SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.20.02.06
Vol.20 (33), No.2 (2024), 241–248

SOME (p,q)-INTEGRAL INEQUALITIES

İLKER GENÇTÜRK

ABSTRACT. In this paper, we obtain a (p,q)-analogue of an open problem represented by Q. A. Ngô et al. in the paper, *Notes on an integral inequality*, J. Inequal. Pure and Appl. Math., 7(4) (2006), Art. 120, by using analytic and elementary methods in (p,q)-calculus.

1. INTRODUCTION

In [19], Q. A. Ngô et al. proposed the following problem:

Problem 1.1. *Let* $f(t)$ *be a continuous function on* $[0,1]$ *satisfying*

$$\int_x^1 f(t)dt \geq \int_x^1 tdt, \quad \forall x \in [0,1].$$

Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(t)dt \geq \int_0^1 t^{\alpha} f^{\beta}(t)dt$$

hold for α *and* β*?*

In view of the interest of this type of inequality, a series of papers obtaining various extensions and generalizations was studied in the literature [3], [4], [13], [14], [17]. In [5], authors studied a q-analogue of Ngô's problem by using analytic and elementary methods in quantum calculus.

(p,q)-calculus, a generalization of q-calculus, is known as two parameter quantum calculus (p and q numbers) which are independent. It was first introduced by R. Chakrabarti and R. Jagannathan [8] in 1991. The applications of (p,q)-calculus play important roles in physical and mathematical sciences, see [1], [6], [7], [9], [11], [12]. Moreover, a lot of literature about (p,q)-calculus and (p,q)-integral inequalities was studied by many authors, see [2], [10], [15], [16], [18], [20], [21], [23], [24], [25], [26] and the references cited therein.

2020 *Mathematics Subject Classification.* 26D10, 26D15, 05A30.
Key words and phrases. inequalities, (p,q)-derivative, (p,q)-integral.

In this position, this paper will focus on obtaining some new integral inequalities in (p,q)-calculus. This paper begins by definitions and facts in (p,q)-calculus. It will then go on to give our main results.

2. PRELIMINARIES

From now on, we will let p,q be two constants which satisfy $0 < q < p \leq 1$. For the convenience of the reader, we repeat some notation and definitions from (p,q)-calculus. For more details, we refer the reader to [22].

Definition 2.1. *The (p,q)-derivative of the function f is defined as*

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0,$$

and $(D_{p,q}f)(0) = f^{'}(0)$, provided that f is differentiable at 0.

Definition 2.2. *(p,q)-analogue is defined as*

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}, \text{ for any number } n.$$

Definition 2.3. *Let f be an arbitrary function and a be a real number, we define the definite (p,q)-integral as*

$$\int_0^a f(x)d_{p,q}x = (p-q)a\sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}a\right), \text{ if } \left|\frac{p}{q}\right| > 1.$$

Let f be an arbitrary function and a and b be two non-negative numbers such that $a < b$, then we have

$$\int_a^b f(x)d_{p,q}x = \int_0^b f(x)d_{p,q}x - \int_0^a f(x)d_{p,q}x.$$

For any function f, we have

$$D_{p,q}\left(\int_a^x f(x)d_{p,q}x\right) = f(x).$$

If $F(x)$ is a (p,q)-antiderivative of $f(x)$ and $F(x)$ is continuous at $x = 0$, we have

$$\int_a^b f(x)d_{p,q}x = F(b) - F(a),$$

where $0 \leq a < b \leq \infty$.

The formula of (p,q)-integration by part is given by

$$\int_a^b f(px)(D_{p,q}g(x))d_{p,q}x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)(D_{p,q}f(x))d_{p,q}x.$$

Finally, we denote

$$[0,1]_{p,q} = \left\{ \frac{q^k}{p^k} : k = 0,1,2,...,\infty \right\}.$$

We note that in the case of $p = 1$, above definitions and notation are reduced to the definitions and notation in q-calculus.

3. Main Results

First, we give following inequality:

Lemma 3.1 (General Cauchy Inequality). *Let α and β be positive real numbers satisfying $\alpha + \beta = 1$. Then for every positive real numbers x and y, we always have*

$$\alpha x + \beta y \geq x^\alpha y^\beta.$$

The following lemma is necessary for obtaining our main results.

Lemma 3.2. *Let f be a non-negative function defined on $[0,1]_{p,q}$ satisfying*

$$\int_{px}^1 f^\beta(t) d_{p,q}t \geq \int_{px}^1 t^\beta d_{p,q}t, \text{ for every } x \in [0,1]_{p,q}\ . \tag{3.1}$$

Then we have

$$\int_0^1 x^\alpha f^\beta(x) d_{p,q}x \geq \frac{1}{[\alpha+\beta+1]_{p,q}}, \tag{3.2}$$

for all positive real numbers $\alpha > 0$ and $\beta > 0$.

Proof. By using (p,q)-integration by parts, we obtain

$$\begin{aligned}\int_0^1 x^{\alpha-1} \left(\int_{px}^1 f^\beta(t) d_{p,q}t \right) d_{p,q}x &= \frac{1}{[\alpha]_{p,q}} \left[x^\alpha \left(\int_x^1 f^\beta(t) d_{p,q}t \right) \right]_{x=0}^1 \\ &\quad + \frac{q^\alpha}{[\alpha]_{p,q}} \int_0^1 x^\alpha f^\beta(x) d_{p,q}x \\ &= \frac{q^\alpha}{[\alpha]_{p,q}} \int_0^1 x^\alpha f^\beta(x) d_{p,q}x,\end{aligned}$$

which yields

$$\int_0^1 x^\alpha f^\beta(x) d_{p,q}x = \frac{[\alpha]_{p,q}}{q^a} \int_0^1 x^{\alpha-1} \left(\int_{px}^1 f^\beta(t) d_{p,q}t \right) d_{p,q}x. \tag{3.3}$$

On the other hand, by condition (3.1), we then have

$$\begin{aligned} \int_0^1 x^{\alpha-1} \left(\int_{px}^1 f^\beta(t) d_{p,q}t \right) d_{p,q}x &\geq \int_0^1 x^{\alpha-1} \left(\int_{px}^1 t^\beta d_{p,q}t \right) d_{p,q}x \\ &= \frac{1}{[\beta+1]_{p,q}} \int_0^1 x^{\alpha-1} (1 - p^{\beta+1} x^{\beta+1}) d_{p,q}x \\ &= \frac{1}{[\beta+1]_{p,q}} \int_0^1 \left(x^{\alpha-1} - p^{\beta+1} x^{\alpha+\beta} \right) d_{p,q}x \\ &= \frac{1}{[\beta+1]_{p,q}} \left[\frac{1}{[\alpha]_{p,q}} - \frac{p^{\beta+1}}{[\alpha+\beta+1]_{p,q}} \right] \\ &= \frac{q^\alpha}{[\alpha+\beta+1]_{p,q}[\alpha]_{p,q}}. \end{aligned}$$

Therefore, by (3.3), we conclude that

$$\int_0^1 x^\alpha f^\beta(x) d_{p,q}x \geq \frac{1}{[\alpha+\beta+1]_{p,q}}.$$

□

Theorem 3.1. *Under the assumptions of Lemma 3.2, we have*

$$\int_0^1 f^{\alpha+\beta}(x) d_{p,q}x \geq \int_0^1 x^\alpha f^\beta(x) d_{p,q}x,$$

for all positive real numbers $\alpha > 0$ *and* $\beta > 0$.

Proof. Using Lemma 3.1, we obtain

$$\frac{\beta}{\alpha+\beta} f^{\alpha+\beta}(x) + \frac{\alpha}{\alpha+\beta} x^{\alpha+\beta} \geq x^\alpha f^\beta(x), \tag{3.4}$$

then, by (p,q)-integrating, we obtain

$$\beta \int_0^1 f^{\alpha+\beta}(x) d_{p,q}x + \alpha \int_0^1 x^{\alpha+\beta} d_{p,q}x \geq (\alpha+\beta) \int_0^1 x^\alpha f^\beta(x) d_{p,q}x. \tag{3.5}$$

Moreover, by using Lemma 3.2, we get

$$(\alpha+\beta)\int_0^1 x^\alpha f^\beta(x)d_{p,q}x = \alpha\int_0^1 x^\alpha f^\beta(x)d_{p,q}x + \beta\int_0^1 x^\alpha f^\beta(x)d_{p,q}x$$
$$\geq \frac{\alpha}{[\alpha+\beta+1]_{p,q}} + \beta\int_0^1 x^\alpha f^\beta(x)d_{p,q}x.$$

Then, by (3.5), we have

$$\beta\int_0^1 f^{\alpha+\beta}(x)d_{p,q}x + \frac{\alpha}{[\alpha+\beta+1]_{p,q}} \geq \frac{\alpha}{[\alpha+\beta+1]_{p,q}} + \beta\int_0^1 x^\alpha f^\beta(x)d_{p,q}x,$$

which completes the proof. □

Corollary 3.1. *Let f be a non-negative function defined on $[0,1]_{p,q}$ satisfying*

$$\int_{px}^1 f^\beta(t)d_{p,q}t \geq \int_{px}^1 t^\beta d_{p,q}t, \text{ for every } x \in [0,1]_{p,q}\ . \tag{3.6}$$

Then we have

$$\int_0^1 f^{\alpha+1}(x)d_{p,q}x \geq \int_0^1 x^\alpha f(x)d_{p,q}x,$$

for every positive real number $\alpha > 0$.

Proof. It suffices to take $\beta = 1$ in Theorem 3.1 and the rest of the proof runs as before. □

Theorem 3.2. *Let f be a non-negative function defined on $[0,1]_{p,q}$ satisfying*

$$\int_{px}^1 f(t)d_{p,q}t \geq \int_{px}^1 t\,d_{p,q}t, \text{ for every } x \in [0,1]_{p,q}\ . \tag{3.7}$$

Then we have

$$\int_0^1 f^{\alpha+1}(x)d_{p,q}x \geq \int_0^1 x f^\alpha(x)d_{p,q}x,$$

for every positive real number $\alpha > 0$.

Proof. We have for every $x \in [0,1]_{p,q}$,

$$(f^\alpha(x)-x^\alpha)(f(x)-x) \geq 0,$$

and so we obtain

$$f^{\alpha+1}(x)+x^{\alpha+1} \geq x^\alpha f(x)+xf^\alpha(x).$$

By (p,q)-integrating with some simple calculations, we obtain that

$$\int_0^1 f^{\alpha+1}(x)d_{p,q}x+\frac{1}{[\alpha+2]_{p,q}} \geq \int_0^1 x^{\alpha}f(x)d_{p,q}x+\int_0^1 xf^{\alpha}(x)d_{p,q}x.$$

Then, from Lemma 3.2 for $\beta=1$, we obtain

$$\int_0^1 f^{\alpha+1}(x)d_{p,q}x+\frac{1}{[\alpha+2]_{p,q}} \geq \frac{1}{[\alpha+2]_{p,q}}+\int_0^1 xf^{\alpha}(x)d_{p,q}x,$$

which completes the proof. □

Lemma 3.3. *Let f be a non-negative function defined on $[0,1]_{p,q}$ satisfying*

$$\int_{px}^1 f^{\beta}(t)d_{p,q}t \geq \int_{px}^1 t^{\beta}d_{p,q}t, \text{ for every } x\in[0,1]_{p,q}\,. \tag{3.8}$$

Then we have

$$\int_0^1 x^{\alpha}f^{\beta}(x)d_{p,q}x \geq \frac{1}{[\alpha+\beta+1]_{p,q}},$$

for all real numbers $\alpha>0$ and $\beta\geq 1$.

Proof. Using Lemma 3.1, we obtain

$$\frac{1}{\beta}f^{\beta}(x)+\frac{\beta-1}{\beta}x^{\beta} \geq x^{\beta-1}f(x),$$

by multiplying x^{α} and (p,q)-integrating, we get

$$\int_0^1 x^{\alpha}f^{\beta}(x)d_{p,q}x+(\beta-1)\int_0^1 x^{\alpha+\beta}d_{p,q}x \geq \beta\int_0^1 x^{\alpha+\beta-1}f(x)d_{p,q}x.$$

Therefore, from Lemma 3.2, it follows that

$$\int_0^1 x^{\alpha}f^{\beta}(x)d_{p,q}x+\frac{\beta-1}{[\alpha+\beta+1]_{p,q}} \geq \frac{\beta}{[\alpha+\beta+1]_{p,q}}.$$

Thus, the lemma is proved. □

Theorem 3.3. *Under the conditions of Lemma 3.3, we have*

$$\int_0^1 f^{\alpha+\beta}(x)d_{p,q}x \geq \int_0^1 x^{\alpha}f^{\beta}(x)d_{p,q}x,$$

for all real numbers $\alpha>0$ and $\beta\geq 1$.

Proof. By using Lemma 3.1, we obtain

$$\frac{\beta}{\alpha+\beta}f^{\alpha+\beta}(x)+\frac{\alpha}{\alpha+\beta}x^{\alpha+\beta}\geq x^{\alpha}f^{\beta}(x),$$

which implies that

$$\beta\int_0^1 f^{\alpha+\beta}(x)d_{p,q}x+\frac{\alpha}{[\alpha+\beta+1]_{p,q}}\geq(\alpha+\beta)\int_0^1 x^{\alpha}f^{\beta}(x)d_{p,q}x.$$

Then, from Lemma 3.3, it may be concluded that

$$\beta\int_0^1 f^{\alpha+\beta}(x)d_{p,q}x+\frac{\alpha}{[\alpha+\beta+1]_{p,q}}\geq\frac{\alpha}{[\alpha+\beta+1]_{p,q}}+\beta\int_0^1 x^{\alpha}f^{\beta}(x)d_{p,q}x,$$

which completes the proof. □

ACKNOWLEDGEMENTS

The author would like to express sincere gratitude to the reviewers for his/her valuable suggestions.

REFERENCES

[1] A. Aral and V. Gupta, *Applications of (p,q)-gamma function to Szász Durrmeyer operators*, Publ. Inst. Math. (Beograd) (N.S.), 102 (2017), 211–220.

[2] N. Arunrat, K. M. Nakprasit, K. Nonlaopon, J., Tariboon and S. K. Ntouyas, *On Fejér Type Inequalities via (p,q)-calculus*, Symmetry, 13(6)(2021), 953.

[3] L. Bougoffa, *Note on an open problem*, JIPAM. J. Inequal. Pure Appl. Math., 8 (2)(2007), Art. 58.

[4] K. Boukerrioua and A. Guezane-Lakoud, *On an open question regarding an integral inequality*, JIPAM. J. Inequal. Pure Appl. Math., 8 (3) (2007), Art. 77.

[5] K. Brahim, *On some q-integral inequalities*, JIPAM. J. Inequal. Pure Appl. Math., 9 (4)(2008), Art. 106.

[6] I. M. Burban, *Two-parameter deformation of the oscillator algebra and (p,q)-analog of two-dimensional conformal field theory*, J. Nonlinear Math. Phys., 2 (1995), 384–391.

[7] I. M. Burban and A. U. Klimyk, *(p,q)-differentiation, (p,q)-integration and (p,q)-hypergeometric functions related to quantum groups*, Integral Transform. Spec. Funct., 2 (1994), 15–36.

[8] R. Chakrabarti and R. Jagannathan, *A (p,q)-oscillator realization of two-parameter quantum algebras*, J. Phys. A: Math. Gen., 24 (1991), 711–718.

[9] U. Duran, M. Acikgoz, A. Esi, and S. Araci, *A note on the (p,q)-Hermite polynomials*, Appl. Math. Inf. Sci., 12 (2018), 227–231.

[10] İ. Gençtürk, *Some Feng Qi type (p,q)-integral inequalities*, Journal of Balikesir University Institute of Science and Technology, 23(1)(2021), 366–376. (Turkish).

[11] İ. Gençtürk, *Boundary value problems for a second-order (p,q)-difference equation with integral conditions*, Turkish J. Math., 46(2)(2022), 499–515.

[12] M. N. Hounkonnou, J. Désiré and B. Kyemba, *$R(p,q)$-calculus: Differentiation and integration*, SUT J. Math., 49 (2013), 145–167.

[13] W. J. Liu, C. C. Li and J. Dong, *On an open problem concerning an integral inequality*, JIPAM. J. Inequal. Pure Appl. Math., 8 (3) (2007), Art. 74.

[14] W. J. Liu, G.S. Cheng and C. C. Li, *Further development of an open problem concerning an integral inequality*, JIPAM. J. Inequal. Pure Appl. Math., 9(1) (2008), Art.14.

[15] W. Luangboon, K. Nonlaopon, J. Tariboon and S. K. Ntouyas, *On Simpson type inequalities for generalized strongly preinvex functions via* (p,q)*-calculus and applications*, AIMS Mathematics, 6(9)(2021), 9236–9261.

[16] W. Luangboon, K. Nonlaopon, J. Tariboon and S. K. Ntouyas, *Simpson-and Newton-Type Inequalities for Convex Functions via* (p,q)*-calculus*, Mathematics, 9(12)(2021), 1338.

[17] T. F. Móri, *A general inequality of Ngô-Thang-Dat-Tuan type*, JIPAM. J. Inequal. Pure Appl. Math., 10(1) (2009), Art. 10.

[18] M. D. Nasiruzzaman, A. Mukheimer and M. Mursaleen, *Some Opial-type integral inequalities via* (p,q)*-calculus*, J. Inequal. Appl., 2019 (2019), 1–11.

[19] Q. A. Ngo, D. Thang, T. Dat and D. Tuan, *Notes on an integral inequality*, JIPAM. J. Inequal. Pure Appl. Math., 7(4)(2006), Art. 120.

[20] J. Prabseang, K. Nonlaopon, J. Tariboon and S. K. Ntouyas, (p,q)*-Hermite–Hadamard Inequalities for Double Integral and* (p,q)*-Differentiable Convex Functions*, Axioms, 8(2)(2019), 68.

[21] J. Prabseang, K. Nonlaopon, J. Tariboon and S. K. Ntouyas, *Refinements of Hermite-Hadamard Inequalities for Continuous Convex Functions via* (p,q)*-calculus*, Mathematics, 9(4)(2021), 446.

[22] P. N. Sadjang, *On the fundamental theorem of* (p,q)*-calculus and some* (p,q)*-Taylor formulas*, Results Math., 73(1) (2018), 1–21.

[23] S. Thongjob, K. Nonlaopon, J. Tariboon and S. K. Ntouyas, *Generalizations of some integral inequalities related to Hardy type integral inequalities via* (p,q)*-calculus*, J. Inequal. Appl. , 2021(1) (2021), 1–17.

[24] M. Tunç and E. Göv, *Some integral inequalities via* (p,q)*-calculus on finite intervals*, Filomat, 35(5) (2021), 1421–1430.

[25] M. Tunç and E. Göv, (p,q)*-integral inequalities*, RGMIA Res. Rep. Coll., 19 (2016), Art. 97, 1–13.

[26] F. Wannalookkhee, K. Nonlaopon, J. Tariboon and S. K. Ntouyas, *On Hermite-Hadamard type inequalities for coordinated convex functions via* (p,q)*-calculus*, Mathematics, 9(7)(2021), 698.

(Received: December 24, 2020)
(Revised: January 19, 2022)

İlker GENÇTÜRK
Kırıkkale University
Department of Mathematics
71450, Yahsihan, Kırıkkale, Turkey
e-mail: *ilkergencturk@gmail.com*