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# INEQUALITIES FOR RATIONAL FUNCTIONS WITH $s$-FOLD ZEROS AT THE ORIGIN

B. A. ZARGAR, M. H. GULZAR, AND RUBIA AKHTER

ABSTRACT. In this paper, we establish some inequalities for rational functions with prescribed poles having $s$ fold zeros at the origin. Our results generalize and refine the results of A. Aziz and W. M. Shah [2], A. Aziz and B. A. Zargar [3] and other known rational inequalities.

## 1. INTRODUCTION

Let $\mathcal{P}_n$ be the class of polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree at most $n$. Let $D_{k-}$ denote the region inside the circle $T_k = \{z; |z| = k > 0\}$ and $D_{k+}$ the region outside $T_k$. For $a_j \in \mathbb{C}$ with $j = 1, 2, ..., n$, we write

$$W(z) = \prod_{j=1}^{n}(z - a_j) \qquad ; \qquad B(z) = \prod_{j=1}^{n}\left(\frac{1 - \overline{a_j}z_j}{z - a_j}\right)$$

and

$$\mathcal{R}_n = \mathcal{R}_n(a_1, a_2, ..., a_n) = \left\{\frac{P(z)}{W(z)} : P \in \mathcal{P}_n\right\}.$$

Then the elements of $\mathcal{R}_n$ are rational functions having poles at the points $a_1, a_2, ..., a_n$ and with a finite limit at infinity. We observe that $B(z) \in \mathcal{R}_n$. For $f$ defined on $T_k$ in the complex plane, we set

$$M(f, k) = \sup_{z \in T_k} |f(z)|.$$

Throughout this paper, we also assume that all poles $a_1, a_2, ..., a_n$ are in $D_{1+}$ . The following famous result is due to Bernstein [4].

**Theorem 1.1.** *If $P \in \mathcal{P}_n$ then $M(P', 1) \leq nM(P, 1)$.*

The following result was conjectured by Erdös and later proved by Lax [7].

**Theorem 1.2.** *If $P \in \mathcal{P}_n$ and all the zeros of $P(z)$ lie in $T_1 \cup D_{1+}$, then*

$$M(P', 1) \leq \frac{n}{2} M(P, 1). \tag{1.1}$$

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Equality in (1.1) holds for $P(z) = \alpha z^n + \beta$ with $|\alpha| = |\beta|$.

Li, Mohapatra and Rodriguez [10] have proved Bernstein-type inequalities similar to Theorem 1.1 and Theorem 1.2 for rational functions with prescribed poles where they replaced $z^n$ by Blaschkes product $B(z)$. Among other things they proved the following generalizations of Theorem 1.2.

**Theorem 1.3.** *Suppose $r \in \mathcal{R}_n$ and all zeros of $r$ lie in $T_1 \cup D_{1+}$, then for $z \in T_1$,*

$$|r'(z)| \leq \frac{1}{2}|B'(z)|M(r,1). \tag{1.2}$$

Equality in (1.2) holds for $r(z) = \alpha B(z) + \beta$ with $|\alpha| = |\beta| = 1$.

**Theorem 1.4.** *Suppose $r \in \mathcal{R}_n$, where $r$ has exactly $n$ poles at $a_1, a_2, ..., a_n$ and all the zeros of $r$ lie in $T_1 \cup D_{1-}$, then for $z \in T_1$,*

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| - (n-m)\right\}|r(z)|, \tag{1.3}$$

*where $m$ is number of zeros of $r$.*

A. Aziz and B. A. Zargar [3] considered a class of rational functions $\mathcal{R}_n$ having all zeros in $T_k \cup D_{k+}$, where $k \geq 1$ and proved the following generalisation of Theorem 1.3.

**Theorem 1.5.** *Suppose $r \in \mathcal{R}_n$ and all zeros of $r$ lie in $T_k \cup D_{k+}$ where $k \geq 1$, then for $z \in T_1$,*

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - \frac{n(k-1)}{(k+1)}\frac{|r(z)|^2}{(M(r,1))^2}\right\}M(r,1). \tag{1.4}$$

Equality in (1.4) holds for $r(z) = \left(\frac{z+k}{z-a}\right)^n$ where $a > 1, k \geq 1$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$.

A. Aziz and W. M. Shah [2] considered a class of rational functions $\mathcal{R}_n$ not vanishing in $T_k \cup D_{k+}$, where $k \leq 1$ and proved the following generalisation of Theorem 1.4.

**Theorem 1.6.** *Suppose $r \in \mathcal{R}_n$, where $r$ has exactly $n$ poles at $a_1, a_2, ..., a_n$ and all zeros of $r$ lie in $T_k \cup D_{k-}$ where $k \leq 1$, then for $z \in T_1$,*

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| + \frac{2m - n(1+k)}{(k+1)}\right\}|r(z)|, \tag{1.5}$$

*where $m$ is the number of zeros of $r(z)$.*

The result is best possible and equality holds for $r(z) = \frac{(z+k)^m}{(z-a)^n}$ where $a > 1, k \leq 1$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$.

## 2. Lemmas

We need the following lemmas for the proofs of our main results.

**Lemma 2.1.** *If $r \in \mathcal{R}_n$ and $r^*(z) = B(z)\overline{r(\frac{1}{\bar{z}})}$ then for $z \in T_1$,*

$$|(r^*(z))'| + |r'(z)| \leq |B'(z)|M(r,1). \tag{2.1}$$

Equality in (2.1) holds in $r(z) = uB(z)$ with $u \in T_1$.

**Lemma 2.2.** *Suppose that $\lambda \in T_1$, then the equation $B(z) - \lambda = 0$ has exactly $n$ simple roots (say) $t_1, t_2, ..., t_n$ and all of them lie on the unit circle $T_1$ and if $r \in \mathcal{R}_n$ and $z \in T_1$, then*

$$B'(z)r(z) - r'(z)[B(z) - \lambda] = \frac{B(z)}{2} \sum_{k=1}^{n} C_k r(t_k) \left| \frac{B(z) - \lambda}{z - t_k} \right|^2,$$

*where $C_k = C_k(\lambda)$ is defined by*

$$C_k^{-1} = \sum_{j=1}^{n} \frac{|a_j|^2 - 1}{|t_k - a_j|^2} \quad \text{for} \quad k = 1, 2, ..., n.$$

*Moreover for $z \in T_1$,*

$$z\frac{B'(z)}{B(z)} = \sum_{k=1}^{n} C_k \left| \frac{B(z) - \lambda}{z - t_k} \right|^2$$

*and also*

$$|B'(z)| = z\frac{B'(z)}{B(z)} = \sum_{k=1}^{n} \frac{|a_k|^2 - 1}{|z - a_k|^2}. \tag{2.2}$$

Lemmas 2.1 and 2.2 are due to Xin Li, R. N. Mohapatra and R. S. Rodriguez [10].

**Lemma 2.3.** *Suppose $t_1, t_2, ..., t_n$ are the zeros of $B(z) - \lambda$ and $s_1, s_2, ..., s_n$ are the zeros of $B(z) + \lambda$, where $\lambda \in T_1$. If $r \in \mathcal{R}_n$ and $z \in T_1$, then*

$$|r'(z)|^2 + |(r^*(z))'|^2 < \frac{1}{2}|B'(z)|^2 \left(M_1^2 + M_2^2\right), \tag{2.3}$$

*where $M_1 = \max_{1 \leq i \leq n} |r(t_i)|$ and $M_2 = \max_{1 \leq i \leq n} |r(s_i)|$.*

The above lemma is due to A. Aziz and W. M.Shah [1].

**Lemma 2.4.** *If $z \in T_1$, then*

$$Re\left(\frac{zW'(z)}{W(z)}\right) = \frac{n - |B(z)|}{2}, \tag{2.4}$$

*where $W(z) = \prod_{j=1}^{n} (z - a_j)$ and $W^*(z) = z^n \overline{W\left(\frac{1}{\bar{z}}\right)}$.*

This Lemma is due to A. Aziz and B. A. Zargar [3].

**Lemma 2.5.** *Assume that $r(z) \in \mathcal{R}_n$, where $r(z)$ has exactly $n$ poles at $a_1, a_2, ..., a_n$.*

(i) *Suppose r has s zeros at origin and all other zeros in* $T_k \cup D_{k+}$ *where* $k \geq 1$, *then for* $z \in T_1$,
$$Re\frac{zr'(z)}{r(z)} \leq \frac{|B'(z)|}{2} - \frac{n(k-1)+2sk}{2(1+k)}.$$

(ii) *Suppose r has s zeros at the origin and all other zeros in* $T_k \cup D_{k-}$, $\quad k \leq 1$, *then for* $z \in T_1$,
$$Re\frac{zr'(z)}{r(z)} \geq \frac{|B'(z)|}{2} + \frac{2m+2sk-n(1+k)}{2(1+k)},$$
*where m is the number of zeros of r, each zero being counted according to its multiplicity.*

*Proof of Lemma 2.5.* Let $r(z) = \frac{P(z)}{W(z)} \in \mathcal{R}_n$. If $b_1, b_2, ..., b_m$ are the zeros of $P(z)$, then $m \leq n$.

(i) Assume that $|b_j| \geq k > 1, j = 1, 2, ..., m$. Then
$$\frac{zr'(z)}{r(z)} = \frac{zP'(z)}{P(z)} - \frac{zW'(z)}{W(z)}. \tag{2.5}$$

Since $P(z)$ has $s$-fold zeros at the origin,
$$P(z) = z^s H(z),$$
where $H(z) = \sum_{j=0}^{m-s} a_j z^j$. Also
$$\frac{zP'(z)}{P(z)} = \frac{zH'(z)}{H(z)} + s. \tag{2.6}$$

Using (2.6) in (2.5) we get
$$\frac{zr'(z)}{r(z)} = s + \frac{zH'(z)}{H(z)} - \frac{zW'(z)}{W(z)}$$
$$= s + \sum_{t=1}^{m-s} \frac{z}{z-b_t} - \frac{zW'(z)}{W(z)}.$$

For $z \in T_1$, we obtain with the help of Lemma 2.4,
$$Re\frac{zr'(z)}{r(z)} = s + Re\sum_{t=1}^{m-s} \frac{z}{z-b_t} - Re\frac{zW'(z)}{W(z)} \tag{2.7}$$
$$= s + Re\sum_{t=1}^{m-s} \frac{z}{z-b_t} - \left(\frac{n-|B'(z)|}{2}\right). \tag{2.8}$$

Now for the points $e^{\iota\theta}$, $\quad 0 \leq \theta < 2\pi$, we have
$$Re\left(\frac{z}{z-b_t}\right) = Re\left(\frac{e^{\iota\theta}}{e^{\iota\theta} - ke^{\iota\phi}}\right)$$
$$= Re\left(\frac{1}{1-ke^{\iota(\phi-\theta)}}\right)$$

$$
\begin{aligned}
&= Re\left(\frac{1-ke^{-\iota(\phi-\theta)}}{(1-ke^{\iota(\phi-\theta)})(1-ke^{-\iota(\phi-\theta)})}\right)\\
&= Re\left(\frac{1-ke^{-\iota(\phi-\theta)}}{1-2k\cos(\phi-\theta)+k}\right)\\
&\le \frac{1}{1+k}
\end{aligned}
$$

if $\frac{1-k\cos(\phi-\theta)}{1-2k\cos(\phi-\theta)+k} \le \frac{1}{1+k}$. That is, if $k \ge 1$. Which is true.

Hence,

$$Re\left(\frac{z}{z-b_t}\right) \le \frac{1}{1+k}. \tag{2.9}$$

Using (2.9) in (2.7), we get

$$
\begin{aligned}
Re\frac{zr'(z)}{r(z)} &= s+\frac{m-s}{1+k}-\left(\frac{n-|B'(z)|}{2}\right)\\
&\le s+\frac{n-s}{1+k}-\left(\frac{n-|B'(z)|}{2}\right)\\
&= \frac{|B'(z)|}{2}-\frac{n(k-1)}{2(1+k)}+\frac{sk}{1+k}.
\end{aligned}
$$

(ii) Assume that $|b_j| \le k \le 1, j = 1,2,...,m$. Then

$$\frac{zr'(z)}{r(z)} = \frac{zP'(z)}{P(z)}-\frac{zW'(z)}{W(z)}. \tag{2.10}$$

Since $P(z)$ has $s$-fold zeros at the origin,

$$P(z) = z^s H(z),$$

where $H(z) = \sum_{j=0}^{m-s} a_j z^j$.

$$\frac{zP'(z)}{P(z)} = \frac{zH'(z)}{H(z)}+s. \tag{2.11}$$

Using (2.11) in (2.10) we get

$$
\begin{aligned}
\frac{zr'(z)}{r(z)} &= s+\frac{zH'(z)}{H(z)}-\frac{zW'(z)}{W(z)}\\
&= s+\sum_{t=1}^{m-s}\frac{z}{z-b_t}-\frac{zW'(z)}{W(z)}.
\end{aligned}
$$

For $z \in T_1$, this gives with the help of Lemma 2.4

$$
\begin{aligned}
Re\frac{zr'(z)}{r(z)} &= s+Re\sum_{t=1}^{m-s}\frac{z}{z-b_t}-Re\frac{zW'(z)}{W(z)}\\
&= s+Re\sum_{t=1}^{m-s}\frac{z}{z-b_t}-\left(\frac{n-|B'(z)|}{2}\right).
\end{aligned} \tag{2.12}
$$

Now it can be easily seen that for $z \in T_1$, $|b_t| \leq k \leq 1$,

$$Re\Big(\frac{z}{z-b_t}\Big) \geq \frac{1}{1+k}. \tag{2.13}$$

Using (2.13) in (2.7) we get for $z \in T_1$,

$$\begin{aligned} Re\frac{zr'(z)}{r(z)} &= s + \frac{m-s}{1+k} - \left(\frac{n-|B'(z)|}{2}\right) \\ &= \frac{sk+m}{1+k} - \left(\frac{n-|B'(z)|}{2}\right) \\ &= \frac{|B'(z)|}{2} + \frac{2m+2sk-n(1+k)}{2(1+k)}. \end{aligned}$$

That proves Lemma 2.5. □

## 3. Main result

The main aim of this paper is to obtain inequalities similar to (1.4) and (1.5) for the rational functions having $s$-fold zeros at the origin. In this direction, we first prove the following result:

**Theorem 3.1.** *Suppose $r \in \mathcal{R}_n$ has $s$ zeros at the origin and all other zeros in $T_k \cup D_{k+}$ where $k \geq 1$, then for $z \in T_1$,*

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - \left(\frac{n(k-1)}{(k+1)} - \frac{2sk}{k+1}\right)\frac{|r(z)|^2}{M(r,1)^2}\right\}M(r,1). \tag{3.1}$$

Equality in (3.1) holds for $r(z) = z^s\frac{(z+k)^{n-s}}{(z-a)^n}$ where $k \geq 1$, $a > 1$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ when evaluated at $z = 1$.

*Proof of theorem* 3.1 For $z \in T_1$ we have as in ( [10], p.529),

$$\begin{aligned} \left|\frac{z(r^*(z))'}{r(z)}\right|^2 &= \left||B'(z)| - \frac{zr'(z)}{r(z)}\right|^2 \\ &= |B'(z)|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2 - 2|B'(z)|Re\left(\frac{zr'(z)}{r(z)}\right). \end{aligned}$$

Using $(i)$ of Lemma 2.5 we have for $z \in T_1$,

$$\begin{aligned} \left|\frac{z(r^*(z))'}{r(z)}\right|^2 &\geq |B'(z)|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2 - 2|B'(z)|\left\{\frac{|B'(z)|}{2} - \frac{n(k-1)}{2(1+k)} + \frac{sk}{1+k}\right\} \\ &= \left|\frac{zr'(z)}{r(z)}\right|^2 + \left\{\frac{n(k-1)}{(1+k)} + \frac{2sk}{1+k}\right\}|B'(z)|. \end{aligned}$$

This implies for $z \in T_1$,

$$\left[|r'(z)|^2 + \left\{\frac{n(k-1)}{(1+k)} + \frac{2sk}{1+k}\right\}|r(z)|^2|B'(z)|\right]^{\frac{1}{2}} \leq |(r^*(z))'|.$$

This gives with the help of Lemma 2.1,

$$|r'(z)|+\left[|r'(z)|^2+\left\{\frac{n(k-1)}{(1+k)}+\frac{2sk}{1+k}\right\}|r(z)|^2|B'(z)|\right]^{\frac{1}{2}}\leq |B'(z)|M(r,1).$$

After a short simplification, this yields for $z\in T_1$,

$$|r'(z)|\leq\frac{1}{2}\left\{|B'(z)|-\left(\frac{n(k-1)}{(k+1)}-\frac{2sk}{k+1}\right)\frac{|r(z)|^2}{M(r,1)^2}\right\}M(r,1). \qquad \square$$

***Remark*** 3.1. Taking $s=0$, we get Theorem 1.5.

Taking $s=1$, we get the following result.

**Corollary 3.1.** *Suppose $r\in\mathcal{R}_n$ has only one zero at the origin and all other zeros in $T_k\cup D_{k+}$ where $k\geq 1$, then for $z\in T_1$,*

$$|r'(z)|\leq\frac{1}{2}\left\{|B'(z)|-\left(\frac{n(k-1)-2k}{(k+1)}\right)\frac{|r(z)|^2}{M(r,1)^2}\right\}M(r,1). \tag{3.2}$$

Equality in (3.2) holds for $r(z)=z\frac{(z+k)^{n-1}}{(z-a)^n}$ where $k\geq 1$, $a>1$ and $B(z)=\left(\frac{1-az}{z-a}\right)^n$ when evaluated at $z=1$.

Next we prove the following result which is a generalization of (1.5).

**Theorem 3.2.** *Suppose $r(z)=\frac{P(z)}{W(z)}\in\mathcal{R}_n$, where $r$ has exactly $n$ poles at $a_1,a_2,\dots,a_n$ and $s$-fold zeros at the origin and all other zeros in $T_k\cup D_{k-}$ $k\leq 1$, then for $z\in T_1$,*

$$|r'(z)|\geq\frac{1}{2}\left\{|B'(z)|+\frac{2m+2sk-n(k+1)}{2(1+k)}\right\}|r(z)|. \tag{3.3}$$

Equality in (3.3) holds for $r(z)=z^s\frac{(z+k)^{m-s}}{(z-a)^n}$ where $k\leq 1$, $a>1$ and $B(z)=\left(\frac{1-az}{z-a}\right)^n$ when evaluated at $z=1$.

*Proof of theorem 3.2.* Using the fact that

$$\left|\frac{r'(z)}{r(z)}\right|\geq Re\frac{zr'(z)}{r(z)}, \tag{3.4}$$

from $(ii)$ of Lemma 2.5 we have for $z\in T_1$,

$$Re\frac{zr'(z)}{r(z)}\geq\frac{|B'(z)|}{2}+\frac{2m+2sk-n(1+k)}{2(1+k)}. \tag{3.5}$$

Combining (3.4) and (3.5), we get (3.3) and the proof is complete. $\square$

As an immediate consequence of Theorem 3.2, we have the following generalization of inequality (12) in [ [10], p. 526], where $r$ has $s$-fold zeros at the origin and $n-r$ zeros in $T_k\cup D_{k-}$.

**Corollary 3.2.** *Suppose $r(z) = \frac{P(z)}{W(z)} \in \mathcal{R}_n$, where $r$ has exactly $n$ poles at $a_1, a_2, ..., a_n$ and $s$-fold zeros at the origin and all other zeros in $T_k \cup D_{k-}$, $k \le 1$. Then for $z \in T_1$,*

$$|r'(z)| \ge \frac{1}{2}\left\{|B'(z)| + \frac{n(1-k)+2sk}{2(1+k)}\right\}|r(z)|. \tag{3.6}$$

Equality in (3.6) holds for $r(z) = z^s \frac{(z+k)^{n-s}}{(z-a)^n}$ where $k \le 1$, $a > 1$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ when evaluated at $z = 1$.

Finally, we prove the following result.

**Theorem 3.3.** *Let $r \in \mathcal{R}_n$ have $s$-fold zeros at the origin and all other zeros in $T_k \cup D_{k+}$. Let $t_1, t_2, ..., t_n$ be the zeros of $B(z) - \lambda$ and $s_1, s_2, ..., s_n$, the zeros of $B(z) + \lambda$, where $\lambda \in T_1$, then for $z \in T_1$*

$$|r'(z)| \le \frac{1}{2}\left\{|B'(z)|^2 - \frac{n(k-1)+2sk}{(1+k)}\frac{|r(z)|^2|B'(z)|}{M_1^2 + M_2^2}\right\}^{\frac{1}{2}} \times \left(M_1^2 + M_2^2\right)^{\frac{1}{2}}, \tag{3.7}$$

*where $M_1 = \max_{1 \le i \le n} |r(t_i)|$ and $M_2 = \max_{1 \le i \le n} |r(s_i)|$.*

*Proof of theorem 3.3.* We have

$$r^*(z) = B(z)\overline{r\left(\frac{1}{\bar{z}}\right)}.$$

Therefore

$$(r^*(z))' = B'(z)\overline{r\left(\frac{1}{\bar{z}}\right)} - B(z)\overline{r'\left(\frac{1}{\bar{z}}\right)}\frac{1}{z^2}.$$

Since $z \in T_1$, we have $\bar{z} = \frac{1}{z}$ and therefore

$$\begin{aligned} |(r^*(z))'| &= |zB'(z)\overline{r(z)} - B(z)\overline{zr'(z)}| \\ &= \left|z\frac{B'(z)}{B(z)}\overline{r(z)} - \overline{zr'(z)}\right|. \end{aligned} \tag{3.8}$$

Using (2.2) of Lemma 2.2 we get for $z \in T_1$,

$$|(r^*(z))'| = ||B'(z)|r(z) - \overline{zr'(z)}|.$$

Hence it follows that for $z \in T_1$

$$\begin{aligned} \left|\frac{z(r^*(z))'}{r(z)}\right|^2 &= \left||B'(z)| - \frac{zr'(z)}{r(z)}\right|^2 \\ &= |B'(z)|^2 + \left|\frac{zr'(z)}{r(z)} - 2B'(z)Re\left(\frac{zr'(z)}{r(z)}\right)\right|. \end{aligned} \tag{3.9}$$

Using $(i)$ of Lemma 2.5 in (3.9) we get

$$\left|\frac{z(r^*(z))'}{r(z)}\right|^2 \geq |B'(z)|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2 - 2B'(z)\left\{\frac{|B'(z)|}{2} - \frac{n(k-1)+2sk}{2(1+k)}\right\}$$
$$= \left|\frac{zr'(z)}{r(z)}\right|^2 + \frac{n(k-1)+2sk}{2(1+k)}|B'(z)|,$$

which implies for $z \in T_1$

$$|r'(z)|^2 + \frac{n(k-1)+2sk}{2(1+k)}|r(z)|^2|B'(z)| \leq |(r^*(z))'|^2. \tag{3.10}$$

Therefore using Lemma 2.3 we get

$$2|r'(z)|^2 + \frac{n(k-1)+2sk}{2(1+k)}|r(z)|^2|B'(z)| \leq |(r^*(z))'|^2 + |r'(z)|^2$$
$$\leq \frac{1}{2}|B'(z)|^2\left\{M_1^2 + M_2^2\right\}.$$

Equivalently,

$$4|r'(z)|^2 \leq |B'(z)|^2\left\{M_1^2 + M_2^2\right\} - \frac{n(k-1)+2sk}{(1+k)}|r(z)|^2|B'(z)|$$
$$= \left\{|B'(z)|^2 - \frac{n(k-1)+2sk}{(1+k)}\frac{|r(z)|^2|B'(z)|}{M_1^2+M_2^2}\right\}\left(M_1^2+M_2^2\right),$$

which immediately leads to the inequality (3.7). □

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B A Zargar
University Of Kashmir
Department of Mathematics
e-mail: *bazargar@gmail.com*
*and*
M H Gulzar
University Of Kashmir
Department of Mathematics
e-mail: *gulzarmh@gmail.com*
*and*
Rubia Akhter (Corresponding Author)
University of Kashmir
Department of Mathematics
e-mail: *rubiaakhter039@gmail.com*