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COMPLEX JACOBSTHAL NUMBERS IN TWO DIMENSION

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ABSTRACT. In this paper, we present a new approach to the generalization of Jacobsthal sequences to the complex plane. It is shown that the Jacobsthal numbers are generalized to two dimensions. For special entries of this new sequence, some relations to the classical Jacobsthal sequences are constructed. Binet formula, the generating function, the explicit closed formula, the sum formula for the new two dimensional Gaussian Jacobsthal sequence are investigated. The relation with classical Jacobsthal Lucas numbers and two dimensional Gaussian Jacobsthal numbers are obtained by using Binet formula. From matrix algebra, the matrix representation of two dimensional Gaussian Jacobsthal sequences is obtained.

1. INTRODUCTION AND PRELIMINARIES

Gaussian numbers are complex numbers $z = a + ib$, $a, b \in \mathbb{Z}$ with $i^2 = -1$ and were investigated by Gauss in 1832. In [8], Horadam introduced the concept the complex Fibonacci number called as the Gaussian Fibonacci number. Jordan considered two of the complex Fibonacci sequences and extented some relationships which are known about the common Fibonacci sequences in [10]. In [3] Berzsenyi, presented a natural manner of extension of the Fibonacci numbers into the complex plane and obtained some interesting identities for the classical Fibonacci numbers. Moreover, the author gave a closed form to Gaussian Fibonacci numbers by the Fibonacci Q matrix. In [7], Harman gave an extension of Fibonacci numbers into the complex plane and generalized the methods given by Horadam (1963); Berzsenyi (1977). In [15], Pethe and Horadam studied generalized Gaussian Fibonacci numbers. In [7], Ascı, Gurel described the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers. Then, they generalized the results for the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers to Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials in [2]. Similarly, Halici, Oz generalized Gaussian Pell numbers to Gaussian Pell polynomials in [6]. Soykan studied summing formulas for generalized Fibonacci and gaussian generalized Fibonacci numbers in [16]. The authors studied a new families of Gauss k-Jacobsthal numbers and Gauss k -Jacobsthal-

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Lucas numbers and their polynomials in [13]. Gauss third-order Jacobsthal numbers and their applications were studied in [4]. Tasci found some properties of Gaussian Padovan and Gaussian Pell-Padovan sequences in [17]. Kartal gave the definitions of Gaussian Padovan and Gaussian Perrin numbers in [12]. The authors established the properties of complex k-Horadam and Gaussian k-Horadam sequences in [12]. Kaplan, Ozturk studied Gaussian quadra Fibona-Pell sequence in [11]. The authors investigated new families of Gauss (k, t)-Horadam numbers in [18]. Ozkan, Uysal denoted d-Gaussian Jacobsthal, d-Gaussian Jacobsthal-Lucas polynomials and their matrix representations.

The classic Jacobsthal and Jacobsthal Lucas sequences are defined by the second order homogeneous linear recurrence relations for $n \in \mathbb{Z}$, respectively in [9] $J_n = J_{n-1} + 2J_{n-2},\ \ J_0 = 0,\ \ J_1 = 1$ and $C_n = C_{n-1} + 2C_{n-2},\ \ C_0 = 2,\ C_1 = 1$ for $n \geq 2$, respectively.

The elements of the classic Jacobsthal and Jacobsthal Lucas numbers with negative subscript are defined as

$$J_{-n} = \frac{(-1)^{n+1}}{2^n} J_n, \quad C_{-n} = \frac{(-1)^n}{2^n} C_n.$$

The authors, in [1], defined the Gaussian Jacobsthal $\{GJ_n\}_{n=0}^{\infty}$ sequence by the following recurrence relation

$$GJ_n = GJ_{n-1} + 2GJ_{n-2},\ \ GJ_0 = \frac{i}{2},\ \ GJ_1 = 1.$$

It can be easily seen that

$$GJ_n = J_n + iJ_{n-1}.$$

The Gaussian Jacobsthal Lucas sequence $\{GC_n\}_{n=0}^{\infty}$ is defined by the following recurrence relation

$$GC_n = GC_{n-1} + 2GC_{n-2},\ \ GC_0 = 2 - \frac{i}{2},\ \ GC_1 = 1 + 2i.$$

Similarly, it is obtained that

$$GC_n = C_n + iC_{n-1}.$$

2. Two Dimensional Gaussian Jacobsthal and Gaussian Jacobsthal Lucas Sequences

Definition 2.1. *Let n, m be any integers. Two dimensional Gaussian Jacobsthal sequence is defined by the following recurrence relations*

$$G(n+2,m) = G(n+1,m) + 2G(n,m),$$
$$G(n,m+2) = G(n,m+1) + 2G(n,m),$$

$$G(0,0) = 0, \quad G(1,0) = 1, \quad G(0,1) = i, \quad G(1,1) = 1+i.$$

If we use Definition 1 for $m = 0$ *and* $n = 0$ *respectively, we get*

$$G(n+2,0) = G(n+1,0) + 2G(n,0),$$
$$G(0,m+2) = G(0,m+1) + 2G(0,m).$$

By initial conditons and the induction method, it is easily obtained that

$$G(n,0) = J_n, \tag{2.1}$$
$$G(0,m) = iJ_m. \tag{2.2}$$

Theorem 2.1. *Let n be any integer, then we get*

$$G(n,1) = J_n G(1,1) + 2J_{n-1} G(0,1)$$

and also we can show in another way that

$$G(n,1) = J_n + iJ_{n+1}. \tag{2.3}$$

Proof. The induction method is used to prove this theorem. For $n = 1$, we have

$$G(1,1) = J_1(1+i) + 2iJ_0.$$

Suppose that the claim is true for $n \leq k$, so

$$G(k,1) = J_k G(1,1) + 2J_{k-1} G(0,1).$$

It is showed that the claim is true for $n = k+1$,

$$\begin{aligned} G(k+1,1) &= G(k,1) + 2G(k-1,1) \\ &= J_k G(1,1) + 2J_{k-1} G(0,1) + 2\left[J_{k-1} G(1,1) + 2J_{k-2} G(0,1)\right] \\ &= G(0,1)\left[2J_{k-1} + 4J_{k-2}\right] + G(1,1)\left[2J_{k-1} + J_k\right] \\ &= 2J_k G(0,1) + J_{k+1} G(1,1). \end{aligned}$$

□

Theorem 2.2. *Let m be any integer, then the following properties are obtained:*

$$G(1,m) = J_m G(1,1) + 2J_{m-1} G(1,0),$$
$$G(1,m) = J_{m+1} + iJ_m. \tag{2.4}$$

Proof. Let us use the induction method to prove this theorem. For $m = 1$,

$$G(1,1) = J_1(1+i) + 2J_0.$$

Suppose that the claim is true for $m \leq k$, so

$$G(1,k) = J_k G(1,1) + 2J_{k-1} G(1,0).$$

It is showed that the claim is true for $m = k+1$

$$\begin{aligned} G(1,k+1) &= G(1,k) + 2G(1,k-1) \\ &= J_k(1+i) + 2J_{k-1} + 2\left[J_{k-1}(1+i) + 2J_{k-2}\right] \\ &= (1+i)J_{k+1} + 2J_k. \end{aligned}$$

□

Theorem 2.3. *Let n,m be any integers, then the following is obtained*

$$G(n,m) = J_m G(n,1) + 2J_{m-1} G(n,0).$$

Proof. We use the induction method to prove this theorem. For $m = 1$,

$$G(n,1) = J_1 G(n,1) + 2J_0 G(n,0).$$

Suppose that the claim is correct for $k \leq m$, so

$$G(n,k) = J_k G(n,1) + 2J_{k-1} G(n,0).$$

It is showed that the claim is true for $m = k+1$.

$$\begin{aligned} G(n,k+1) &= G(n,k) + 2G(n,k-1) \\ &= J_k G(n,1) + 2J_{k-1} G(n,0) + 2\left[J_{k-1} G(n,1) + 2J_{k-2} G(n,0)\right] \\ &= J_{k+1} G(n,1) + 2J_k G(n,0). \end{aligned}$$ □

In the following corollary, we give another formula for $G(n,m)$.

Corollary 2.1. *Let n,m be any integers, then we get a relation between the two dimensional Gaussian Jacobsthal sequence and the Jacobsthal sequence:*

$$G(n,m) = J_{m+1} J_n + iJ_m J_{n+1}. \tag{2.5}$$

Proof. By Theorem 2.2, we have

$$\begin{aligned} G(n,m) &= J_m G(n,1) + 2J_{m-1} G(n,0) \\ &= J_m (J_n + iJ_{n+1}) + 2J_{m-1} J_n \\ &= J_{m+1} J_n + iJ_m J_{n+1}. \end{aligned}$$ □

Corollary 2.2. *In the following formula we get a relation between the two dimensional Gaussian Jacobsthal sequence and the Jacobsthal Lucas sequence:*

$$G(n,m) + G(m,n) = \frac{2C_{m+n+1} - (-2)^m C_{n-m}}{9}(1+i).$$

Proof. By the property of Jacobsthal numbers in [9], we get

$$J_{m+1} J_n + J_m J_{n+1} = \frac{2C_{m+n+1} - (-2)^m C_{n-m}}{9}.$$

Then adding both sides of the following equalities, the proof is easily obtained:

$$\begin{aligned} G(n,m) &= J_{m+1} J_n + iJ_m J_{n+1} \\ G(m,n) &= J_{n+1} J_m + iJ_n J_{m+1}. \end{aligned}$$

It is seen that the commutative property is not satisfied by $G(n,m)$. □

Theorem 2.4. *The elements of two dimensional Gaussian Jacobsthal sequence with negative indices are demonstrated by*

$$G(-n,-m) = \left(-\frac{1}{2}\right)^{n+m-1} G(m-1,n-1).$$

Proof. By equality (2.5), we get

$$\begin{aligned} G(-n,-m) &= J_{-m+1}J_{-n} + iJ_{-m}J_{-n+1} \\ &= \frac{(-1)^{n+1}}{2^n}J_n\frac{(-1)^m}{2^{m-1}}J_{m-1} + i\frac{(-1)^n}{2^{n-1}}J_{n-1}\frac{(-1)^{m-1}}{2^m}J_m \\ &= \frac{(-1)^{m+n+1}(J_nJ_{m-1} + iJ_{n-1}J_m)}{2^{m+n-1}}. \end{aligned}$$

□

Corollary 2.3. *The following results are satisfied by the two dimensional Gaussian Jacobsthal sequence*

$$\begin{aligned} G(n+1,m+1) &= J_{n+1}J_{m+1}(1+i) + 2G(m,n) \\ G(n+2,m+2) &= (1+i)(3J_{n+1}J_{m+1} + 4J_nJ_m + 6G(n,m) + 2G(m,n) \\ G(n+2,m+2) &= 6G(n+1,m+1) + 2G(m+1,n+1) \\ &\quad + 3(1+i)J_{n+2}J_{m+2} + 4(1+i)J_{n+1}J_{m+1}. \end{aligned}$$

Proof. By equality (2.5), we have

$$\begin{aligned} G(n+1,m+1) &= J_{n+1}J_{m+2} + iJ_{n+2}J_{m+1} \\ &= J_{n+1}(J_{m+1} + 2J_m) + iJ_{m+1}(J_{n+1} + 2J_n) \\ &= J_{n+1}J_{m+1}(1+i) + 2J_{n+1}J_m + 2iJ_nJ_{m+1} \\ &= J_{n+1}J_{m+1}(1+i) + 2G(m,n) \end{aligned}$$

The other equalities are established in a similar way. □

Corollary 2.4. *By Definition 2.1, it is obtained that*

$$\begin{aligned} G(n,m) &= G(n-1,m) + 2G(n-2,m) \\ &= G(n-1,m-1) + 2G(n-1,m-2) + 2G(n-2,m-1) + 4G(n-2,m-2). \end{aligned}$$

Theorem 2.5. ***(Generating Function)*** *The generating function for the two dimensional Gaussian Jacobsthal sequence is given by*

$$G(x,y) = \sum_{m,n=0}^{\infty} G(n,m)x^ny^m = \frac{x + iy + (1+i)xy + \frac{2ixy^3}{1-y-2y^2} + \frac{2yx^3}{1-x-2x^2}}{1 - 4xy - 2x^2y - 2xy^2 - x^2y^2}.$$

Proof. By [9], we know that $\sum_{n=0}^{\infty} J_nx^n = \frac{x}{1-x-2x^2}$. By Corollary 2.4 and this property, we obtain

$$\begin{aligned} G(x,y) &= G(0,0) + G(1,0)x + G(0,1)y + G(1,1)xy + \sum_{m,n=2}^{\infty} G(n,m)x^ny^m, \\ -xyG(x,y) &= -xyG(0,0) - \sum_{m,n=2}^{\infty} G(n-1,m-1)x^ny^m, \\ -4x^2y^2G(x,y) &= -4\sum_{m,n=2}^{\infty} G(n-2,m-2)x^ny^m, \end{aligned}$$

and

$$
\begin{aligned}
-2x^2yG(x,y) &= -2\sum_{n=2}^{\infty} G(n-2,0)x^n y - 2\sum_{m,n=2}^{\infty} G(n-2,m-1)x^n y^m \\
&= -2y\sum_{n=2}^{\infty} J_{n-2}x^n - 2\sum_{m,n=2}^{\infty} G(n-2,m-1)x^n y^m \\
&= \frac{-2yx^3}{1-x-2x^2} - 2\sum_{m,n=2}^{\infty} G(n-2,m-1)x^n y^m
\end{aligned}
$$

and similarly

$$
\begin{aligned}
-2xy^2G(x,y) &= -2\sum_{m=2}^{\infty} G(0,m-2)xy^m - 2\sum_{m,n=2}^{\infty} G(n-1,m-2)x^n y^m \\
&= -2ix\sum_{m=2}^{\infty} J_{m-2}x^n - 2\sum_{m,n=2}^{\infty} G(n-1,m-2)x^n y^m \\
&= \frac{-2ixy^3}{1-y-2y^2} - 2\sum_{m,n=2}^{\infty} G(n-1,m-2)x^n y^m.
\end{aligned}
$$

By the above equalities, it is established that

$$
\begin{aligned}
&\left[1-xy-2x^2y-2xy^2-4x^2y^2\right]G(x,y) \\
&= x+iy+(1+i)xy - \frac{2ixy^3}{1-y-2y^2} - \frac{2yx^3}{1-x-2x^2}.
\end{aligned}
$$

□

Theorem 2.6. ***(Binet Formula)*** *The Binet Formula for the two dimensional Gaussian Jacobsthal sequence is given as*

$$
\begin{aligned}
G(n,m) &= \frac{2^{m+n+1}+(-1)^{m+n+1}-(-2)^n(2^{m-n+1}+(-1)^{m-n+1})}{9} \\
&\quad + i\frac{2^{m+n+1}+(-1)^{m+n+1}-(-2)^{n+1}(2^{m-n+1}+(-1)^{m-n+1})}{9}.
\end{aligned}
$$

Proof. By equality (2.5) and the Binet formula for the Jacobsthal sequence, it is obtained that

$$
\begin{aligned}
G(n,m) &= J_{m+1}J_n + iJ_mJ_{n+1} \\
&= \frac{2^{m+1}-(-1)^{m+1}}{3}\frac{2^n-(-1)^n)}{3} + i\frac{2^m-(-1)^m}{3}\frac{2^{n+1}-(-1)^{n+1})}{3}.
\end{aligned}
$$

□

Corollary 2.5. *By the Binet Formula for the two dimensional Gaussian Jacobsthal sequence, the following relation between Jacobsthal Lucas numbers and two dimensional Gaussian Jacobsthal numbers is obtained:*

$$
G(n,m) = \frac{C_{m+n+1}-(-2)^nC_{m-n+1}}{9} + i\frac{C_{m+n+1}-(-2)^mC_{n-m+1}}{9}.
$$

Theorem 2.7. *The explicit closed formula for the two dimensional Gaussian Jacobsthal sequence is in the following*

$$\begin{aligned} G(n,m) = \frac{1}{9} \sum_{k=0}^{\lfloor \frac{m+n+1}{2} \rfloor} \frac{m+n+1}{m+n+1-k} \binom{m+n+1-k}{k} 2^k (1+i) \\ - \frac{(-2)^n}{9} \sum_{k=0}^{\lfloor \frac{m-n+1}{2} \rfloor} \frac{m-n+1}{m-n+1-k} \binom{m-n+1-k}{k} 2^k \\ - \frac{i(-2)^m}{9} \sum_{k=0}^{\lfloor \frac{m-n+1}{2} \rfloor} \frac{n-m+1}{n-m+1-k} \binom{n-m+1-k}{k} 2^k. \end{aligned}$$

Proof. The explicit closed formula for Jacobsthal Lucas numbers is $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} 2^k$. Then by Corallary 2.5, we get

$$\begin{aligned} G(n,m) &= \frac{C_{m+n+1} - (-2)^n C_{m-n+1}}{9} + i \frac{C_{m+n+1} - (-2)^m C_{n-m+1}}{9} \\ &= \frac{\sum_{k=0}^{\lfloor \frac{m+n+1}{2} \rfloor} \frac{m+n+1}{m+n+1-k} \binom{m+n+1-k}{k} 2^k - (-1)^n \sum_{k=0}^{\lfloor \frac{m-n+1}{2} \rfloor} \frac{m-n+1}{m-n+1-k} \binom{m-n+1-k}{k} 2^{k+n}}{9} \\ &+ i \frac{\sum_{k=0}^{\lfloor \frac{m+n+1}{2} \rfloor} \frac{m+n+1}{m+n+1-k} \binom{m+n+1-k}{k} 2^k - (-1)^m \sum_{k=0}^{\lfloor \frac{m-n+1}{2} \rfloor} \frac{m-n+1}{n-m+1-k} \binom{n-m+1-k}{k} 2^{k+m}}{9} \\ &= \frac{\sum_{k=0}^{\lfloor \frac{m+n+1}{2} \rfloor} \frac{m+n+1}{m+n+1-k} \binom{m+n+1-k}{k} 2^k (1+i)}{9} - \frac{(-2)^n \sum_{k=0}^{\lfloor \frac{m-n+1}{2} \rfloor} \frac{m-n+1}{m-n+1-k} \binom{m-n+1-k}{k} 2^k}{9} \\ &- \frac{i(-2)^m \sum_{k=0}^{\lfloor \frac{m-n+1}{2} \rfloor} \frac{n-m+1}{n-m+1-k} \binom{n-m+1-k}{k} 2^k}{9}. \end{aligned}$$

□

Theorem 2.8. ***(Sum Formula)*** *The sum formula for two dimensional Gaussian Jacobsthal sequence is*

$$\begin{aligned} \sum_{k=0}^{n} \sum_{j=0}^{m} G(j,k) &= \left[\frac{c_{m+n+5} - c_{m+4} - c_{n+4} + 7}{36} \right] (1+i) \\ &+ \frac{2^{m+2}[(-1)^{n+1} - 1] + (-1)^{m+n} - (-1)^{m+1} - (-1)^{n+1} + 1}{-18} (1+i) \\ &+ \left[\frac{((-1)^{n+1} - 1)(-1+4i)}{-18} \right] \end{aligned}$$

Proof. By Corollary 2.5, we get

$$
\begin{aligned}
G(n,m) &= J_{m+1}J_n + iJ_mJ_{n+1} \\
&= \frac{2^{m+1}-(-1)^{m+1}}{3}\frac{2^n-(-1)^n}{3} + i\frac{2^m-(-1)^m}{3}\frac{2^{n+1}-(-1)^{n+1}}{3}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{n}\sum_{j=0}^{m} G(j,k) &= \sum_{k=0}^{n}\sum_{j=0}^{m}\left[\frac{2^{k+1}-(-1)^{k+1}}{3}\frac{2^j-(-1)^j}{3} + i\frac{2^k-(-1)^k}{3}\frac{2^{j+1}-(-1)^{j+1}}{3}\right] \\
&= \sum_{k=0}^{n}\sum_{j=0}^{m} . \frac{2^{j+k+1}+(-1)^{j+k+1}-(-2)^k[-2^{j-k}+2(-1)^{j-k}]}{9} \\
&\quad + i\sum_{k=0}^{n}\sum_{j=0}^{m} . \frac{2^{j+k+1}+(-1)^{j+k+1}-(-2)^k[2^{j-k+1}-(-1)^{j-k}]}{9}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{n}\sum_{j=0}^{m} G(j,k) &= \sum_{k=0}^{n}\left[\frac{2^{m+k+2}-2^{k+1}}{9} + \frac{(-1)^{m+k+2}-(-1)^{k+1}}{9(-2)} + \frac{(-2)^k\left(2^{m+1-k}-2^{-k}\right)}{9}\right. \\
&\quad \left. -2(-2)^k\frac{(-1)^{m-k+1}-(-1)^{-k}}{9(-2)}\right] \\
&\quad + i\left[\frac{2^{m+k+2}-2^{k+1}}{9} + \frac{(-1)^{m+k+2}-(-1)^{k+1}}{9(-2)} - \frac{(-2)^k\left(2^{m+2-k}-2^{-k+1}\right)}{9}\right. \\
&\quad \left. +(-2)^k\frac{(-1)^{m-k+1}-(-1)^{-k}}{9(-2)}\right]
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{n}\sum_{j=0}^{m} G(j,k) &= \frac{2^{m+n+3}-2^{m+2}}{9} - \frac{2^{n+2}-2}{9} + \frac{(-1)^{m+n+3}-(-1)^{m+2}}{-18(-2)} - \frac{(-1)^{n+2}-(-1)}{-18(-2)} + \\
&\quad + 2^{m+1}\frac{(-1)^{n+1}-1}{-18} - \frac{(-1)^{n+1}-1}{-18} + (-1)^{m+1}\frac{2^{n+1}-1}{9} - \frac{2^{n+1}-1}{9} + \\
&\quad + i\left[\frac{2^{m+n+3}-2^{m+2}}{9} - \frac{2^{n+2}-2}{9} + \frac{(-1)^{m+n+3}-(-1)^{m+2}}{-18(-2)} - \frac{(-1)^{n+2}-(-1)}{-18(-2)}\right] + \\
&\quad + i\left[2^{m+2}\frac{(-1)^{n+1}-1}{-18} + 2\frac{(-1)^{n+1}-1}{-18} + (-1)^{m+1}\frac{2^{n+1}-1}{-18} - \frac{2^{n+1}-1}{-18}\right]
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{n}\sum_{j=0}^{m} G(j,k) &= \left[\frac{2^{m+n+3}-2^{m+2}}{9} - \frac{2^{n+2}-2}{9} + \frac{(-1)^{m+n+3}-(-1)^{m+2}}{-18(-2)} - \frac{(-1)^{n+2}-(-1)}{-18(-2)}\right](1+i) \\
&\quad + 2^{m+2}\frac{(-1)^{n+1}-1}{-18} - \frac{(-1)^{n+1}-1}{-18} + (-1)^{m+1}\frac{(-1)^{n+1}-1}{9(-2)} - \frac{(-1)^{n+1}-1}{9(-2)} \\
&\quad + i\left[2^{m+2}\frac{(-1)^{n+1}-1}{-18} + 4\frac{(-1)^{n+1}-1}{-18} + (-1)^{m+1}\frac{(-1)^{n+1}-1}{9(-2)} - \frac{(-1)^{n+1}-1}{9(-2)}\right]
\end{aligned}
$$

$$
\sum_{k=0}^{n}\sum_{j=0}^{m} G(j,k) = \left[\frac{2^{m+n+3}4-2^{m+2}4-2^{n+2}4+8}{36} + \frac{(-1)^{m+n+3}-(-1)^{m+2}-(-1)^{n+2}-1}{36}\right](1+i)
$$

$$+\frac{2^{m+2}(-1)^{n+1}-2^{m+2}-(-1)^{n+1}+1+(-1)^{m+n}-(-1)^{m+1}-(-1)^{n+1}+1}{-18}$$
$$+i\left[\frac{2^{m+2}(-1)^{n+1}-2^{m+2}+4(-1)^{n+1}-4+(-1)^{m+n}-(-1)^{m+1}-(-1)^{n+1}+1}{-18}\right]$$

$$\sum_{k=0}^{n}\sum_{j=0}^{m}G(j,k)=\left[\frac{c_{m+n+5}-c_{m+4}-c_{n+4}}{36}+\frac{7}{36}\right](1+i)$$
$$+\frac{2^{m+2}(-1)^{n+1}-2^{m+2}+(-1)^{m+n}-(-1)^{m+1}-(-1)^{n+1}+1}{-18}(1+i)$$
$$+\left[\frac{((-1)^{n+1}-1)(-1+4i)}{-18}\right]$$

$$\sum_{k=0}^{n}\sum_{j=0}^{m}G(j,k)=\left[\frac{c_{m+n+5}-c_{m+4}-c_{n+4}+7}{36}\right](1+i)$$
$$+\frac{2^{m+2}[(-1)^{n+1}-1]+(-1)^{m+n}-(-1)^{m+1}-(-1)^{n+1}+1}{-18}(1+i)$$
$$+\left[\frac{((-1)^{n+1}-1)(-1+4i)}{-18}\right]. \qquad \square$$

Theorem 2.9. *The elements of the two dimensional Gaussian Jacobsthal sequence are found using the product of the following matrices*

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1+i & 1 \\ i & 0 \end{bmatrix} = \begin{bmatrix} G(n+1,1) & G(n+1,0) \\ G(n,1) & G(n,0) \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1+i & i \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} G(1,n+1) & G(0,n+1) \\ G(1,n) & G(0,n) \end{bmatrix}.$$

Proof. The mathematical induction method is used for the proof. □

Theorem 2.10. *The elements of the two dimensional Gaussian Jacobsthal sequence are also found by the product of the following matrices*

$$\begin{bmatrix} G(n+1,m+1) \\ G(n,m+1) \\ G(n+1,m) \\ G(n,m) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^n \begin{bmatrix} 1+i \\ i \\ 1 \\ 0 \end{bmatrix}.$$

Proof. The mathematical induction method is used for the proof. The assertion is true for $n=1$. Now assume that it is true for $k \le n$. For $k=n+1$,

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{n+1} \begin{bmatrix} 1+i \\ i \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G(n+1,m+1) \\ G(n,m+1) \\ G(n+1,m) \\ G(n,m) \end{bmatrix}$$

$$= \begin{bmatrix} G(n+2,m+2) \\ G(n+1,m+2) \\ G(n+2,m+1) \\ G(n+1,m+1) \end{bmatrix}.$$

Thus, the proof is completed. □

3. CONCLUSION

In conclusion, we firstly consider the generalized Gaussian Jacobsthal sequences with two dimensions. Then, we introduce the relations between the two dimensional Gaussian Jacobsthal sequence and the Jacobsthal and Jacobsthal Lucas sequences. We give the generating functions and Binet formula of this sequence. Furthermore, we obtain some important identities involving the terms of the two dimensional Gaussian Jacobsthal sequence.

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