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ON BORNOLOGICAL SPACES OF SERIES IN SYSTEMS OF FUNCTIONS

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ABSTRACT. Let f be an entire transcendental function, $M_f(r) = \max\{|f(z)| : |z| = r\}$, (λ_n) be a sequence of positive numbers increasing to $+\infty$ and suppose that the series $A(z) = \sum\limits_{n=1}^{\infty} a_n f(\lambda_n z)$ regularly converges in $\mathbb{C}$, i. e. $\sum\limits_{n=1}^{\infty} |a_n| M_f(r\lambda_n) < +\infty$ for all $r \in [0, +\infty)$. Bornology is introduced on a set of such series as a system of functions $f(\lambda_n z)$ and its connection with Frechet spaces is studied.

1. INTRODUCTION

Let $\Lambda = (\lambda_n)$ be a sequence of positive numbers increasing to $+\infty$,

$$f(z) = \sum_{k=0}^{\infty} f_k z^k \tag{1.1}$$

be an entire transcendental function and $M_f(r) = \max\{|f(z)| : |z| = r\}$. Suppose that the series

$$A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z) \tag{1.2}$$

in the system $f(\lambda_n z)$ regularly converges in $\mathbb{C}$, i. e. for all $r \in [0, +\infty)$

$$\mathfrak{M}(r, A) := \sum_{n=1}^{\infty} |a_n| M_f(r\lambda_n) < +\infty. \tag{1.3}$$

Many authors have studied the representation of analytic functions by series in the system $f(\lambda_n z)$. We will focus here on the monographs of A.F. Leont'ev [1] and B.V. Vinnitsky [8], where references to other works can be found.

Since series (1.2) regularly converges in $\mathbb{C}$, the function A is entire. We remark that the function $\ln M_f(r)$ is logarithmically convex and, therefore,

$$\Gamma_f(r) := \frac{d \ln M_f(r)}{d \ln r} \nearrow +\infty, \quad r \to +\infty,$$

(in points where the derivative does not exist, $\dfrac{d \ln M_f(r)}{d \ln r}$ denotes theW right-hand derivative).

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We remark also that if $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$, then [5] series (1.2) regularly converges in $\mathbb{C}$ if and only if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} M_f^{-1}\left(\frac{1}{|a_n|}\right) = +\infty, \tag{1.4}$$

where $M_f^{-1}(x)$ is the function inverse to $M_f(r)$.

The growth of entire functions given by regularly convergent series (1.2) was studied in articles [5], [6] and [7]. In addition, in [7] the belonging of the entire functions (1.2) to a certain Banach space is investigated. For entire functions of a finite generalized order the belonging to the Frechet space is investigated in [7].

In the second half of the last century, the concept of bornological space appeared (see, for example, [2], [3] and [4]). Here we will define bornology on the set $\mathfrak{A}$ of all entire functions represented by series (1.2) regularly converging in $\mathbb{C}$ and prove some of its properties. Clearly, $\mathfrak{A}$ is vector space.

2. Bornology on $\mathfrak{A}$

A bornology on a set X is a family $\mathfrak{B}$ of subsets of X such that: a) $X = \bigcup\limits_{B\in\mathfrak{B}} B$; b) if $A \subset \mathfrak{B}$ and $B \subset A$ then $B \subset \mathfrak{B}$; c) if $A \subset \mathfrak{B}$ and $B \subset \mathfrak{B}$ then $A \bigcup B \subset \mathfrak{B}$. A pair $(X, \mathfrak{B})$ is called a *bornological* space, and the elements of $\mathfrak{B}$ are called the bounded subset of X.

A *base* of a bornology $\mathfrak{B}$ on X is any subfamily $\mathfrak{B}_0$ of $\mathfrak{B}$ such that every element of $\mathfrak{B}$ is contained in an element of $\mathfrak{B}_0$. A family $\mathfrak{B}_0$ of subsets of X is a base for a bornology $\mathfrak{B}$ on X if and only if $X = \bigcup\limits_{B\in\mathfrak{B}_0} B$ and every finite union of element of $\mathfrak{B}_0$ is contained in a member of $\mathfrak{B}_0$. Then the collection of these subsets of X, which are contained in an element of $\mathfrak{B}_0$, defines a bornology $\mathfrak{B}$ on X having $\mathfrak{B}_0$ as a base. A bornology is said to be a bornology with a *countable* base if it possesses a countable base $\mathfrak{B}_0 = \{B_n\}_{n=1}^{\infty}$.

For a vector space E over the complex field $\mathbb{C}$, a bornology $\mathfrak{B}$ on E is said to be a vector bornology on E if $\mathfrak{B}$ is stable under vector addition, homothetic transformations and the formation of circled hulls, i. e. the sets $A+B$, λA, $\bigcup\limits_{|\eta|\le 1} \eta A$ belong to $\mathfrak{B}$, whenever A and B belong to $\mathfrak{B}$ and $\lambda \in \mathbb{C}$. A pair $(E, \mathfrak{B})$ is called a bornological vector space.

A vector bornology on E is called a *convex vector bornology* if it is stable under the formation of convex hulls. Such a bornology is also stable under the formation of disked hulls, since the convex hull of a circled set is circled. A bornological vector space $(E, \mathfrak{B})$ whose bornology $\mathfrak{B}$ is convex is called a *convex bornological vector space*.

A separated bornological vector space $(E, \mathfrak{B})$ is one where $\{0\}$ is the only bounded vector subspace of E.

Suppose that $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$. In view of (1.4)

$$\exp\left\{-\frac{1}{\lambda_n}M_f^{-1}\left(\frac{1}{|a_n|}\right)\right\}\to 0, \quad n\to\infty,$$

and therefore, there exists

$$\gamma(A)=\sup_{n\geq 1}\exp\left\{-\frac{1}{\lambda_n}M_f^{-1}\left(\frac{1}{|a_n|}\right)\right\}.$$

Note that $M_f^{-1}(x)$ can be <0 on $(0,x_0]$ (for example, if $M_f(r)=e^r$, $M_f^{-1}(x)=\ln x<0$ for $0<x<1$). Thus, $\gamma(A)$ can be a sufficiently large positive number. Therefore, we denote

$$B_k=\{A\in\mathfrak{A}:\gamma(A)\leq k\}=\left\{A\in\mathfrak{A}:\ \sup_{n\geq 1}\exp\left\{-\frac{1}{\lambda_n}M_f^{-1}\left(\frac{1}{|a_n|}\right)\right\}\leq k\right\}, \quad k\in\mathbb{N}.$$

Then $B_k\subset B_{k+1}$ and for every $A\in\mathfrak{A}$ there exists $k\in\mathbb{N}$ such that $A\in B_k$. Thus, the family $\mathfrak{B}_0=\{B_k: k=1,2,\ldots\}$ forms a base for a bornology $\mathfrak{B}$ on $\mathfrak{A}$.

Theorem 2.1. *If* $\ln n=o(\Gamma_f(\lambda_n))$ *as* $n\to\infty$*, then* $(\mathfrak{A},\mathfrak{B})$ *is a separated convex bornological vector space with a countable base.*

Proof. Since the vector bornology $\mathfrak{B}$ on the vector space $\mathfrak{A}$ is stable under the formation of the convex hulls, it is a convex vector bornology. Hence it follows that $(\mathfrak{A},\mathfrak{B})$ is a convex bornological vector space.

To show that $\{0\}$ is the only bounded vector subspaces on $\mathfrak{A}$, we must show that $\mathfrak{A}$ contains no bounded open set.

Let $U(\varepsilon)=\{A\in\mathfrak{A}:\gamma(A)<\varepsilon\}$. It is enough to show that no $U(\varepsilon)$ is bounded, that is given $U(\varepsilon)$ there exists $U(\eta)$, for which there is no $c>0$ such that $U(\varepsilon)\subset cU(\eta)$.

Since $\Gamma_f(r)\nearrow+\infty$ as $r\to+\infty$, for every $\varepsilon>0$ we have

$$\begin{aligned}\frac{M_f(\lambda_n\ln(4/\varepsilon))}{M_f(\lambda_n\ln(2/\varepsilon))}&=\exp\left\{\ln M_f\left(\lambda_n\ln\frac{4}{\varepsilon}\right)-\ln M_f\left(\lambda_n\ln\frac{2}{\varepsilon}\right)\right\}\\&=\exp\left\{\int_{\lambda_n\ln(2/\varepsilon)}^{\lambda_n\ln(4/\varepsilon)}\frac{d\ln M_f(x)}{d\ln x}d\ln x\right\}=\exp\left\{\int_{\lambda_n\ln(2/\varepsilon)}^{\lambda_n\ln(4/\varepsilon)}\Gamma_f(x)d\ln x\right\}\\&\geq\Gamma_f(\lambda_n\ln(2/\varepsilon))\ln\frac{\ln(4/\varepsilon)}{\ln(2/\varepsilon)}\to+\infty, \quad n\to\infty.\end{aligned}$$

Therefore, for any $c>0$ we can choose $m\in\mathbb{N}$ such that for $\eta=\varepsilon/4$

$$c<\frac{M_f(\lambda_n\ln(4/\varepsilon))}{M_f(\lambda_n\ln(2/\varepsilon))}=\frac{M_f(\lambda_m\ln(1/\eta))}{M_f(\lambda_m\ln(2/\varepsilon))}.$$

Now we put $a_m=\dfrac{1}{M_f(\lambda_m\ln(2/\varepsilon))}$ and consider a function $A_m(z)=a_mf(\lambda_m z)$. Then

$$\gamma(A_m)=\exp\left\{-\frac{1}{\lambda_m}M_f^{-1}\left(\frac{1}{|a_m|}\right)\right\}=\varepsilon/2<\varepsilon.$$

On the other hand, for every $c>0$

$$\gamma(A_m/c)=\exp\left\{-\frac{1}{\lambda_m}M_f^{-1}\left(\frac{c}{|a_m|}\right)\right\}$$
$$>\exp\left\{-\frac{1}{\lambda_m}M_f^{-1}\left(\frac{M_f\,(\lambda_m\ln(1/\eta))}{|a_m|M_f\,(\lambda_m\ln(2/\varepsilon))}\right)\right\}=\eta$$

must hold, that is $A_m(z)/c)$ does not belong to $U(\eta)$, i. e. $A_m(z)$ does not belong to $cU(\eta)$. This indicates that $U(\varepsilon)$ is not bounded. Thus, $\{0\}$ is the only bounded vector subspace on $\mathfrak{A}$ and $(\mathfrak{A},\mathfrak{B})$ is a separated vector space.

Finally, since $\mathfrak{B}$ possesses a base consisting of an increasing sequence of bounded sets, $(\mathfrak{A},\mathfrak{B})$ is a bornological vector space with a countable base. Theorem 2.1 is proved. □

3. Bornology and Frechet Spaces

For $q\in\mathbb{N}$ we define

$$\|A\|_q=\sum_{n=1}^{\infty}|a_n|M_f\,(q\lambda_n)\,. \tag{3.1}$$

It is easily seen that, for every $q>0$, $\|A\|_q$ defines a norm on the set of series (1.2) regularly convergent in $\mathbb{C}$. Let $\mathfrak{B}_q$ be the bornology on $(\mathfrak{A},\|\cdot\|_q)$ consisting of the sets bounded in the sense of the norm $\|\cdot\|_q$.

Proposition 3.1. $\bigcup_{q\geq 1}\mathfrak{B}_q\subset\mathfrak{B}$.

Indeed, if $A\in\mathfrak{B}_q$ *for* $q\geq 1$, *then* $\sum_{n=1}^{\infty}|a_n|M_f\,(q\lambda_n)\leq Q<+\infty$, *whence* $|a_n|\leq Q/M_f\,(q\lambda_n))$ *for all* $n\geq 1$ *and, thus,*

$$\exp\left\{-\frac{1}{\lambda_n}M_f^{-1}\left(\frac{1}{|a_n|}\right)\right\}\leq\exp\left\{-\frac{1}{\lambda_n}M_f^{-1}\left(\frac{M_f\,(q\lambda_n)}{Q}\right)\right\}=$$
$$=\exp\left\{-(1+o(1))q\right\},\quad n\to\infty,$$

because $M_f^{-1}(x)$ *is a slowly increasing function. Hence it follows that* $\gamma(A)\leq k$, *i. e.* $A\in B_k$ *for some* k *and, thus,* $\bigcup_{q\geq 1}\mathfrak{B}_q\subset\mathfrak{B}$.

Conjecture 3.1. *If* $\ln n=o(\Gamma_f(\lambda_n))$ *as* $n\to\infty$, *then* $\bigcup_{q\geq 1}\mathfrak{B}_q=\mathfrak{B}$.

The family $||A||_q:q\in\mathbb{N}$ induces on $\mathfrak{A}$ a unique topology given by the metric d, where

$$d(A_1,A_2)=\sum_{q=1}^{\infty}\frac{1}{2^q}\frac{||A_1-A_2||_q}{1+||A_1-A_2||_q}. \tag{3.2}$$

The space with the metric d we denote by $\mathfrak{A}_d$.

Theorem 3.1. *If* $\ln n=o(\Gamma_f(\lambda_n))$ *as* $n\to\infty$, *then* $\mathfrak{A}_d$ *is a Frechet space.*

Proof. It is sufficient to show that $\mathfrak{A}_d$ is complete. Let (A_j) be a d-Cauchy sequence in $\mathfrak{A}_d$ and so for a given $\varepsilon > 0$ there corresponds a $m = m(\varepsilon)$ such that $||A_j - A_k||_q < \varepsilon$ for all j, $k \geq m$ and $q \in \mathbb{N}$; consequently for these j, k and each $q \geq 1$ we have

$$\sum_{n=1}^{\infty} |a_n^{(j)} - a_n^{(k)}| M_f(q\lambda_n) < \varepsilon, \tag{3.3}$$

i. e. $|a_n^{(j)} - a_n^{(k)}| < \varepsilon$ and $(a_n^{(j)})_{j\geq 1})$ is a Cauchy sequence. Therefore, $a_n^{(j)} \to a_n$ as $j \to \infty$. Letting $k \to \infty$ in (3.3) one has for $j \geq j_0$

$$\sum_{n=1}^{\infty} |a_n^{(j)} - a_n| M_f(q\lambda_n) < \varepsilon, \tag{3.4}$$

and consequently taking $j = j_0$ in (3.4) we get $\sum\limits_{n=1}^{\infty} |a_n^{(j_0)} - a_n| M_f(q\lambda_n) < \varepsilon$ for a fixed q, i. e. $|a_n^{(j_0)} - a_n| M_f(q\lambda_n) < \varepsilon$.

Since $\varliminf\limits_{n\to\infty} \dfrac{1}{\lambda_n} M_f^{-1}\left(\dfrac{1}{|a_n^{(j_0)}|}\right) = +\infty$, for every $K > q$ and all $n \geq n_0(K)$ we have $|a_n^{(j_0)}| \leq 1/M_f(K\lambda_n)$. Using this inequality we obtain

$$|a_n| M_f(q\lambda_n) \leq |a_n^{(j_0)}| M_f(q\lambda_n) + \varepsilon \leq \frac{M_f(q\lambda_n)}{M_f(K\lambda_n)} + \varepsilon \leq 2\varepsilon$$

i.e.

$$\varliminf_{n\to\infty} \frac{1}{\lambda_n} M_f^{-1}\left(\frac{1}{|a_n|}\right) \geq \varliminf_{n\to\infty} \frac{1}{\lambda_n} M_f^{-1}\left(\frac{M_f(q\lambda_n)}{2\varepsilon}\right) = q,$$

because $M_f^{-1}(x)$ is a slowly increasing function, whence in view of the arbitrariness of q it follows that series (1.2) with such coefficients a_n regularly converges in $\mathbb{C}$ and, thus, using (3.4), again we see that $||A_j - A||_q < \varepsilon$ for $j \geq j_0$ and the result is proved. □

For $\mathfrak{A}_d$ by $\mathfrak{A}_d^*$ we denote the dual space, i. e. $\mathfrak{A}_d^*$ is the family of all continuous linear functionals $L(A)$ on $\mathfrak{A}_d$.

Theorem 3.2. *If* $\ln n = o(\Gamma_f(\lambda_n))$ *as* $n \to \infty$, *then the continuous linear functional* L *on* $\mathfrak{A}_d$ *is of the form*

$$L(A) = \sum_{n=1}^{\infty} a_n g_n, \quad A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z), \tag{3.5}$$

if and only if for all $n \in \mathbb{N}$ *and* $q \in \mathbb{N}$.

$$|g_n| \leq QM_f(q\lambda_n)), \quad Q = const > 0. \tag{3.6}$$

Proof. Let $L \in \mathfrak{A}_d^*$, then clearly if $A_m \to A$ in $\mathfrak{A}_d$, then $L(A_m) \to L(A)$.

Now let $A_m(s) = \sum\limits_{n=1}^{m} a_n f(z\lambda_n)$. Then we claim that $A_m \to A$ in $\mathfrak{A}_d$ (observe that $A_m \in \mathfrak{A}_d$). To ascertain this, it is sufficient to prove that $A_m \to A$ under the norm $||\cdot||_q$ for every $q \in \mathbb{N}$.

So let q be a fixed integer. Choose $K > q$. Then as above we can determine an integer $m = m(\varepsilon)$ such that $|a_n| \le 1/M_f(K\lambda_n)$ for $n \ge m+1$, and it follows from above that

$$||A_m - A||_q = ||\sum_{n=m+1}^{\infty} a_n f(z\lambda_n)||_q = \sum_{n=m+1}^{\infty} |a_n| M_f(q\lambda_n)) \le \sum_{n=m+1}^{\infty} \frac{M_f(q\lambda_n)}{M_f(K\lambda_n)} \to 0, \ m \to \infty,$$

and this ascertains our claim. From that and the continuity of L, we have $\lim\limits_{m\to\infty} L(A_m) = L(A)$ in the topology given by d.

Note that $L(A_m) = \sum\limits_{n=1}^{m} d_n g_n$, where $g_n = L(f(z\lambda_n))$ for each n. Since L is continuous on $(\mathfrak{A}_d, ||\cdot||_q)$, there exists a $Q > 0$ such that $|g_n| = |L(f(z\lambda_n))| \le Q||f(z\lambda_n))||_q$ for each $q \in \mathbb{N}$ and so, using the definition of the norm $||f(z\lambda_n))||_q$, we get (3.6).

To prove the other part, let now g_n satisfy (3.6). Then

$$|L(A)| \le Q \sum_{n=1}^{\infty} |a_n| M_f(q\lambda_n)), \quad q \in \mathbb{N},$$

and hence $|L(A)| \le Q||A||_q$ for all $q \in \mathbb{N}$. Therefore, $L \in (\mathfrak{A}_d, ||\cdot||_q)^*$ for all $q \in \mathbb{N}$. Since $\mathfrak{A}_d^* = \bigcup\limits_{q\ge 1} (\mathfrak{A}_d, ||\cdot||_q)^*$, we get $L \in \mathfrak{A}_d^*$. Theorem 3.2 is proved. ☐

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