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VARIATIONAL APPROXIMATION FOR MODIFIED MEYER-KÖNIG AND ZELLER OPERATORS

EMRE TAŞ AND TUĞBA YURDAKADIM

Abstract. In the present paper we introduce modified Meyer-König and Zeller operators which coincide with the classical Meyer-König and Zeller operators if $\omega(x) = x$. We provide sufficient conditions on the boundedness of the total variation of these operators and we also present a result which deals with the variational approximation of the new modified operators.

1. Introduction

The classical Meyer-König and Zeller (MKZ) operators [5], [10] are defined by

$$(M_n f)(x) = (1-x)^{n+1} \sum_{r=0}^{\infty} \binom{r+n}{r} x^r f(\frac{r}{r+n}),$$

and $(M_n f)(1) = f(1)$ where $f \in C[0,1]$, the space of all continuous functions on $[0,1]$ and $x \in [0,1]$. These operators have been discussed by means of convergence in variation in [9]. There are some generalizations and many studies on the MKZ operators [1], [6], [11]. A real valued function f is said to be of bounded variation on $[0,1]$ if its total variation is finite, i.e.

$$\|f\|_V := V_0^1[f] = \sup \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)|,$$

where the supremum is taken over all sequences $0 \leq x_0 \leq x_1 \leq ... \leq x_n \leq 1$. Many well known operators such as Bernstein have been studied by using convergence in variation [3], [12].

The classical MKZ operators have been evolved as follows

$$(M_n^{\omega} f)(x) = (1-\omega(x))^{n+1} \sum_{r=0}^{\infty} \binom{r+n}{r} \omega^r(x) (f \circ \omega^{-1})(\frac{r}{r+n}), \quad \text{for } f \in C[0,1], x \in [0,1), \tag{1.1}$$

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and

$$(M_n^\omega f)(1) = f(1)$$

where ω is an absolutely continuous and infinitely differentiable function on $[0,1]$. We have also assumed two conditions to overcome some challenges in the results.

- $(C1)$ $\inf \omega^{'}(x) \geq m > 0$ for almost every x,
- $(C2)$ $\omega(0) = 0$ *and* $\omega(1) = 1$.

Our generalization of the MKZ operators is related to the recently published papers [2], [7], [8], [9].

Proposition 1.1 ([4]). *Let f be an absolutely continuous function on $[0,1]$ with positive derivative almost everywhere. Then f is strictly increasing and the inverse function is absolutely continuous on $[f(0), f(1)]$.*

By taking derivative of (1.1), we get

$$\begin{aligned}
&(M_n{}^\omega f)'(x) = \\
&\quad = (n+1)(1-\omega(x))^{n+1}\omega'(x)\sum_{r=0}^{\infty}\binom{r+n+1}{r}(f\circ\omega^{-1})(\frac{r+1}{r+n+1})\omega^r(x) \\
&\quad - (n+1)(1-\omega(x))^{n}\omega'(x)\sum_{r=0}^{\infty}\binom{r+n}{r}(f\circ\omega^{-1})(\frac{r}{r+n})\omega^r(x) \\
&\quad = (n+1)(1-\omega(x))^{n}\omega'(x)\sum_{r=0}^{\infty}\binom{r+n+1}{r}(f\circ\omega^{-1})(\frac{r+1}{r+n+1})\omega^r(x) \\
&\quad - (n+1)(1-\omega(x))^{n}\omega'(x)\Bigg\{\sum_{r=1}^{\infty}\binom{r+n}{r-1}(f\circ\omega^{-1})(\frac{r}{r+n})\omega^r(x) \\
&\quad - \sum_{r=0}^{\infty}\binom{r+n}{r}(f\circ\omega^{-1})(\frac{r}{r+n})\omega^r(x)\Bigg\} \\
&\quad = (n+1)(1-\omega(x))^{n}\omega'(x)\cdot S,
\end{aligned}$$

where

$$S := \sum_{r=0}^{\infty}\binom{r+n+1}{r}\left[(f\circ\omega^{-1})(\frac{r+1}{r+n+1}) - (f\circ\omega^{-1})(\frac{r}{r+n})\right]\omega^r(x).$$

2. Main Results

In this section we give two main results. The first one provides sufficient conditions on the boundedness of $\|M_n^\omega f\|_V$. The second one is on the variational approximation of the MKZ operators.

Let us give our first main theorem.

Theorem 2.1. *Let f be a function of bounded variation on $[0,1]$ and ω be an absolutely continuous function on $[0,1]$ satisfying conditions $(C1)$, $(C2)$. Then we have $M_n^\omega f$ is an absolutely continuous function on $[0,1]$ and also*

$$\|M_n^\omega f\|_V \leq \|f\|_V.$$

Proof. By Proposition 1, (C1) and (C2), we have the absolute continuity of ω^{-1} on $[0,1]$. Since absolute continuity implies bounded variation, ω^{-1} is a function of bounded variation on $[0,1]$. Hence we obtain $f \circ \omega^{-1}$ is a function of bounded variation which implies $(M_n^\omega f)$ is an absolutely continuous function on $[0,1]$. Then

$$\|M_n^\omega f\|_V = \int_0^1 |(M_n^\omega f)'(x)|dx$$
$$\leq (n+1)\sum_{r=0}^{\infty}\binom{r+n+1}{r}\left|(f\circ\omega^{-1})(\frac{r+1}{r+n+1}) - (f\circ\omega^{-1})(\frac{r}{r+n})\right| \cdot I_r,$$

where

$$I_r := \int_0^1 \omega^r(x)(1-\omega(x))^n\omega'(x)dx.$$

By substituting $t=\omega(x)$ and using the beta function, we obtain

$$\begin{aligned}\|M_n^\omega f\|_V \leq &(n+1)\sum_{r=0}^{\infty}\binom{r+n+1}{r}\Big|(f\circ\omega^{-1})(\frac{r+1}{r+n+1}) \\ &- (f\circ\omega^{-1})(\frac{r}{r+n})\Big|\int_0^1 t^r(1-t)^n dt \\ &= (n+1)\sum_{r=0}^{\infty}\binom{r+n+1}{r}\Big|(f\circ\omega^{-1})(\frac{r+1}{r+n+1}) \\ &- (f\circ\omega^{-1})(\frac{r}{r+n})\Big|\beta(r+1,n+1) \\ &= \sum_{r=0}^{\infty}\left|(f\circ\omega^{-1})(\frac{r+1}{r+n+1}) - (f\circ\omega^{-1})(\frac{r}{r+n})\right| \\ &\leq \|f\|_V\end{aligned}$$

which completes the proof. □

Now, one can also rewrite the operators $(M_n^\omega)'$ via a direct computation in the following form

$$(M_n^\omega f)'(x) = \frac{(1-\omega(x))^n}{\omega(x)}\omega'(x)\cdot T(x),\ x\in(0,1),$$

where

$$T(x) := \sum_{r=0}^{\infty}\binom{r+n}{r}(f\circ\omega^{-1})(\frac{r}{r+n})[r-(r+n+1)\omega(x)]\omega^r(x),\ x\in(0,1).$$

If we take

$$L^{\omega}_{s,n}(\omega(x)) := (1-\omega(x))^{n+1}\sum_{r=0}^{\infty}\binom{r+n}{r}[r-(r+n+1)\omega(x)]^{s}\omega^{r}(x),$$

then we have

$$L^{\omega}_{s,n}(\omega(x)) = \begin{cases} 1 & ,s=0 \\ 0 & ,s=1 \\ (n+1)\omega(x) & ,s=2 \\ (n+1)\omega(x)(1+\omega(x)) & ,s=3 \\ (n+1)\omega(x)[1+3(n+2)\omega(x)+\omega(x)(1+\omega(x))] & ,s=4. \end{cases} \tag{2.1}$$

On the other hand with the use of Taylor's formula for the function $f\circ\omega^{-1}$ at $\omega(x)$ as long as ω'' and f'' exist, we get

$$\begin{aligned} f(t) &= (f\circ\omega^{-1})(\omega(t)) \\ &= (f\circ\omega^{-1})(\omega(x)) + (\omega(t)-\omega(x))(f\circ\omega^{-1})'(\omega(x)) \\ &+ \frac{1}{2}(\omega(t)-\omega(x))^{2}(f\circ\omega^{-1})''(\omega(x)) \\ &+ \frac{1}{2}\int_{0}^{\omega(t)-\omega(x)}\Big[\omega(t)-\omega(x)-v\Big]^{2}(f\circ\omega^{-1})'''(\omega(x)+v)dv. \end{aligned} \tag{2.2}$$

If we apply the operators $(M^{\omega}_{n})'$ to (2.2), we obtain

$$\begin{aligned} (M^{\omega}_{n}f)'(x) &= f(x)\frac{(1-\omega(x))^{n}}{\omega(x)}\omega'(x)\sum_{r=0}^{\infty}\binom{r+n}{r}[r-(r+n+1)\omega(x)]\omega^{r}(x) \\ &+f'(x)\frac{(1-\omega(x))^{n}}{\omega(x)}\sum_{r=0}^{\infty}\binom{r+n}{r}[r-(r+n+1)\omega(x)]\left(\frac{r}{r+n}-\omega(x)\right)\omega^{r}(x) \\ &+\frac{1}{2}(f\circ\omega^{-1})''(\omega(x))\frac{(1-\omega(x))^{n}}{\omega(x)}\omega'(x) \\ &\cdot\sum_{r=0}^{\infty}\binom{r+n}{r}[r-(r+n+1)\omega(x)]\left(\frac{r}{r+n}-\omega(x)\right)^{2}\omega^{r}(x) \\ &+\frac{1}{2}\frac{(1-\omega(x))^{n}}{\omega(x)}\omega'(x)\sum_{r=0}^{\infty}\frac{1}{(r+n)^{2}}\binom{r+n}{r}[r-(r+n+1)\omega(x)]\omega^{r}(x) \\ &\cdot\int_{0}^{\frac{r}{r+n}-\omega(x)}\Big[r-(r+n)(\omega(x)+v)\Big]^{2}(f\circ\omega^{-1})'''(\omega(x)+v)dv \\ &=f(x)A_{n}(x)+f'(x)B_{n}(x)+\frac{1}{2}(f\circ\omega^{-1})''(\omega(x))\omega'(x)C_{n}(x)+D_{n}(x). \end{aligned} \tag{2.3}$$

Note that $A_n(x)=0$ and $B_n(x)=1$ for every $n\in\mathbb{N}$ by (2.1).

Now we give some lemmas which are used in the proof of the second main theorem.

Lemma 2.2. *Let ω be an absolutely continuous function satisfying conditions $(C1)$ and $(C2)$. Then we have*

$$|C_n(x)| \leq \frac{2}{n} + \frac{2}{n(n-1)} + \frac{2\sqrt{21}}{n\sqrt{n-1}},\ n \geq 2.$$

Proof. Using basic computation, we get

$$\begin{aligned}
C_n(x) &= \\
&= \frac{(1-\omega(x))^n}{\omega(x)} \sum_{r=0}^{\infty} \frac{(r+n)!}{r!n!}[U(x)-2\omega(x)]\left(\frac{U(x)-\omega(x)}{r+n}\right)^2 \omega^r(x) \\
&= \frac{(1-\omega(x))^n}{\omega(x)} \sum_{r=0}^{\infty} \frac{(r+n-1)!}{r!n!}[U(x)-2\omega(x)]\frac{[U(x)-\omega(x)]^2}{r+n}\omega^r(x) \\
&= \frac{(1-\omega(x))^n}{\omega(x)} \sum_{r=0}^{\infty} \frac{(r+n-2)!}{r!n!}\left(1-\frac{1}{r+n}\right)[U(x)-2\omega(x)][U(x)-\omega(x)]^2\omega^r(x) \\
&= \frac{(1-\omega(x))^n}{\omega(x)} \sum_{r=0}^{\infty} \binom{r+n-2}{r}\frac{1}{n(n-1)}\left(1-\frac{1}{r+n}\right) \\
&\cdot [U(x)-2\omega(x)][U(x)-\omega(x)]^2\omega^r(x) \\
&= \frac{(1-\omega(x))^n}{\omega(x)} \sum_{r=0}^{\infty} \binom{r+n-2}{r}\frac{1}{n(n-1)}[U(x)-2\omega(x)][U(x)-\omega(x)]^2\omega^r(x) \\
&- \frac{(1-\omega(x))^n}{\omega(x)} \sum_{r=0}^{\infty} \binom{r+n-2}{r}\frac{1}{n(n-1)(r+n)}[U(x)-2\omega(x)][U(x)-\omega(x)]^2\omega^r(x) \\
&= \frac{(1-\omega(x))^n}{\omega(x)} \sum_{r=0}^{\infty} \binom{r+n-2}{r}\frac{1}{n(n-1)} \\
&\cdot \left\{U(x)^3(x) - 4\omega(x)U(x)^2(x) + 5\omega^2(x)U(x) - 2\omega^3(x)\right\}\omega^r(x) - K
\end{aligned} \tag{2.4}$$

where $U(x) := r-(r+n-1)\omega(x)$ and

$$K := \frac{(1-\omega(x))^n}{\omega(x)} \sum_{r=0}^{\infty} \binom{r+n-2}{r}\frac{1}{n(n-1)(r+n)} \cdot [U(x)-2\omega(x)][U(x)-\omega(x)]^2\omega^r(x).$$

By using the equations given in (2.1), we have

$$C_n(x) = \frac{(1-\omega(x))(1-3\omega(x))}{n} - 2\frac{\omega^2(x)(1-\omega(x))}{n(n-1)} - K.$$

Applying the Cauchy-Schwartz inequality, we get

$$|K| \leq \frac{1}{n^2(n-1)}\frac{1-\omega(x)}{\omega(x)}$$
$$\cdot\left\{(1-\omega(x))^{n-1}\sum_{r=0}^{\infty}\Big(r-(r+n-1)\omega(x)-2\omega(x)\Big)^2\binom{r+n-2}{r}\omega^r(x)\right\}^{\frac{1}{2}}$$
$$\cdot\left\{(1-\omega(x))^{n-1}\sum_{r=0}^{\infty}\Big(r-(r+n-1)\omega(x)-\omega(x)\Big)^4\binom{r+n-2}{r}\omega^r(x)\right\}^{\frac{1}{2}}$$
$$\leq \frac{1}{n^2(n-1)}\frac{(1-\omega(x))}{\omega(x)}\left\{(n-1)\omega(x)+4\omega^2(x)\right\}^{\frac{1}{2}}$$
$$.\Big\{(n-1)\omega(x)[1+3n\omega(x)+\omega(x)(1+\omega(x))]$$
$$-4\omega(x)(n-1)\omega(x)(1+\omega(x))+6\omega^2(x)(n-1)\omega(x)+\omega^4(x)\Big\}^{\frac{1}{2}}$$
$$\leq \frac{1}{n^2(n-1)}\left\{n+3\right\}^{\frac{1}{2}}\left\{(n-1)(3n+3)+8(n-1)+6(n-1)+1\right\}^{\frac{1}{2}}$$
$$\leq \frac{1}{n^2(n-1)}\left\{n+3\right\}^{\frac{1}{2}}\left\{(n-1)(3n+17)+n-1\right\}^{\frac{1}{2}}$$
$$\leq \frac{1}{n^2(n-1)}\left\{4n\right\}^{\frac{1}{2}}\left\{(n-1)(3n+18)\right\}^{\frac{1}{2}}$$
$$\leq \frac{2\sqrt{n}\sqrt{n-1}\sqrt{21}\sqrt{n}}{n^2(n-1)}=\frac{2\sqrt{21}}{n\sqrt{n-1}},\ n\geq 2. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain for $n \geq 2$

$$|C_n(x)| \leq \frac{2}{n}+\frac{2}{n(n-1)}+\frac{2\sqrt{21}}{n\sqrt{n-1}}. \qquad \square$$

Lemma 2.3. *Let $f \in C[0,1]$ and ω be an absolutely continuous function satisfying conditions $(C1)$ and $(C2)$. If ω'' and f'' are absolutely continuous on $[0,1]$, then we have*

$$\|D_n\|_1 \leq \left\{\frac{7\pi^2}{36n}+\frac{2}{n\sqrt{n}}+\frac{1}{(n-1)}+\frac{1}{2n(n-1)}\right\}\int_0^1\Big|(f\circ\omega^{-1})'''(u)\Big|du,$$

where $\|.\|_1$ is usual norm in the space of all absolutely Lebesque integrable functions on $[0,1]$.

Proof.

$$D_n(x)=\frac{(1-\omega(x))^n}{2\omega(x)}\omega'(x)\sum_{r=0}^{\infty}\frac{1}{(r+n)^2}[r-(r+n+1)\omega(x)]\binom{r+n}{r}\omega^r(x)$$

$$\cdot\int_0^{\frac{r}{r+n}-\omega(x)}[r-(r+n)(\omega(x)+v)]^2(f\circ\omega^{-1})'''(\omega(x)+v)dv$$
$$= E_1 - E_2, \tag{2.6}$$

where

$$E_1 := \frac{(1-\omega(x))^n}{2\omega(x)}\omega'(x)\sum_{r=0}^{\infty}\frac{r-(r+n)\omega(x)}{(r+n)^2}\binom{r+n}{r}\omega^r(x)$$
$$\cdot\int_0^{\frac{r}{r+n}-\omega(x)}[r-(r+n)(\omega(x)+v)]^2(f\circ\omega^{-1})'''(\omega(x)+v)dv$$

and

$$E_2 := \frac{(1-\omega(x))^n}{2}\omega'(x)\sum_{r=0}^{\infty}\frac{1}{(r+n)^2}\binom{r+n}{r}\omega^r(x)$$
$$\cdot\int_0^{\frac{r}{r+n}-\omega(x)}[r-(r+n)(\omega(x)+v)]^2(f\circ\omega^{-1})'''(\omega(x)+v)dv.$$

Let us give upper bounds for $\|E_1\|_1$ *and* $\|E_2\|_1$*.*

$$\|E_1\|_1 \le$$
$$\le \frac{1}{2}\int_0^1\frac{(1-\omega(x))^n}{\omega(x)}\omega'(x)\sum_{r=0}^{\infty}\frac{1}{(r+n)^2}[r-(r+n)\omega(x)]\binom{r+n}{r}\omega^r(x)$$
$$\cdot\int_0^{\frac{r}{r+n}-\omega(x)}[r-(r+n)(\omega(x)+v)]^2|(f\circ\omega^{-1})'''(\omega(x)+v)|dvdx$$
$$\le \frac{1}{2}\int_0^1\frac{(1-\omega(x))^n}{\omega(x)}\omega'(x)\sum_{r=0}^{\infty}[r-(r+n)\omega(x)]\binom{r+n}{r}\omega^r(x)$$
$$\cdot\int_0^{\frac{r}{r+n}-\omega(x)}[\frac{r}{r+n}-(\omega(x)+v)]^2|(f\circ\omega^{-1})'''(\omega(x)+v)|dvdx.$$

After making the substitution $\omega(x)+v=u$ *and taking into account* $\left|\frac{r}{r+n}-(\omega(x)+v)\right|^2 \le \left|\frac{r}{r+n}-\omega(x)\right|^2$ *for* $\frac{r}{r+n}\le v\le\omega(x)$ *or* $\omega(x)\le v\le\frac{r}{r+n}$, *one can observe that*

$$\|E_1\|_1 \le \frac{1}{2}\int_0^1\frac{(1-\omega(x))^n}{\omega(x)}\omega'(x)\sum_{r=0}^{\infty}\frac{[r-(r+n)\omega(x)]}{(r+n)^2}\binom{r+n}{r}\omega^r(x)$$
$$\cdot\int_{\omega(x)}^{\frac{r}{r+n}}[r-(r+n)u]^2|(f\circ\omega^{-1})'''(u)|dudx$$

$$\leq \frac{1}{2}\int_0^1 \frac{(1-\omega(x))^n}{\omega(x)}\omega'(x)\sum_{r=0}^{\infty}\frac{[r-(r+n)\omega(x)]^3}{(r+n)^2}\binom{r+n}{r}\omega^r(x)$$
$$\cdot\left(\int\limits_0^{\frac{r}{r+n}} - \int\limits_0^{\omega(x)}\right)|(f\circ\omega^{-1})'''(u)|dudx$$
$$= E_1^2 - E_1^1$$

where

$$E_1^1 := \frac{1}{2}\int_0^1 \omega'(x)\frac{(1-\omega(x))^n}{\omega(x)}\sum_{r=0}^{\infty}\frac{[r-(r+n)\omega(x)]^3}{(r+n)^2}\binom{r+n}{r}\omega^r(x)$$
$$\cdot\int_0^{\omega(x)}|(f\circ\omega^{-1})'''(u)|dudx$$
$$E_1^2 := \frac{1}{2}\int_0^1 \omega'(x)\frac{(1-\omega(x))^n}{\omega(x)}\sum_{r=0}^{\infty}\frac{[r-(r+n)\omega(x)]^3}{(r+n)^2}\binom{r+n}{r}\omega^r(x)$$
$$\cdot\int_0^{\frac{r}{r+n}}|(f\circ\omega^{-1})'''(u)|dudx.$$

In order to obtain upper bounds for $\|E_1^1\|_1$ *and* $\|E_1^2\|_1$ *we need the following:*

$$X = \frac{(1-\omega(x))^n}{\omega(x)}\sum_{r=0}^{\infty}\frac{[r-(r+n)\omega(x)]^3}{(r+n)^2}\binom{r+n}{r}\omega^r(x)$$
$$= \frac{(1-\omega(x))^n}{\omega(x)}\sum_{r=0}^{\infty}\frac{[r-(r+n)\omega(x)]^3}{(r+n)}\frac{(r+n-1)!}{r!n!}\omega^r(x)$$
$$= \frac{(1-\omega(x))^n}{n\omega(x)}\sum_{r=0}^{\infty}\binom{r+n-1}{r}\left(\frac{1}{r+n-1}-\frac{1}{(r+n)(r+n-1)}\right)$$
$$\cdot[r-(r+n)\omega(x)]^3\omega^r(x) \tag{2.7}$$

$$= \frac{(1-\omega(x))^n}{n(n-1)\omega(x)}\sum_{r=0}^{\infty}\binom{r+n-2}{r}[r-(r+n-1)\omega(x)-\omega(x)]^3\omega^r(x)$$
$$- \frac{(1-\omega(x))^n}{n(n-1)\omega(x)}\sum_{r=0}^{\infty}\frac{1}{r+n}\binom{r+n-2}{r}[r-(r+n-1)\omega(x)-\omega(x)]^3\omega^r(x)$$
$$= \frac{1}{n(n-1)}\frac{1-\omega(x)}{\omega(x)}\Big[(n-1)\omega(x)(1-2\omega(x))-\omega^3(x)\Big]-M$$
$$= \frac{1-\omega(x)}{n(n-1)}\Big[(n-1)(1-2\omega(x))-\omega^2(x)\Big]-M. \tag{2.8}$$

Using the Cauchy-Schwartz inequality for M, we get

$$\begin{aligned}|M| \le& \frac{1-\omega(x)}{n^2(n-1)\omega(x)}\Bigg[(1-\omega(x))^{n-1}\sum_{r=0}^{\infty}\binom{r+n-2}{r}\\
&\cdot[r-(r+n-1)\omega(x)-\omega(x)]^4\omega^r(x)\Bigg]^{\frac{1}{2}}\\
&.\Bigg[(1-\omega(x))^{n-1}\sum_{r=0}^{\infty}\binom{r+n-2}{r}[r-(r+n-1)\omega(x)-\omega(x)]^2\omega^r(x)\Bigg]^{\frac{1}{2}}\\
&=\frac{1}{n^2(n-1)}(1-\omega(x))\Bigg[3(n-1)^2\omega(x)+(n-1)(1+3\omega^2(x))+\omega^3(x)\Bigg]^{\frac{1}{2}}\\
&\cdot\Bigg[n-1+\omega(x)\Bigg]^{\frac{1}{2}}\\
&=\frac{1}{n^2}(1-\omega(x))\Bigg[3\omega(x)+\frac{1+3\omega^2(x)}{n-1}+\frac{\omega^3(x)}{(n-1)^2}\Bigg]^{\frac{1}{2}}\sqrt{n-1}\Bigg[1+\frac{\omega(x)}{n-1}\Bigg]^{\frac{1}{2}}\\
&\le\frac{1}{n\sqrt{n}}(1-\omega(x))\Bigg[3\omega(x)+1+3\omega^2(x)+\omega^3(x)\Bigg]^{\frac{1}{2}}\Bigg[1+\omega(x)\Bigg]^{\frac{1}{2}}\\
&\le\frac{4}{n\sqrt{n}},\ n\ge 2. \qquad (2.9)\end{aligned}$$

From (2.8) and (2.9), we have

$$|X| \le \frac{1}{n(n-1)}\{(n-1)+1\}+\frac{4}{n\sqrt{n}}=\frac{n}{n(n-1)}+\frac{4}{n\sqrt{n}}\le\frac{1}{n-1}+\frac{4}{n\sqrt{n}}.$$

$$\begin{aligned}|E_1^1| &\le \frac{1}{2}\left(\frac{1}{n-1}+\frac{4}{n\sqrt{n}}\right)\int_0^1\omega'(x)\int_0^{\omega(x)}\Big|(f\circ\omega^{-1})'''(u)\Big|dudx\\
&\le\frac{1}{2}\left(\frac{1}{n-1}+\frac{4}{n\sqrt{n}}\right)(\omega(1)-\omega(0))\int_0^1\Big|(f\circ\omega^{-1})'''(u)\Big|du \qquad (2.10)\\
&\le\frac{1}{2}\left(\frac{1}{n-1}+\frac{4}{n\sqrt{n}}\right)\int_0^1\Big|(f\circ\omega^{-1})'''(u)\Big|du.\end{aligned}$$

On the other hand

$$\begin{aligned}|E_1^2| =&\frac{1}{2}\sum_{r=0}^{\infty}\frac{1}{(r+n)^2}\Bigg|\int_0^{\frac{r}{r+n}}(f\circ\omega^{-1})'''(u)du\binom{r+n}{r}\\
&\cdot\int_0^1\omega'(x)[r-(r+n)\omega(x)]^3(1-\omega(x))^n\omega^{r-1}(x)dx\Bigg|\end{aligned}$$

and substitute $\omega(x)=y$, then we can rewrite the above equality

$$|E_1^2| =\frac{1}{2}\sum_{r=0}^{\infty}\frac{1}{(r+n)^2}\Bigg|\int_0^{\frac{r}{r+n}}(f\circ\omega^{-1})'''(u)du\binom{r+n}{r}\cdot$$

$$\cdot \int_0^1 [r-(r+n)y]^3(1-y)^n y^{r-1}dy\Big|.$$

Using the beta function, we get

$$\left|\binom{r+n}{r}\int_0^1 [r-(r+n)y]^3(1-y)^n y^{r-1}dy\right| =$$
$$= \left|\frac{5r^2n+3rn^2-2n^3+6r^2}{(r+n+1)(r+n+2)(r+n+3)}\right| \leq 2.$$

Hence we obtain

$$\begin{aligned} |E_1^2| &\leq \sum_{r=0}^{\infty}\frac{1}{(r+n)^2}\int_0^{\frac{r}{r+n}}\left|(f\circ\omega^{-1})'''(u)\right|du \\ &\leq \sum_{r=0}^{\infty}\frac{1}{(r+n)^2}\int_0^{1}\left|(f\circ\omega^{-1})'''(u)\right|du \\ &\leq \frac{\pi^2}{6n}\int_0^{1}\left|(f\circ\omega^{-1})'''(u)\right|du. \end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11), we have

$$\begin{aligned} \|E_1\|_1 &\leq \left\{\frac{1}{2}\left(\frac{1}{n-1}+\frac{4}{n\sqrt{n}}\right)+\frac{\pi^2}{6n}\right\}\int_0^1\left|(f\circ\omega^{-1})'''(u)\right|du \\ &\leq \left\{\frac{\pi^2}{6n}+\frac{1}{2(n-1)}+\frac{2}{n\sqrt{n}}\right\}\int_0^1\left|(f\circ\omega^{-1})'''(u)\right|du. \end{aligned}$$

To give an upper bound for $\|E_2\|_1$, notice that

$$\begin{aligned} &\|E_2\|_1 \\ &\leq \int_0^1 \frac{(1-\omega(x))^n}{2}\omega'(x)\sum_{r=0}^{\infty}\frac{1}{(r+n)^2}\binom{r+n}{r}\omega^r(x) \\ &\cdot \int_0^{\frac{r}{r+n}-\omega(x)} [r-(r+n)(\omega(x)+v)]^2\left|(f\circ\omega^{-1})'''(\omega(x)+v)\right|dvdx \\ &= \frac{1}{2}\int_0^1 (1-\omega(x))^n\omega'(x)\sum_{r=0}^{\infty}\frac{1}{(r+n)^2}\binom{r+n}{r}\omega^r(x) \\ &\int_{\omega(x)}^{\frac{r}{r+n}}[r-(r+n)u]^2\left|(f\circ\omega^{-1})'''(u)\right|dudx \\ &\leq \frac{1}{2}\int_0^1 (1-\omega(x))^n\omega'(x)\sum_{r=0}^{\infty}\frac{1}{(r+n)^2}[r-(r+n)\omega(x)]^2\binom{r+n}{r}\omega^r(x) \end{aligned}$$

$$\cdot \left(\int_0^{\frac{r}{r+n}} - \int_0^{\omega(x)} \right) \left| (f \circ \omega^{-1})'''(u) \right| du dx$$

$$= \frac{1}{2} \int_0^1 (1-\omega(x))^n \omega'(x) \sum_{r=0}^{\infty} \frac{1}{(r+n)^2} [r - (r+n)\omega(x)]^2 \binom{r+n}{r} \omega^r(x)$$

$$\cdot \int_0^{\frac{r}{r+n}} \left| (f \circ \omega^{-1})'''(u) \right| du dx$$

$$- \frac{1}{2} \int_0^1 (1-\omega(x))^n \omega'(x) \sum_{r=0}^{\infty} \frac{1}{(r+n)^2} [r - (r+n)\omega(x)]^2 \binom{r+n}{r} \omega^r(x)$$

$$\cdot \int_0^{\omega(x)} \left| (f \circ \omega^{-1})'''(u) \right| du dx = E_2^2 - E_2^1.$$

Let

$$Q := (1-\omega(x))^n \sum_{r=0}^{\infty} \frac{1}{(r+n)^2} [r - (r+n)\omega(x)]^2 \binom{r+n}{r} \omega^r(x).$$

$$Q = \frac{1}{n} (1-\omega(x))^n \sum_{r=0}^{\infty} \left(\frac{1}{r+n-1} - \frac{1}{(r+n)(r+n-1)} \right)$$

$$\cdot [r - (r+n)\omega(x)]^2 \binom{r+n-1}{r} \omega^r(x)$$

$$= \frac{1}{n(n-1)} (1-\omega(x))^n \sum_{r=0}^{\infty} [r - (r+n-1)\omega(x) - \omega(x)]^2$$

$$\cdot \binom{r+n-2}{r} \omega^r(x)$$

$$- \frac{1}{n(n-1)} (1-\omega(x))^n \sum_{r=0}^{\infty} \frac{1}{r+n} [r - (r+n-1)\omega(x) - \omega(x)]^2$$

$$\cdot \binom{r+n-2}{r} \omega^r(x) = \frac{1}{n(n-1)} (1-\omega(x)) [(n-1)\omega(x) + \omega^2(x)] - P \quad (2.12)$$

where
$P := \frac{1}{n(n-1)} (1-\omega(x))^n \sum_{r=0}^{\infty} \frac{1}{r+n} [r - (r+n-1)\omega(x) - \omega(x)]^2 \binom{r+n-2}{r} \omega^r(x).$
If we consider $\frac{1}{r+n} \leq \frac{1}{n}$, *we have*

$$P \leq \frac{1}{n^2(n-1)} (1-\omega(x)) [(n-1)\omega(x) + \omega^2(x)] \leq \frac{1}{n(n-1)}. \quad (2.13)$$

By (2.12) and (2.13), it is satisfied that

$$|Q| \leq \frac{1}{n-1} + \frac{1}{n(n-1)}.$$

Hence we get from the last inequality

$$|E_2^1| \le \frac{1}{2}\int_0^1 \left(\frac{1}{n-1}+\frac{1}{n(n-1)}\right)\omega'(x)\int_0^{\omega(x)} \left|(f\circ\omega^{-1})'''(u)\right| dudx$$
$$\le \frac{1}{2}\left(\frac{1}{n-1}+\frac{1}{n(n-1)}\right)\int_0^1 \omega'(x)dx\int_0^1 \left|(f\circ\omega^{-1})'''(u)\right| du \qquad (2.14)$$
$$= \frac{1}{2}\left(\frac{1}{n-1}+\frac{1}{n(n-1)}\right)\int_0^1 \left|(f\circ\omega^{-1})'''(u)\right| du.$$

$$|E_2^2| \le \frac{1}{2}\sum_{r=0}^{\infty}\frac{1}{(r+n)^2}\bigg|\int_0^{\frac{r}{r+n}}(f\circ\omega^{-1})'''(u)du\binom{r+n}{r}$$
$$\cdot\int_0^1 [r-(r+n)\omega(x)]^2(1-\omega(x))^n\omega^r(x)\omega'(x)dx\bigg|$$

and again by substituting $\omega(x)=y$

$$|E_2^2| \le \frac{1}{2}\sum_{r=0}^{\infty}\frac{1}{(r+n)^2}\int_0^1\left|(f\circ\omega^{-1})'''(u)\right|du\left|\binom{r+n}{r}\int_0^1[r-(r+n)y]^2(1-y)^n y^r dy\right|.$$

Then using the following inequality

$$\left|\binom{r+n}{r}\int_0^1[r-(r+n)y]^2(1-y)^n y^r dy\right| = \left|\frac{r^2n+rn^2+2r^2-2rn+2n^2}{(r+n+1)(r+n+2)(r+n+3)}\right| \le \frac{1}{3},$$

we obtain

$$|E_2^2| \le \frac{1}{6}\int_0^1\left|(f\circ\omega^{-1})'''(u)\right|du\sum_{r=0}^{\infty}\frac{1}{(r+n)^2} \le \frac{1}{6}\frac{\pi^2}{6n}\int_0^1\left|(f\circ\omega^{-1})'''(u)\right|du. \qquad (2.15)$$

From (2.14) and (2.15), we get

$$\|E_2\|_1 \le \left\{\frac{\pi^2}{36n}+\frac{1}{2(n-1)}+\frac{1}{2n(n-1)}\right\}\int_0^1\left|(f\circ\omega^{-1})'''(u)\right|du.$$

Combining the above inequalities, we have

$$\|D_n\|_1 \le \left\{\frac{\pi^2}{6n}+\frac{1}{2(n-1)}+\frac{2}{n\sqrt{n}}\right\}\int_0^1\left|(f\circ\omega^{-1})'''(u)\right|du$$
$$+\left\{\frac{\pi^2}{36n}+\frac{1}{2(n-1)}+\frac{1}{2n(n-1)}\right\}\int_0^1\left|(f\circ\omega^{-1})'''(u)\right|du$$
$$=\left\{\frac{7\pi^2}{36n}+\frac{1}{n-1}+\frac{2}{n\sqrt{n}}+\frac{1}{2n(n-1)}\right\}\int_0^1\left|(f\circ\omega^{-1})'''(u)\right|du. \qquad \square$$

Theorem 2.4. *Let* $f\in C[0,1]$ *and* ω *be an absolutely continuous function satisfying conditions* $(C1)$ *and* $(C2)$*. If* ω'' *and* f'' *are absolutely continuous on* $[0,1]$*, then we have*

$$\|(M_n^\omega f)-f\|_V \leq \tfrac{S}{n-1}\left\{\|f\|_V+\|f''\|_V\right\},\ n\geq 2,$$

where $S := 7R\max\{H+\|\omega'''\|_\infty+3\|\omega''\|_\infty^2+(3H+1)\|\omega''\|_\infty, H+1+3H\|\omega''\|_\infty\}$ *such that* $R:=\max\{\frac{1}{m^r},\ r=2,3,4\}$, $H>1$ *and* $\|.\|_\infty$ *is usual supremum norm.*

Proof. By using Lemma 2.2, Lemma 2.3, (2.3) and condition $(C1)$, we obtain

$$\begin{aligned}
&\|M_n^\omega f-f\|_V=\|(M_n^\omega f)'-f'\|_1\\
&\leq \frac{1}{2}\int_0^1\left|(f\circ\omega^{-1})''(\omega(x))\omega'(x)C_n(x)\right|dx+\|D_n\|_1\\
&\leq \left[\frac{1}{n}+\frac{1}{n(n-1)}+\frac{\sqrt{21}}{\sqrt{n}(n-1)}\right]\int_0^1\left|(f\circ\omega^{-1})''(\omega(x))\omega'(x)\right|dx\\
&+\left\{\frac{7\pi^2}{36n}+\frac{1}{n-1}+\frac{2}{n\sqrt{n}}+\frac{1}{2n(n-1)}\right\}\int_0^1\left|(f\circ\omega^{-1})'''(u)\right|du\\
&=\left[\frac{1}{n}+\frac{1}{n(n-1)}+\frac{\sqrt{21}}{\sqrt{n}(n-1)}\right]\int_0^1\left|\frac{f''(x)}{(\omega'(x))^2}-\frac{f'(x)\omega''(x)}{(\omega'(x))^3}\right|\omega'(x)dx\\
&+\left\{\frac{7\pi^2}{36n}+\frac{1}{n-1}+\frac{2}{n\sqrt{n}}+\frac{1}{2n(n-1)}\right\}\int_0^1\left|(f\circ\omega^{-1})'''(\omega'(x))\right|dx\\
&=\left[\frac{1}{n}+\frac{1}{n(n-1)}+\frac{\sqrt{21}}{\sqrt{n}(n-1)}\right]\int_0^1\left|\frac{f''(x)}{(\omega'(x))^2}-\frac{f'(x)\omega''(x)}{(\omega'(x))^3}\right|\omega'(x)dx\\
&+\left\{\frac{7\pi^2}{36n}+\frac{1}{n-1}+\frac{2}{n\sqrt{n}}+\frac{1}{2n(n-1)}\right\}\\
&\cdot\int_0^1\left|\frac{f'''(x)}{(\omega'(x))^3}-3\frac{f''(x)\omega''(x)}{(\omega'(x))^4}-\frac{f'(x)\omega'''(x)}{(\omega'(x))^4}+\frac{f'(x)(\omega''(x))^2}{(\omega'(x))^5}\right|\omega'(x)dx\\
&\leq\left[\frac{1}{n}+\frac{1}{n(n-1)}+\frac{\sqrt{21}}{2\sqrt{n}(n-1)}\right]\left\{\frac{\|f''\|_1}{m}+\frac{\|f'\|_1\|\omega''\|_\infty}{m^2}\right\}\\
&+\left\{\frac{7\pi^2}{36n}+\frac{1}{n-1}+\frac{2}{n\sqrt{n}}+\frac{1}{2n(n-1)}\right\}\left\{\frac{\|f'''\|_1}{m^2}+3\frac{\|f''\|_1\|\omega''\|_\infty}{m^3}+\right.\\
&\left.\frac{\|f'\|_1\|\omega'''\|_\infty}{m^3}+3\frac{\|f'\|_1\|\omega''\|_\infty}{m^4}\right\}\\
&\leq\frac{7}{n-1}\left\{\frac{\|f''\|_1}{m}+\frac{\|f'\|_1\|\omega''\|_\infty}{m^2}\right\}\\
&+\frac{6}{n-1}\left\{\frac{\|f'''\|_1}{m^2}+3\frac{\|f''\|_1\|\omega''\|_\infty}{m^3}+\frac{\|f'\|_1\|\omega'''\|_\infty}{m^3}+3\frac{\|f'\|_1\|\omega''\|_\infty}{m^4}\right\}\\
&\leq\frac{7}{n-1}R\left\{\|f''\|_1+\|f'\|_1\|\omega''\|_\infty+\|f'''\|_1+\right.\\
&\left.3\|f''\|_1\|\omega''\|_\infty+\|f'\|_1\|\omega'''\|_\infty+3\|f'\|_1\|\omega''\|_\infty\right\},
\end{aligned}$$

where $R:=\max\{\frac{1}{m^r},\ r=2,3,4\}$.

Following the Stein's inequality [13], we write

$$\|f''\|_1 \leq H\sqrt{\|f'\|_1\|f'''\|_1} \leq H(\|f'\|_1 + \|f'''\|_1)$$

such that $H > 1$. Let $S = 7R\max\{H + \|\omega'''\|_\infty + 3\|\omega''\|_\infty^2 + (3H + 1)\|\omega''\|_\infty, H + 1 + 3H\|\omega''\|_\infty\}$. Then we have

$$\|(M_n^\omega f)' - f'\|_1 \leq \frac{S}{n-1}\left\{\|f'\|_1 + \|f'''\|_1\right\}$$

and hence

$$\|(M_n^\omega f) - f\|_V \leq \frac{S}{n-1}\left\{\|f\|_V + \|f''\|_V\right\}.$$

This completes the proof. □

References

[1] U. Abel, V. Gupta and M. Ivan, *The complete asymptotic expansion for a general Durrmeyer variant of the Meyer-König and Zeller operators*, Math. Comput. Modelling 40 (2004), 867–875.

[2] O. Agratini, *On the variation detracting property of a class of operators*, Appl. Math. Lett. 19 (2006), 1261–1264.

[3] C. Bardaro, P.L. Butzer, R.L. Stens and G. Vinti, *Convergence in variation and rates of approximation for Bernstein-type polynomials and singular convolution integrals*, Analysis 23 (2003), 299–340.

[4] V.I. Bogachev, *Measure Theory*, vol. I. Springer, Berlin, 2007.

[5] E. W. Cheney and A. Sharma, *Bernstein power series*, Canad. J. Math. 16 (1964), 241–253.

[6] O. Dogru and O. Duman, *Statistical approximation of Meyer-König and Zeller operators based on q-integers*, Publ. Math. Debrecen 68 (2006), 199–214.

[7] H.G. Ince Ilarslan and G. Bascanbaz Tunca, *Convergence in variation for Bernstein-type operators*, Mediterr. J. Math. 13 (2016), 2577–2592.

[8] H. Karsli, *On convergence of Chlodovsky and Chlodovsky-Kantorovich polynomials in the variation seminorm*, Mediterr. J. Math. 10 (2013), 41–56.

[9] A. Kivinukk and T. Metsmagi, *Approximation in variation by the Meyer-König and Zeller operators*, Proc. Estonian Acad. Sci. 60 (2011), 88–97.

[10] W. Meyer-König and K. Zeller. *Bernsteinische potenzreihen*, Studia Math. 19 (1960), 89–94.

[11] M.W. Müller, *L_p-approximation by the method of integral Meyer-König and Zeller operators*, Studia Math. 63 (1978), 81–88.

[12] Ö. Öksüzer, H. Karsli and F. Tasdelen Yesildal, *On the convergence of Bernstein-Stancu polynomials in the variation seminorm*, Numer. Funct. Anal. Optim. 37 (2016), 583–602.

[13] E.M. Stein, *Functions of exponential type*, Ann. Math. 65 (1957), 582–592.

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Emre Taş
Kırşehir Ahi Evran University
Department of Mathematics
Kırşehir
Turkey
emretas86@hotmail.com

Tuğba Yurdakadim
Hitit University
Department of Mathematics
Çorum
Turkey
tugbayurdakadim@hotmail.com