept of an *ideal* in a quasimodule. This concept is completely different from the concept of an ideal in a ring. We shall discuss several properties of ideals and construct the ideal generated by any subset of a quasimodule. We shall define a *minimal ideal* and find a necessary and sufficient condition for a proper ideal to be a minimal ideal.

1. INTRODUCTION

Module theory is a very common and established theory in mathematics. We have introduced in our paper [3] the concept of a *quasimodule* which is a generalisation of a module that speaks of a topological hyperspace structure as well as a module structure in some sense. If M is a topological module over some topological unitary ring R, then the hyperspace $\mathscr{C}(M)$ consisting of all nonempty compact subsets of M enjoys the following properties: For $A,B\in\mathscr{C}(M)$ and $r\in R$, the sets $A+B:=\{a+b : a\in A, b\in B\}, rA:=\{ra : a\in A\}$ are compact. Moreover for $A\subseteq B$, $rA\subseteq rB$, for any $r\in R$. Also if θ is the additive identity of M, then $A-A=\{\theta\}$ iff A is a singleton set. The collection $\big\{\{m\} : m\in M\big\}$ is precisely the set of all minimal elements of $\mathscr{C}(M)$ with respect to the usual set-inclusion as partial order. These properties are axiomatised in a quasimodule. The definition of a quasimodule is as follows.

Definition 1.1. [3] *Let $(X,\leq)$ be a partially ordered set, '$+$' be a binary operation on X [called* addition*] and '$\cdot$': $R\times X\longrightarrow X$ be another composition [called* ring multiplication*, R being a unitary ring]. If the operations and partial order satisfy the following axioms, then $(X,+,\cdot,\leq)$ is called a* quasimodule *(in short* qmod*) over R.*

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$A_1 : (X,+)$ *is a commutative semigroup with identity* θ.
$A_2 : x \leq y\ (x,y \in X) \Rightarrow x+z \leq y+z,\ r\cdot x \leq r\cdot y,\ \forall z \in X, \forall r \in R$.
$A_3 :$ *(i)* $r\cdot(x+y) = r\cdot x + r\cdot y$,
(ii) $r\cdot(s\cdot x) = (rs)\cdot x$,
(iii) $(r+s)\cdot x \leq r\cdot x + s\cdot x$,
(iv) $1\cdot x = x$, *'1' being the multiplicative identity of* R
(v) $0\cdot x = \theta$ *and* $r\cdot\theta = \theta$
$\forall x,y \in X,\ \forall r,s \in R$.
$A_4 : x+(-1)\cdot x = \theta$ *if and only if* $x \in X_0 := \{z \in X : y \nleq z, \forall y \in X \smallsetminus \{z\}\}$.
$A_5 :$ *For each* $x \in X, \exists\, y \in X_0$ *such that* $y \leq x$.

In the above definition the set X_0 is precisely the set of all minimal elements of X with respect to the partial order of X. The elements of the set X_0 are called *primitive elements* of X. In our paper [3] we have shown that this set X_0 is a module over R. Conversely, given any module M over some unitary ring R, a quasimodule X can be constructed such that M is isomorphic with X_0 as module. This is shown in the same paper [3]. In this sense a quasimodule can be considered as a generalisation of module. As the definition shows, the partial order has a significant role in the entire structure; it actually reflects the hyperspace structure within a quasimodule. In our papers [3], [2], [1] and [4], we have introduced various notions like exact sequences, ascending and descending chain conditions, order-morphisms etc., to name a few. Using these notions we have established a number of important theorems in the line of module theory.

In the present paper we shall introduce the concept of an *ideal* in a quasimodule. This concept is completely different from the concept of an ideal in a ring. We shall discuss several properties of ideal and construct the ideal generated by any subset of a quasimodule. We shall define a *minimal ideal* and find a necessary and sufficient condition for a proper ideal to be a minimal ideal.

Before introducing ideals let us first give some examples of qmod for lucid understanding.

Example 1.1. *Let* X *be a topological space and* $C^+(X)$ *be the set of all continuous functions from* X *to* $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$. *We define addition on* $C^+(X)$ *by* $(f+g)(x) := f(x)+g(x)$, *for all* $x \in X$ *and ring multiplication on* $C^+(X)$ *by the ring* $\mathbb{Z}$ *by* $(r\cdot f)(x) := |r|f(x)$, *for all* $x \in X$, *for all* $r \in \mathbb{Z}$. *We define the partial order '*$\leq$*' on* $C^+(X)$ *by* $f \leq g \Leftrightarrow f(x) \leq g(x)$, *for all* $x \in X$. *We now show that* $C^+(X)$ *is a quasimodule over the ring* $\mathbb{Z}$.

$\mathbf{A_1}$: *Clearly* $(C^+(X),+)$ *is a commutative semigroup with identity* θ, *where* $\theta(x) := 0$, *for all* $x \in X$.

A$_2$: $f \leq g \Rightarrow f(x) \leq g(x)$, $\forall x \in X \Rightarrow f(x)+h(x) \leq g(x)+h(x) \Rightarrow f+h \leq g+h$, $\forall h \in C^+(X)$. *Again* $|r|f(x) \leq |r|g(x)$, $\forall r \in \mathbb{Z} \Rightarrow (r \cdot f)(x) \leq (r \cdot g)(x)$, $\forall x \in X$ $\Rightarrow r \cdot f \leq r \cdot g$.

A$_3$: *Let* $f, g \in C^+(X)$ *and* $r, s \in \mathbb{Z}$. *Then*

(i) $(r \cdot (f+g))(x) = |r|(f(x)+g(x)) = |r|f(x)+|r|g(x) = (r \cdot f)(x)+(r \cdot g)(x)$, $\forall x \in X \Rightarrow r \cdot (f+g) = r \cdot f + r \cdot g$.

(ii) $(r \cdot (s \cdot f))(x) = |r||s|f(x) = |rs|f(x) = ((rs) \cdot f)(x)$, $\forall x \in X$. *So* $r \cdot (s \cdot f) = (rs) \cdot f$.

(iii) $((r+s) \cdot f)(x) = |r+s|f(x) \leq (|r|+|s|)f(x) = |r|f(x)+|s|f(x) = (r \cdot f)(x) + (s \cdot f)(x)$, $\forall x \in X$. *Thus* $(r+s) \cdot f \leq r \cdot f + s \cdot f$.

(iv) $(1 \cdot f)(x) = f(x)$, $\forall x \in X \Rightarrow 1 \cdot f = f$.

(v) $(0 \cdot f)(x) = |0|f(x) = 0 = \theta(x)$, $\forall x \in X \Rightarrow 0 \cdot f = \theta$. *Also for any* $r \in \mathbb{Z}$, $(r \cdot \theta)(x) = |r|\theta(x) = 0 = \theta(x)$, $\forall x \in X \Rightarrow r \cdot \theta = \theta$.

A$_4$: $[C^+(X)]_0 = \{g \in C^+(X) : f \nleq g, \forall f \in C^+(X) \setminus \{g\}\} = \{\theta\}$, *since for any* $g \neq \theta$, $\exists y \in X$ *such that* $g(y) > 0 = \theta(y)$ *and hence* $\theta \leq g$ *with* $\theta \neq g$. *Now* $f(x)+((-1) \cdot f)(x) = \theta(x)$, $\forall x \in X \Leftrightarrow 2f(x) = \theta(x) \Leftrightarrow f(x) = \theta(x)$, $\forall x \in X \Leftrightarrow f = \theta$. *Thus* $f - f = \theta \Leftrightarrow f \in [C^+(X)]_0$.

A$_5$: *For each* $f \in C^+(X)$, $\theta(x) = 0 \leq f(x)$, $\forall x \in X \Rightarrow \theta \leq f$, *where* $\theta \in [C^+(X)]_0$. *Therefore* $(C^+(X), +, ., \leq)$ *is a quasimodule over* $(\mathbb{Z}, +, .)$.

Example 1.2. *Let* $(X, \mathscr{A}, \mu)$ *be a measure space and*

$$L_p^+ := \left\{ f : X \to \mathbb{R}^+ \;\middle|\; f \text{ is measurable and } \int_X |f|^p d\mu < \infty \right\}.$$

Then L_p^+ *is a quasimodule over the unitary ring* $(\mathbb{Z}, +, .)$ *with respect to the operations and partial order defined as follows: For* $f, g \in L_p^+$ *and* $r \in \mathbb{Z}$,

(i) $(f+g)(x) := f(x)+g(x)$, $\forall x \in X$,

(ii) $(r \cdot f)(x) := |r|f(x)$, $\forall x \in X$,

(iii) $f \leq g$ $(f, g \in L_p^+)$ *if* $f(x) \leq g(x)$ *almost everywhere on X with respect to* μ *i.e if* $\mu(\{x \in X : f(x) > g(x)\}) = 0$.

Here the important point that should be noted is that two functions $f, g \in L_p^+$ *are considered to be "*same*" if* $f(x) = g(x)$ *almost everywhere on X with respect to* μ *i.e if* $\mu(\{x \in X : f(x) \neq g(x)\}) = 0$. *Such a consideration is reasonable since for such two functions* $f, g \in L_p^+$ *which are equal almost everywhere on X, we have* $\int_X f d\mu = \int_X g d\mu$. *Now clearly, the relation '$\leq$' is reflexive. To show that '$\leq$' is antisymmetric let* $f \leq g$ *and* $g \leq f$. *Then* $\mu(\{x \in X : f(x) > g(x)\}) = 0$ *and* $\mu(\{x \in X : f(x) < g(x)\}) = 0$. *So*

$$\begin{aligned}
&\mu(\{x \in X : f(x) \neq g(x)\}) \\
&= \mu(\{x \in X : f(x) > g(x)\} \cup \{x \in X : f(x) < g(x)\}) \\
&= \mu(\{x \in X : f(x) > g(x)\}) + \mu(\{x \in X : f(x) < g(x)\}) \\
&= 0.
\end{aligned}$$

This justifies that $f(x) = g(x)$ almost everywhere on X with respect to μ $\big($a.e. $[\mu]$, in short$\big)$. Again if $f \leq g$ and $g \leq h$ ($f,g,h \in L_p^+$), then $\mu\big(\{x \in X : f(x) > g(x)\}\big) = 0$ and $\mu\big(\{x \in X : g(x) > h(x)\}\big) = 0$. Since $\{x \in X : f(x) > h(x)\} \subseteq \{x \in X : f(x) > g(x)\} \cup \{x \in X : g(x) > h(x)\}$ we must have

$$\begin{aligned}
&\mu\big(\{x \in X : f(x) > h(x)\}\big) \\
&\leq \mu\big(\{x \in X : f(x) > g(x)\} \cup \{x \in X : g(x) > h(x)\}\big) \\
&\leq \mu\big(\{x \in X : f(x) > g(x)\}\big) + \mu\big(\{x \in X : g(x) > h(x)\}\big) \\
&= 0.
\end{aligned}$$

This shows that the relation '$\leq$' is transitive and hence is a partial order on L_p^+. Then by similar arguments as in the Example 1.1 we can verify all the axioms of quasimodule except the axiom A_4. To verify this let us first determine the minimal elements of L_p^+ with respect to '$\leq$'.

First of all, the zero function $O(x) := 0$, $\forall x \in X$ is a minimal element. If $f \in L_p^+$ be any other element which is different from the zero function i.e $f \neq O$ a.e. $[\mu]$, then $\mu\big(\{x \in X : f(x) \neq 0\}\big) > 0$. So clearly $\frac{1}{2}f \leq f$ but $\frac{1}{2}f \neq f$. This justifies that the zero function O is the only minimal element of L_p^+. In our notation, $[L_p^+]_0 = \{O\}$.

Now $g + (-1) \cdot g = O \Longleftrightarrow g(x) + |-1|g(x) = 0$ a.e. $[\mu] \Longleftrightarrow 2g(x) = 0$ a.e. $[\mu] \Longleftrightarrow \mu\big(\{x \in X : g(x) \neq 0\}\big) = 0 \Longleftrightarrow g = O$ a.e. $[\mu]$. Thus it follows that $g + (-1) \cdot g = O \Longleftrightarrow g \in [L_p^+]_0$.

Axiom A_5 is obvious, since $O \leq f$ for any $f \in L_p^+$. Thus it follows that L_p^+ is a quasimodule over $\mathbb{Z}$.

Example 1.3. *Let X be any set and $\mathcal{P}(X)$ be the power set of X. We know that $(\mathcal{P}(X), \Delta, \bigcap)$ is a commutative ring with unity X, where 'Δ' denotes the symmetric difference $\big[A \Delta B := (A \setminus B) \bigcup (B \setminus A)$, for $A, B \in \mathcal{P}(X)\big]$ and is considered as the "addition" to form the ring structure, whereas '$\bigcap$' is used as the "multiplication".*

Now we consider the set $\mathcal{P}(X)$ again and define "addition" on $P(X)$ as the 'union of two sets' and external ring multiplication by elements of the ring $\mathcal{P}(X)$ as the usual 'intersection of two sets'. Then considering the usual set-inclusion as the partial order we show that $\big(\mathcal{P}(X), \bigcup, \bigcap, \subseteq\big)$ forms a quasimodule over the unitary ring $\big(\mathcal{P}(X), \Delta, \bigcap\big)$.

$\mathbf{A_1}$: *It is easy to check that $\big(\mathcal{P}(X), \bigcup\big)$ is a commutative semigroup with identity $\emptyset$.*

$\mathbf{A_2}$: *Let $A_1 \subseteq A_2$. Then $A_1 \bigcup A_3 \subseteq A_2 \bigcup A_3$ and $P_1 \bigcap A_1 \subseteq P_1 \bigcap A_2$, for any $P_1, A_1, A_2, A_3 \in \mathcal{P}(X)$.*

$\mathbf{A_3}$: *Let $A, A_1, A_2 \in \mathcal{P}(X)$ and P, P_1, P_2 be any three members of the ring $\mathcal{P}(X)$. Then*

(i) $P_1 \bigcap (A_1 \bigcup A_2) = (P_1 \bigcap A_1) \bigcup (P_1 \bigcap A_2)$.

(ii) $P_1 \bigcap (P_2 \bigcap A_1) = (P_1 \bigcap P_2) \bigcap A_1$.

(iii) $(P_1 \Delta P_2) \bigcap A = \big((P_1 \setminus P_2) \bigcup (P_2 \setminus P_1)\big) \bigcap A \subseteq (P_1 \bigcap A) \bigcup (P_2 \bigcap A)$ $\big[\because x \in \big((P_1 \setminus P_2) \bigcup (P_2 \setminus P_1)\big) \bigcap A \Rightarrow$ *either* $x \in (P_1 \setminus P_2) \bigcap A$ *or* $x \in (P_2 \setminus P_1) \bigcap A \Rightarrow$ *either* $x \in P_1 \bigcap A \setminus P_2$ *or* $x \in P_2 \bigcap A \setminus P_1 \Rightarrow x \in (P_1 \bigcap A) \bigcup (P_2 \bigcap A)\big]$.

(iv) $X \bigcap A = A$.
(v) $\emptyset \bigcap A = \emptyset$ *and* $P \bigcap \emptyset = \emptyset$.

A_4 : *Let us first observe that the only minimal element in* $\mathcal{P}(X)$ *with respect to the set-inclusion order is the empty set* $\emptyset$. *So in our notation,* $[\mathcal{P}(X)]_0 = \{\emptyset\}$. *Again in the ring* $\big(\mathcal{P}(X), \Delta, \bigcap\big)$, *the additive inverse of any element* $P \in \mathcal{P}(X)$ *is* P *itself* [$\because P \Delta P = \emptyset$]. *So to fulfill our requirement let us consider the equation* $(X \bigcap A) \bigcup (X \bigcap A) = \emptyset$ *which is equivalent to* $A = \emptyset$. *Thus* $(X \bigcap A) \bigcup (X \bigcap A) = \emptyset \Longleftrightarrow A \in [\mathcal{P}(X)]_0$.
A_5 : *For each* $A \in \mathcal{P}(X)$, $\exists \emptyset \in [\mathcal{P}(X)]_0$ *such that* $\emptyset \subseteq A$.
Thus $\big(\mathcal{P}(X), \bigcup, \bigcap, \subseteq\big)$ *is a quasimodule over the ring* $\big(\mathcal{P}(X), \Delta, \bigcap\big)$.

The following notation and concepts will be needed in the sequel.
Notation: Let X be a quasimodule. For any $A \subseteq X$ we denote
$\uparrow A := \{x \in X : x \geq a$ for some $a \in A\}$ and $\downarrow A := \{x \in X : x \leq a$ for some $a \in A\}$.

Definition 1.2. *[3] A nonempty subset* Y *of a quasimodule* X *over some unitary ring* R *is called a sub quasimodule (subqmod in short) if* Y *itself is a quasimodule with respect to the operations and partial order of* X *being restricted to* Y.

Definition 1.3. *[3] A mapping* $f : X \longrightarrow Y$ *(*X, Y *being two quasimodules over a unitary ring* R*) is called an* order-morphism *if*
(i) $f(x+y) = f(x) + f(y)$, $\forall x, y \in X$
(ii) $f(rx) = rf(x)$, $\forall r \in R$, $\forall x \in X$
(iii) $x \leq y$ $(x, y \in X) \Rightarrow f(x) \leq f(y)$
(iv) $p \leq q$ $\big(p, q \in f(X)\big) \Rightarrow f^{-1}(p) \subseteq \downarrow f^{-1}(q)$ *and* $f^{-1}(q) \subseteq \uparrow f^{-1}(p)$.

An order-morphism is called order-epimorphism if it is surjective; it will be called order-monomorphism if it is injective.

2. Ideal of a quasimodule

In this section we shall introduce the concept of an ideal of a quasimodule. This concept is completely different from the concept of an ideal in ring theory; while in ring theory, an ideal is necessarily a subring, in our case an ideal of a qmod can never be a sub quasimodule if it is to be a proper ideal. We shall introduce the concept of minimal ideal and find a necessary and sufficient condition for a proper ideal to be minimal. Let us start with the definition of an ideal of a quasimodule.

Definition 2.1. *A nonempty subset* I *of a quasimodule* X *(over a unitary ring* R*) is said to be an* ideal *of* X *if, (i)* $X + I \subseteq I$, *(ii)* $\alpha I \subseteq I$, *for all* $\alpha \in R^* \big(\equiv R \smallsetminus \{0\}\big)$ *and (iii)* $\uparrow I = I$.

Result 2.1. Let X be a quasimodule over a unitary ring R and I be an ideal of X. If $\theta \in I$, then $I = X$.

Proof. Clearly $I \subseteq X$ and we are only to show that $X \subseteq I$. Let $x \in X$. Then $x = x + \theta \in X + I \subseteq I$ [since $\theta \in I$]. $\square$

Result 2.2. Any proper ideal of a quasimodule cannot contain any primitive element of the quasimodule.

Proof. Let I be an ideal of a quasimodule X. If I contains any primitive element p (say), then $-p+p \in I$ [since $-p \in X$]. This implies $\theta \in I$. Therefore $I = X$ [by Result 2.1]. So contrapositively, any proper ideal cannot contain any primitive element. $\square$

From the above two results it follows that an ideal cannot be a subqmod, if it is to be proper, as we have mentioned above. Thus a proper ideal I of a qmod X is contained in $X \smallsetminus X_0$. Actually a proper ideal reflects the order structure of the qmod, which is increasing in nature [since $\uparrow I = I$]. Moreover the decreasing set of a proper ideal ($\downarrow I$) is the entire qmod, as the following result shows.

Result 2.3. Let I be a proper ideal of a quasimodule X. Then $\downarrow I = X$.

Proof. Since $\downarrow I \subseteq X$ always, we have only to show that $X \subseteq \downarrow I$. For this let $x \in X$. Then for any $i \in I$ we have $i - i \geq \theta$. This implies $x + i - i \geq x$. So $x \leq (x-i)+i \in X + I \subseteq I$. Therefore $x \in \downarrow I$. Thus $X = \downarrow I$. $\square$

Since any proper ideal I is contained in $X \smallsetminus X_0$, I does not contain any invertible element and hence cannot produce any equivalence relation on X, in contrast to the ideal of a ring. Moreover the order structure present in a proper ideal of a qmod has a significant role in an ideal. Thus a quotient structure on X cannot be constructed using an ideal; this can be done in a different way, as we have shown in our paper [1]. Actually the concept of ideal explores the hyperspace nature of a quasimodule.

In next few results we shall show how this ideal behaves under order-morphism.

Result 2.4. Let $f : X \longrightarrow Y$ (X, Y being two quasimodules over a common unitary ring R) be an order-epimorphism and I be an ideal of X. Then $f(I)$ is an ideal of Y.

Proof. Let $i \in I$ and $y \in Y$. Now for this $y \in Y$, $\exists x \in X$ such that $f(x) = y$ [since f is onto]. Now $y + f(i) = f(x) + f(i) = f(x+i) \in f(X+I) \subseteq f(I)$ [since $X + I \subseteq I$ for, I is an ideal].

Now $\alpha f(i) = f(\alpha i) \in f(\alpha I) \subseteq f(I)$, for all $\alpha \in R^*$, for all $i \in I$ [since $\alpha I \subseteq I$].

We now show that $\uparrow f(I) = f(I)$. For this first of all we note that $I = \uparrow I$. So $f(I) = f(\uparrow I)$. We claim that $f(\uparrow I) = \uparrow f(I)$. In fact, $y \in f(\uparrow I)$ implies $\exists x \in \uparrow I$ such that $y = f(x)$. Now $x \in \uparrow I$ implies $\exists i \in I$ such that $x \geq i$. Therefore $y = f(x) \geq f(i)$ [since f is an order-morphism] and hence $y \in \uparrow f(I)$. Thus $f(\uparrow I) \subseteq \uparrow f(I)$. Conversely let, $z \in \uparrow f(I)$. Then $\exists u \in I$ such that $z \geq f(u)$. Now f being an order-epimorphism we have $u \in f^{-1}(f(u)) \subseteq \downarrow f^{-1}(z)$. So $\exists v \in f^{-1}(z)$ such that $u \leq v$ and hence $v \in \uparrow I$. This implies $z = f(v) \in f(\uparrow I)$. Thus $\uparrow f(I) \subseteq f(\uparrow I)$. So our assertion is justified and the result is proved. $\square$

Remark 2.1. In the course of proof of the above result we have proved that for any order-epimorphism $f : X \longrightarrow Y$ and any subset A of X, $f(\uparrow A) = \uparrow f(A)$.

Result 2.5. Let $f : X \longrightarrow Y$ be an order-epimorphism and I' be an ideal of Y. Then $f^{-1}(I')$ is an ideal of X.

Proof. Let $i' \in f^{-1}(I')$, $x \in X$. Then $f(i') \in I'$. Now $f(x+i') = f(x) + f(i') \in I'$ [since I' is an ideal of Y]. So $x+i' \in f^{-1}(I')$, for all $x \in X$, for all $i' \in f^{-1}(I')$.

Now let $q \in f^{-1}(I')$ and $\alpha \in R^*$. Then $f(q) \in I'$. So $f(\alpha q) = \alpha f(q) \in I'$ implies $\alpha q \in f^{-1}(I')$.

To prove that $\uparrow f^{-1}(I') = f^{-1}(I')$ let us choose $a \in \uparrow f^{-1}(I')$. Then $a \geq b$ for some $b \in f^{-1}(I')$ which implies $f(b) \in I'$. Now $a \geq b$ implies $f(a) \geq f(b)$ which further implies $f(a) \in \uparrow I' = I'$ [since I' is an ideal]. Therefore $a \in f^{-1}(I')$. Thus $\uparrow f^{-1}(I') = f^{-1}(I')$ [since $f^{-1}(I') \subseteq \uparrow f^{-1}(I')$ always]. □

In the course of proof of the above result we have obtained the following.

Proposition 2.1. *If $f : X \to Y$ is an order-epimorphism between two qmods X, Y over an unitary ring R, then for any set $B \subseteq Y$,*
(i) $\uparrow f^{-1}(B) = f^{-1}(\uparrow B)$,
(ii) $\downarrow f^{-1}(B) = f^{-1}(\downarrow B)$.

We shall now discuss an ideal generated by an element of a qmod.

Theorem 2.6. *Let X be a quasimodule over a unitary ring R and $a \in X \smallsetminus X_0$. Then*
$$\mathscr{I}(a) := \left\{ y \in X : y \geq x + \alpha a, x \in X, \alpha \in R^* \right\}$$
is a proper ideal of X containing a.

Proof. Clearly $a \in \mathscr{I}(a)$ [since $a = \theta + 1_R \cdot a$, 1_R being the multiplicative identity in R].

(i) Let $x_1 \in X, i_1 \in \mathscr{I}(a)$. Then there exists $x_2 \in X, \alpha \in R^*$ such that $i_1 \geq x_2 + \alpha a$. So $x_1 + i_1 \geq x_1 + x_2 + \alpha a$. Therefore $x_1 + i_1 \in \mathscr{I}(a)$ [since $x_1 + x_2 \in X$]. Thus $X + \mathscr{I}(a) \subseteq \mathscr{I}(a)$.
(ii) Let $\alpha \in R^*$ and $i \in \mathscr{I}(a)$. Then there exists $x \in X, \beta \in R^*$ such that $i \geq x + \beta a$ which implies $\alpha i \geq \alpha x + \alpha\beta a$. So $\alpha i \in \mathscr{I}(a)$ [since $\alpha\beta \in R^*, \alpha x \in X$]. Therefore $\alpha\mathscr{I}(a) \subseteq \mathscr{I}(a)$.
(iii) Let $i' \in \uparrow \mathscr{I}(a)$. Then $\exists\, i'' \in \mathscr{I}(a)$ such that $i' \geq i''$. So $\exists x \in X$ such that $i'' \geq x + \alpha a$, for some $\alpha \in R^*$ which implies $i' \geq i'' \geq x + \alpha a$ and hence $i' \in \mathscr{I}(a)$.

Thus $\uparrow \mathscr{I}(a) = \mathscr{I}(a)$. So $\mathscr{I}(a)$ is an ideal containing a. Since $\theta \notin \mathscr{I}(a)$, $\mathscr{I}(a)$ is proper. □

Note 2.1. The ideal $\mathscr{I}(a)$ in the quasimodule X is said to be the *ideal generated by* a. It is the smallest ideal containing a. In fact, if I' is another ideal containing a such that $I' \subseteq \mathscr{I}(a)$, then $i \in \mathscr{I}(a)$ implies $i \geq x + \alpha a$, for some $x \in X, \alpha \in R^*$. Again $x + \alpha a \in I'$ [as $a \in I'$ and I' is an ideal]. Therefore $i \in \uparrow I' = I'$ implies $\mathscr{I}(a) \subseteq I'$ and hence $\mathscr{I}(a) = I'$.

Theorem 2.7. *If $i' \leq i''$ $(i', i'' \in X \smallsetminus X_0)$, then $\mathscr{I}(i'') \subseteq \mathscr{I}(i')$.*

Proof. Let $x \in \mathscr{I}(i'')$. Then $x \geq y + \alpha i''$, for some $y \in X, \alpha \in R^*$. Now $i' \leq i''$ implies $\alpha i' \leq \alpha i''$ which again implies $y + \alpha i' \leq y + \alpha i'' \leq x$. Thus $x \in \mathscr{I}(i')$ and hence $\mathscr{I}(i'') \subseteq \mathscr{I}(i')$. □

Theorem 2.8. *Any proper ideal I' of a quasimodule X can be represented as*
$$I' = \bigcup_{x \in I'} \mathscr{I}(x).$$

Proof. Let $x \in I'$. Then $x \in \mathscr{I}(x) \subseteq \bigcup_{i \in I'} \mathscr{I}(i)$.

Conversely, let $y \in \bigcup_{x \in I'} \mathscr{I}(x)$. Then $\exists i' \in I'$ such that $y \in \mathscr{I}(i')$ which implies there exists $a \in X, \alpha \in R^*$ such that $y \geq a + \alpha i'$. Now $a + \alpha i' \in I'$ [since $i' \in I'$ and I' is an ideal]. So $y \in \uparrow I' = I'$ and hence $\bigcup_{x \in I'} \mathscr{I}(x) \subseteq I'$. Thus $\bigcup_{x \in I'} \mathscr{I}(x) = I'$. □

Result 2.9. Let $f : X \longrightarrow Y$ be an order-epimorphism. Then $f(\mathscr{I}(a)) = \mathscr{I}(f(a))$, for all $a \in X \smallsetminus X_0$.

Proof. Let $r_1 \in f(\mathscr{I}(a))$. So $r_1 = f(b)$, for some $b \in \mathscr{I}(a)$ which implies $b \geq x + \alpha a$, for some $x \in X, \alpha \in R^*$. Then $f(b) \geq f(x) + \alpha f(a)$. So $r_1 = f(b) \in \mathscr{I}(f(a))$ and hence $f(\mathscr{I}(a)) \subseteq \mathscr{I}(f(a))$.
Conversely, let $r_2 \in \mathscr{I}(f(a))$. So there exists $y \in Y, \alpha \in R^*$ such that $r_2 \geq y + \alpha f(a)$ which implies $r_2 \geq f(x) + f(\alpha a) = f(x + \alpha a)$ [since f is onto so $y = f(x)$ for some $x \in X$]. This implies $x + \alpha a \in \downarrow f^{-1}(r_2)$ [since f is an order-morphism]. So $x + \alpha a \leq r$, for some $r \in X$ such that $f(r) = r_2$. Therefore $r \in \mathscr{I}(a)$ and hence $r_2 = f(r) \in f(\mathscr{I}(a))$. Thus $\mathscr{I}(f(a)) \subseteq f(\mathscr{I}(a))$. Consequently $f(\mathscr{I}(a)) = \mathscr{I}(f(a))$. □

We know that a nonempty intersection of any number of ideals in a ring is again an ideal, although this is not true in the case of union. However in a quasimodule, arbitrary union as well as nonempty intersection of ideals are again ideals of the qmod, as the following theorems show.

Theorem 2.10. *An arbitrary nonempty intersection of ideals of a quasimodule is again an ideal of the quasimodule.*

Proof. Let $\{I_\alpha : \alpha \in \Lambda\}$ be an arbitrary family of ideals of a quasimodule X (over an unitary ring R) such that $\bigcap_{\alpha \in \Lambda} I_\alpha \neq \emptyset$. Now let $I := \bigcap_{\alpha \in \Lambda} I_\alpha$. Then

(i) $i \in I$ implies $i \in I_\alpha$, for all $\alpha \in \Lambda$. So $x + i \in I_\alpha$, for all $x \in X$, for all $\alpha \in \Lambda$ [since I_α is an ideal for all α]. Therefore $x + i \in \bigcap_{\alpha \in \Lambda} I_\alpha = I$, for all $x \in X$ and hence $X + I \subseteq I$.

(ii) Let $i \in I$. Then $i \in I_\alpha$, for all $\alpha \in \Lambda$. So $ri \in I_\alpha$, for all $r \in R^*$, for all $\alpha \in \Lambda$. Therefore $ri \in \bigcap_{\alpha \in \Lambda} I_\alpha = I$, for all $r \in R^*$ and hence $rI \subseteq I$.

(iii) Let $i' \in \uparrow I$. Then $i' \geq i''$, for some $i'' \in I$. This implies $i'' \in I_\alpha$, for all α. So $i' \in \uparrow I_\alpha = I_\alpha$, for all $\alpha \in \Lambda$. Therefore $i' \in \bigcap_{\alpha\in\Lambda} I_\alpha = I$ and hence $\uparrow I = I$.

Thus I is an ideal of X. □

Theorem 2.11. *An arbitrary union of ideals of a quasimodule is also an ideal of the quasimodule.*

Proof. Let $\{I_\alpha : \alpha \in \Lambda\}$ be a nonempty collection of ideals of a quasimodule X (over some unitary ring R) and $I := \bigcup_{\alpha\in\Lambda} I_\alpha$. Then

(i) $i \in I$ implies $i \in I_\beta$, for some $\beta \in \Lambda$. Since I_β is an ideal so $x + i \in I_\beta$, for all $x \in X$. This implies $x + i \in \bigcup_{\alpha\in\Lambda} I_\alpha = I$, for all $x \in X$ and hence $X + I \subseteq I$.

(ii) Let $i \in I$. Then $i \in I_\beta$, for some $\beta \in \Lambda$. Since I_β is an ideal so $ri \in I_\beta$, for all $r \in R^*$. Therefore $ri \in \bigcup_{\alpha\in\Lambda} I_\alpha = I$, for all $r \in R^*$ and hence $rI \subseteq I$.

(iii) Let $i' \in \uparrow I$. Then $i' \geq i''$, for some $i'' \in I$. So $i'' \in I_\beta$, for some $\beta \in \Lambda$. Therefore $i' \in \uparrow I_\beta = I_\beta \subseteq \bigcup_{\alpha\in\Lambda} I_\alpha = I$ and hence $\uparrow I = I$.

Thus I is an ideal of X. □

We shall now introduce the concept of a minimal ideal in a quasimodule.

Definition 2.2. *A proper ideal I of a quasimodule X is said to be a* minimal ideal *of X if there does not exist any proper ideal I' of X such that $I' \subseteq I$.*

The following theorem is a characterisation of a proper ideal to be a minimal ideal.

Theorem 2.12. *A proper ideal I' is a minimal ideal of a quasimodule X if and only if $I' = \mathscr{I}(x)$, for all $x \in I'$.*

Proof. Let I' be a minimal ideal of X and $x \in I'$. Then from the Note 2.1 we see that $\mathscr{I}(x) \subseteq I'$. Also I' being minimal we have $I' \subseteq \mathscr{I}(x)$. Therefore $I' = \mathscr{I}(x)$.

Conversely, let $I' = \mathscr{I}(x)$, for all $x \in I'$. If possible let I'' be another ideal of X such that $I'' \subseteq I'$. Then $i'' \in I''$ implies $i'' \in I'$. So $I' = \mathscr{I}(i'')$, by hypothesis. Now $\mathscr{I}(i'')$ is the smallest ideal containing i'', by Note 2.1. So $\mathscr{I}(i'') \subseteq I''$ which implies $I' \subseteq I''$. Thus $I' = I''$. This justifies that I' is a minimal ideal of X. □

Theorem 2.13. *Let X be a quasimodule over a unitary ring R and $A \subset X \smallsetminus X_0$. Then*

$$\mathscr{I}(A) := \left\{ y \in X : y \geq x + \sum_{i=1}^{n} r_i a_i, r_i \in R^*, a_i \in A, \forall i, x \in X, n \in \mathbb{N} \right\}$$

is a proper ideal of X containing A.

Proof. Let $a \in A$. Then $a = \theta + 1 \cdot a$ implies $a \in \mathscr{I}(A)$ and hence $A \subseteq \mathscr{I}(A)$.

(i) Let $x \in X, y \in \mathscr{I}(A)$. Then $\exists x' \in X$ such that $y \geq x' + \sum_{i=1}^{n} r_i a_i$, where $r_i \in R^*$ and $a_i \in A$ for all i. So for all $x \in X$, $x + y \geq x + x' + \sum_{i=1}^{n} r_i a_i$ which implies $x + y \in \mathscr{I}(A)$ [since $x + x' \in X$] and hence $X + \mathscr{I}(A) \subseteq \mathscr{I}(A)$.

(ii) Let $r \in R^*$ and $y \in \mathscr{I}(A)$. Then $\exists x \in X$ such that $y \geq x + \sum_{i=1}^{n} r_i a_i$, where $r_i \in R^*$ and $a_i \in A$ for all i. This implies $ry \geq rx + \sum_{i=1}^{n} (rr_i) a_i$. This shows that $ry \in \mathscr{I}(A)$. Thus $r\mathscr{I}(A) \subseteq \mathscr{I}(A)$.

(iii) Let $p \in \uparrow \mathscr{I}(A)$. Then $\exists q \in \mathscr{I}(A)$ such that $p \geq q$. So $\exists x \in X$ such that $q \geq x + \sum_{i=1}^{n} r_i a_i$, where $r_i \in R^*$ and $a_i \in A$ for all i. Therefore $p \geq q \geq x + \sum_{i=1}^{n} r_i a_i$ which implies $p \in \mathscr{I}(A)$. This justifies that $\uparrow \mathscr{I}(A) = \mathscr{I}(A)$.

Thus in view of (i), (ii) and (iii) we can say that $\mathscr{I}(A)$ is an ideal containing A. Since $\theta \notin \mathscr{I}(A)$, $\mathscr{I}(A)$ is proper. □

Theorem 2.14. *Let X be a quasimodule over a unitary ring R and $A \subset X \smallsetminus X_0$. Then*
$$\mathscr{I}(A) = \bigcup_{a \in A} \mathscr{I}(a).$$

Proof. Let $a' \in \mathscr{I}(A)$. Then there exist $x \in X, a_1, \ldots, a_n \in A, r_1, \ldots, r_n \in R^*$ such that $a' \geq x + \sum_{i=1}^{n} r_i a_i = x + \sum_{i=1}^{n-1} r_i a_i + r_n a_n$. So $a' \in \mathscr{I}(a_n)$ $\left[\text{since } x + \sum_{i=1}^{n-1} r_i a_i \in X\right]$ and hence $a' \in \bigcup_{a \in A} \mathscr{I}(a)$. Therefore $\mathscr{I}(A) \subseteq \bigcup_{a \in A} \mathscr{I}(a)$.

Conversely, let $a'' \in \bigcup_{a \in A} \mathscr{I}(a)$. Then $\exists a \in A$ such that $a'' \in \mathscr{I}(a)$. So $a'' \geq x + ra$, for some $x \in X, r \in R^*$. Therefore $a'' \in \mathscr{I}(A)$ and hence $\bigcup_{a \in A} \mathscr{I}(a) \subseteq \mathscr{I}(A)$.

Thus $\mathscr{I}(A) = \bigcup_{a \in A} \mathscr{I}(a)$. □

In [3] we have proved that the direct product of an arbitrary family of qmods is again a qusimodule. We shall show below that such a direct product of ideals is also an ideal. To show this let us first describe the direct product of quasimodules.

Definition 2.3. *[3] Let $\{X_\mu : \mu \in \Lambda\}$ be an arbitrary family of quasimodules over a unitary ring R. Let $X := \prod_{\mu \in \Lambda} X_\mu$ be the Cartesian product of X_μ's defined as: $x \in X$ if and only if $x : \Lambda \longrightarrow \bigcup_{\mu \in \Lambda} X_\mu$ is a map such that $x(\mu) \in X_\mu$, for all $\mu \in \Lambda$. Then by*

the axiom of choice we know that X is nonempty, since Λ is nonempty and each X_μ contains at least the additive identity θ_μ (for instance).
Let us denote $x_\mu := x(\mu)$, for all $\mu \in \Lambda$. Also we write each $x \in X$ as $x = (x_\mu)$, where $x_\mu = p_\mu(x)$, $p_\mu : X \longrightarrow X_\mu$ being the projection map, for all $\mu \in \Lambda$. Now we define addition, ring multiplication and partial order as follows: for $x = (x_\mu), y = (y_\mu) \in X$ and $r \in R$, (i) $x+y := (x_\mu + y_\mu)$; (ii) $r \cdot x := (rx_\mu)$; (iii) $x \le y$ if $x_\mu \le y_\mu$, for all $\mu \in \Lambda$.

Theorem 2.15. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of quasimodules over a common unitary ring R, where Λ is some index set and I_α be an ideal of X_α, for each $\alpha \in \Lambda$. Then the direct product $I := \prod_{\alpha\in\Lambda} I_\alpha$ is an ideal of the direct product $X := \prod_{\alpha\in\Lambda} X_\alpha$ of those quasimodules.*

Proof. We have discussed in our paper [3] that this arbitrary product $X := \prod_{\alpha\in\Lambda} X_\alpha$ is a quasimodule over R with respect to the operations and partial order defined above. Any point x of X can be represented as $x = (x_\alpha)$, where $x_\alpha \in X_\alpha$, for all α. We now show that I is an ideal of X.

(i) Let $x = (x_\alpha) \in X$ and $i = (i_\alpha) \in I$. Then $x + i = (x_\alpha + i_\alpha) \in I$, since $x_\alpha + i_\alpha \in I_\alpha$, for all $\alpha \in \Lambda$ as each I_α is an ideal. Thus $X + I \subseteq I$.

(ii) Let $i = (i_\alpha) \in I$ and $r \in R^*$. Then $r \cdot i = (ri_\alpha) \in I$, since $ri_\alpha \in I_\alpha$, for all $\alpha \in \Lambda$. So $rI \subseteq I$, for all $r \in R^*$.

(iii) Since $I \subseteq \uparrow I$ we are only to show that $\uparrow I \subseteq I$. For this let $x \in \uparrow I$. Then there exists $i \in I$ such that $x \ge i$. If $x = (x_\alpha)$ and $i = (i_\alpha)$, then we have $x_\alpha \ge i_\alpha$, for all $\alpha \in \Lambda$ [by the definition of the partial order in the product space, as defined in Definition 2.3]. This shows that $x_\alpha \in \uparrow I_\alpha = I_\alpha$, for all α [since each $i_\alpha \in I_\alpha$ and I_α is an ideal of X_α]. So $x = (x_\alpha) \in I$. Consequently $\uparrow I \subseteq I$.

In view of (i), (ii) and (iii) it follows that I is an ideal of X. □

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