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THE SPATIAL NUMERICAL RANGE IN NON-UNITAL, NORMED ALGEBRAS AND THEIR UNITIZATIONS

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ABSTRACT. Let $(A, \|\cdot\|)$ be any normed algebra (not necessarily complete nor unital). Let $a \in A$ and let $V_A(a)$ denote the spatial numerical range of a in $(A, \|\cdot\|)$. Let $A_e = A + \mathbb{C}1$ be the unitization of A. If A is faithful, then we get two norms on A_e; namely, the operator norm $\|\cdot\|_{op}$ and the ℓ^1-norm $\|\cdot\|_1$. Let $A^{op} = (A, \|\cdot\|_{op})$, $A_e^{op} = (A_e, \|\cdot\|_{op})$, and $A_e^1 = (A_e, \|\cdot\|_1)$. We can calculate the spatial numerical range of a in all three normed algebras. Because the spatial numerical range highly depends on the identity as well as on the completeness and the regularity of the norm, they are different. In this paper, we study the relations among them. Some results that are proved in [2, section 2] and [3, section 10] will become corollaries of our results. We shall also show that the completeness and regularity of the norm is not required in [6, Theorem 2.3].

1. INTRODUCTION

Throughout the paper A is any normed algebra; the (algebra) norm on A will be denoted by $\|\cdot\|$. The algebra A is *faithful* if $a = 0$ whenever $a \in A$ and $aA = \{0\}$. Let $A_e = A + \mathbb{C}1$ be the unitization of A [3, page 15], where 1 is the identity of A_e. If A is faithful, then any norm $\|\cdot\|$ on A induces the following two norms on A_e; namely, the operator norm $\|\cdot\|_{op}$ and the ℓ^1-norm $\|\cdot\|_1$, which are defined as follows.

$$\|a + \lambda 1\|_{op} = \sup\{\|ax + \lambda x\| : x \in A, \|x\| \leq 1\}$$
$$\|a + \lambda 1\|_1 = \|a\| + |\lambda|.$$

In general, $\|\cdot\|_{op} \leq \|\cdot\|$ on A. The norm $\|\cdot\|$ is *regular* if $\|\cdot\|_{op} = \|\cdot\|$ on A. We shall use the notions $A = (A, \|\cdot\|)$, $A^{op} = (A, \|\cdot\|_{op})$, $A_e^{op} = (A_e, \|\cdot\|_{op})$, and $A_e^1 = (A_e, \|\cdot\|_1)$.

Let $S(A) = \{x \in A : \|x\| = 1\}$ be the unit sphere in A. Let A^* be the Banach space dual of A. Let $D_A(x) = \{\varphi \in A^* : \|\varphi\| = 1 = \varphi(x)\}$ for each $x \in S(A)$. Further, let $V_A(a;x) = \{\varphi(ax) : \varphi \in D_A(x)\}$ for $a \in A$ and $x \in S(A)$. Then $V_A(a) = \cup\{V_A(a;x) : x \in S(A)\}$ is the *spatial numerical range (SNR)* and $\nu_A(a) = \sup\{|\lambda| : \lambda \in V_A(a)\}$ is the *spatial numerical radius* of a in $(A, \|\cdot\|)$. It is still an open problem whether $V_A(a)$ is always convex? This problem is discussed in [5].

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We shall see that the spatial numerical range $V_A(a)$ highly depends on both the algebra A and the norm $\|\cdot\|$. Therefore the sets $V_A(a)$, $V_{A^{op}}(a)$, $V_{A_e^{op}}(a)$, and $V_{A_e^1}(a)$ cannot be identical. In this paper, we study the relations among them and exhibit various examples. We have applied this concept in proving some results on the spectral extension property (SEP) in non-unital Banach algebras [4].

2. MAIN RESULTS

Recall that the spatial numerical range $V_A(a) = \cup\{V_A(a;x) : x \in S(A)\}$. Firstly we list some basic properties of the sets $V_A(a;x)$.

Theorem 2.1. *Let A be any normed algebra, $a,b \in A$, $x,y \in S(A)$, and $\alpha \in \mathbb{C}$. Then*

(1) $V_A(\alpha a;x) = \alpha V_A(a;x)$ *and* $V_A(a+b;x) \subset V_A(a;x)+V_A(b;x)$*;*
(2) $V_A(a;x)$ *is compact and convex;*
(3) $V_A(a;x) = V_A(a;y)$ *whenever x and y are linearly dependent;*
(4) *Let B be a subalgebra of A. Then*

$$V_B(a;x) = V_A(a;x)\ (a \in B, x \in S(B));$$

(5) *Let $a_n \longrightarrow a$ in A. Then* $\overline{V_A(a;x)} \subset \overline{\cup_{n=1}^{\infty} V_A(a_n;x)}$.

Proof. (1) This is easy.

(2) Let $\{\lambda_n\} \subset V_A(a;x)$ such that $\lambda_n \longrightarrow \lambda$. Then there exists $\varphi_n \in D_A(x)$ such that $\varphi_n(ax) = \lambda_n$. Since $D_A(x)$ is weak*-compact, we may assume that $\varphi_n \longrightarrow \varphi$ in $D_A(x)$ in weak*-topology. Then $\lambda_n = \varphi_n(ax) \longrightarrow \varphi(ax)$. Hence $\lambda = \varphi(ax) \in V_A(a;x)$. So $V_A(a;x)$ is closed. Clearly, it is a bounded set. Let $\lambda_1, \lambda_2 \in V_A(a;x)$. Then there exist $\varphi_1, \varphi_2 \in D_A(x)$ such that $\varphi_1(ax) = \lambda_1$ and $\varphi_2(ax) = \lambda_2$. Let $r \in [0,1]$ and $\varphi = r\varphi_1 + (1-r)\varphi_2 \in A^*$. Then $\|\varphi\| = \|r\varphi_1 + (1-r)\varphi_2\| \leq |r|\|\varphi_1\| + |1-r|\|\varphi_2\| \leq 1$ and $\varphi(x) = (r\varphi_1 + (1-r)\varphi_2)(x) = 1$. Therefore $\|\varphi\| = 1$. So $\varphi \in D_A(x)$ and $r\lambda_1 + (1-r)\lambda_2 = r\varphi_1(ax) + (1-r)\varphi_2(ax) = \varphi(ax) \in V_A(a;x)$.

(3) Since x and y are linearly dependent, there exists $\alpha \in \mathbb{C}$ such that $y = \alpha x$. Since $\|x\| = \|y\| = 1$, we get $|\alpha| = 1$. Let $\lambda \in V_A(a;x)$. Then there exists $\varphi \in D_A(x)$ such that $\varphi(ax) = \lambda$. Define $\psi(z) = \overline{\alpha}\varphi(z)\,(z \in A)$. Then $\psi \in D_A(y)$, and $\lambda = \varphi(ax) = \varphi(\overline{\alpha}\alpha(ax)) = \overline{\alpha}\varphi(a(\alpha x)) = \overline{\alpha}\varphi(ay) = \psi(ay)$. Hence $V_A(a;x) \subset V_A(a;y)$. Similarly, we can prove that $V_A(a;y) \subset V_A(a;x)$.

(4) Let $\lambda \in V_B(a;x)$. So there exists $\varphi \in D_B(x)$ such that $\lambda = \varphi(ax)$. By the Hahn-Banach extension theorem, there exists $\widetilde{\varphi} \in A^*$ such that $\widetilde{\varphi}|_B = \varphi$ and $\|\widetilde{\varphi}\| = \|\varphi\|$. Therefore $\widetilde{\varphi} \in D_A(x)$. Hence $\lambda = \varphi(ax) = \widetilde{\varphi}(ax) \in V_A(a;x)$. Conversely, let $\lambda \in V_A(a;x)$. So there exists $\widetilde{\varphi} \in D_A(x)$ such that $\lambda = \widetilde{\varphi}(ax)$. Now consider $\varphi = \widetilde{\varphi}|_B$. Since $x \in S(B)$, we have $\varphi(x) = \widetilde{\varphi}|_B(x) = \widetilde{\varphi}(x) = 1$. So $\|\varphi\| = 1$. Thus, $\varphi \in D_B(x)$. Hence $\lambda = \widetilde{\varphi}(ax) = \widetilde{\varphi}|_B(ax) = \varphi(ax) \in V_B(a;x)$.

(5) Let $\lambda \in V_A(a;x)$. There exists $\varphi \in D_A(x)$ such that $\lambda = \varphi(ax)$. Then $\lambda = \varphi(ax) = \lim_{n\to\infty} \varphi(a_n x) \in \overline{\cup_{n=1}^{\infty} V_A(a_n;x)}$. ☐

Theorem 2.2. *Let A be any normed algebra. Let $a,b \in A$, $x,y \in S(A)$, and $\alpha \in \mathbb{C}$. Then*

(1) *$V_A(\alpha a) = \alpha V_A(a)$ and $V_A(a+b) \subset V_A(a)+V_A(b)$;*
(2) *$V_A(a)$ is bounded but need not be closed;*
(3) *If A has a finite dimension, then $V_A(a)$ is compact;*
(4) *Let B be a subalgebra of A. Then $V_B(a) \subset V_A(a)$ $(a \in B)$;*
(5) *Let $a_n \longrightarrow a$ in A. Then $\overline{V_A(a)} \subset \overline{\cup_{n=1}^{\infty} V_A(a_n)}$.*

Proof. (1) This is trivial. Also See [6, Lemma 2.2].

(2) Note that $|\varphi(ax)| \leq \|ax\| \leq \|a\|$ $(x \in S(A), \varphi \in D_A(x))$. Hence $V_A(a)$ is bounded. Let $A = (\ell^1, \|\cdot\|_1)$ with the pointwise product. Let $a = (1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots, \frac{1}{n^2}, \ldots) \in A$. Then by [5, Theorem 2.4], $V_A(a) = (0,1]$. Thus, the spatial numerical range may not be closed.

(3) $V_A(a)$ is bounded by Statement (2) above.
Let $\lambda \in \overline{V_A(a)}$. Then there exists $\{\lambda_n\} \subset V_A(a)$ such that λ_n converges to λ. So there exist $x_n \in S(A)$ and $\varphi_n \in D_A(x_n)$, where $n \in \mathbb{N}$ such that $\varphi_n(ax_n) = \lambda_n$. Since A has finite dimension, $S(A)$ is compact. So without loss of generality, we can assume that $x_n \longrightarrow x$ in $S(A)$. Similarly, $S(A^*)$ is compact. So $\{\varphi_n\}$ has a convergent subsequence, say $\varphi_{n_k} \longrightarrow \varphi$. So we have

$$\begin{aligned}|1-\phi(x)| &= |\varphi_{n_k}(x_{n_k}) - \varphi(x)| \\ &= |\varphi_{n_k}(x_{n_k}) - \varphi_{n_k}(x) + \varphi_{n_k}(x) - \varphi(x)| \\ &= \|\varphi_{n_k}\|\|x_{n_k} - x\| + \|\varphi_{n_k} - \varphi\|\|x\| \\ &\longrightarrow 0.\end{aligned}$$

Therefore $\varphi(x) = 1$. Thus $\|\varphi\| = 1$ and $\|x\| = 1$ and so $\varphi(ax) \in V_A(a)$. Now

$$\begin{aligned}|\lambda_{n_k} - \varphi(ax)| &= |\varphi_{n_k}(ax_{n_k}) - \varphi(ax)| \\ &= |\varphi_{n_k}(ax_{n_k}) - \varphi_{n_k}(ax) + \varphi_{n_k}(ax) - \varphi(ax)| \\ &= \|\varphi_{n_k}\|\|ax_{n_k} - ax\| + \|\varphi_{n_k} - \varphi\|\|ax\| \\ &\longrightarrow 0.\end{aligned}$$

Thus $\lambda = \lim_{k\to\infty} \lambda_{n_k} = \varphi(ax) \in V_A(a)$. Hence $V_A(a)$ is closed and consequently compact.

(4) Let $\lambda \in V_B(a)$. Then $\lambda \in V_B(a;x)$ for some $x \in S(B)$. So there exists $\varphi \in D_B(x)$ such that $\lambda = \varphi(ax)$. By the Hahn-Banach extension theorem, there exists $\widetilde{\varphi} \in A^*$ such that $\widetilde{\varphi}|_B = \varphi$ and $\|\widetilde{\varphi}\| = \|\varphi\|$. Therefore $\widetilde{\varphi} \in D_A(x)$. Hence $\lambda = \varphi(ax) = \widetilde{\varphi}(ax) \in V_A(a;x) \subset V_A(a)$.

(5) Let $\lambda \in V_A(a)$. Choose $x \in S(A)$ and $\varphi \in D_A(x)$ such that $\lambda = \varphi(ax)$. Then $\lambda = \varphi(ax) = \lim_{n\to\infty} \varphi(a_n x) \in \overline{\cup_{n=1}^{\infty} V_A(a_n)}$. □

The well-established results (2), (4) and (5) in Corollary 2.1 follow from Theorems 2.1 and 2.2 above.

Corollary 2.1. [3, Section 10] *Let A be a unital normed algebra and let $a \in A$. Then*

(1) $V_A(a) = V_A(a;1)$;
(2) $V_A(a)$ *is compact and convex;*
(3) $\exp(-1)\|a\| \le \nu_A(a) \le \|a\|$;
(4) *If $B \subseteq A$ is a subalgebra, then* $V_B(b) \subseteq V_A(b;1)$ $(b \in B)$;
(5) *If $B \subseteq A$ is a unital subalgebra with the same identity* 1*, then*

$$V_B(b;1) = V_A(b;1) \ (b \in B).$$

The first statement of the following result is a generalization of [6, Theorem 2.3]; neither the completeness nor the regularity of the norm is required. In the rest of the paper, the set $\overline{co}(K)$ denotes the closed convex hull of $K \subset \mathbb{C}$.

Theorem 2.3. *Let $(A, \|\cdot\|)$ be any non-unital, faithful, normed algebra. Then,*

(1) $\overline{co}V_A(a) = V_{A_e^{op}}(a;1)$;
(2) $\nu_A(a) = \nu_{A_e^{op}}(a;1)$;
(3) $\exp(-1)\|a+\lambda 1\|_{op} \le \nu_{A_e^{op}}(a+\lambda 1;1) \le \|a+\lambda 1\|_{op}$ $(a+\lambda 1 \in A_e)$;
(4) $\exp(-1)\|a\|_{op} \le \nu_A(a) \le \|a\|_{op}$ $(a \in A)$;
(5) *If $\|\cdot\|$ is regular, then* $\exp(-1)\|a\| \le \nu_A(a) \le \|a\|$ $(a \in A)$;
(6) *If $\|\cdot\|$ is regular and complete, then* $0 \in \overline{co}V_A(a)$ $(a \in A)$.

Proof. (1) For each $a+\lambda 1 \in A_e$, define $L_{a+\lambda 1}(x) = ax + \lambda x$ $(a \in A)$. Then $L_{a+\lambda 1} \in BL(A)$, where $BL(A)$ is the set of all bounded linear operators on A. We define $\Phi : A_e^{op} \longrightarrow BL(A)$ by $\Phi(a+\lambda 1) = L_{a+\lambda 1}$. Clearly, the map Φ is an algebra homomorphism. If $\Phi(a+\lambda 1) = 0$, then for all $x \in A$, $ax+\lambda x = 0$ so $x = \frac{-a}{\lambda}x$ if $\lambda \neq 0$. Thus $\frac{-a}{\lambda}$ for $\lambda \neq 0$, is a left identity and, by faithfulness, also a right identity of A, which is ruled out by the hypothesis. If $\lambda = 0$, then $ax = 0$. By the faithfulness, $a = 0$. Therefore $a+\lambda 1 = 0$. Thus Φ is injective.
Also, we have $\|a+\lambda 1\|_{op} = \|L_{a+\lambda 1}\|$. So Φ is an isometric algebra isomorphism. Now consider $L_{A_e} = \{L_{a+\lambda 1} : a+\lambda 1 \in A_e\}$, which is a unital subalgebra of $BL(A)$. Then $\Phi : (A_e, \|\cdot\|_{op}) \longrightarrow (L_{A_e^{op}}, \|\cdot\|)$ is an isometric onto algebra isomorphism. This implies that

$$V_{A_e^{op}}(a+\lambda 1;1) = V_{L_{A_e}}(L_{a+\lambda 1};I) \quad (a+\lambda 1 \in A_e). \tag{2.1}$$

Hence, by [2, Theorem 4(i), page 84]),

$$\begin{aligned}\overline{co}V_A(a) &= V_{BL(A)}(L_a;I)\\ &= V_{L_{A_e}}(L_a;I) \quad \text{(By Corrollary 2.1(5))}\\ &= V_{A_e^{op}}(a;1) \quad \text{(By Equation (2.1))}.\end{aligned}$$

Hence $\overline{co}V_A(a) = V_{A_e^{op}}(a;1)$.

(2) This is clear because $\sup\{|\lambda| : \lambda \in \overline{co}(K)\} = \sup\{|\lambda| : \lambda \in K\}$ for any $K \subset \mathbb{C}$.

(3) This follows from Corollary 2.1(3).

(4) This follows from Statement (2) and (3) above.

(5) This is immediate from Statement (4).

(6) Define $\varphi_\infty : A_e^{op} \longrightarrow \mathbb{C}$ as $\varphi_\infty(x+\lambda 1) = \lambda$. Since $\|\cdot\|$ is complete and regular, $ker\varphi_\infty = A$ is closed in A_e^{op}. Hence φ_∞ is continuous. Therefore we have $0 \in V_{A_e^{op}}(a;1)\,(a \in A)$. By Statement (1), we get $0 \in \overline{co}V_A(a)\,(a \in A)$. □

The reader should compare Theorem 2.3(1) with Theorem 2.4(1) below.

Theorem 2.4. *Let A be any non-unital normed algebra. Then*
(1) $\overline{co}(V_A(a) \cup \{0\}) \subset V_{A_e^1}(a;1)$ $(a \in A)$*;*
(2) $\exp(-1)\|a+\lambda 1\|_1 \leq \nu_{A_e^1}(a+\lambda 1;1) \leq \|a+\lambda 1\|_1$ $(a+\lambda 1 \in A_e)$*;*
(3) $\exp(-1)\|a\| \leq \nu_{A_e^1}(a;1) \leq \|a\|$ $(a \in A)$.

Proof. (1) Define $\varphi_\infty : A_e^1 \longrightarrow \mathbb{C}$ as $\varphi_\infty(x+\lambda 1) = \lambda$. Then $\varphi_\infty \in D_{A_e^1}(1)$ and so, $0 = \varphi_\infty(a) \in V_{A_e^1}(a;1)$. Hence, by Theorem 2.2(4), $V_A(a) \cup \{0\} \subset V_{A_e^1}(a;1)$. Since $V_{A_e^1}(a;1)$ is closed and convex as per Theorem 2.1(2), $\overline{co}(V_A(a) \cup \{0\}) \subset V_{A_e^1}(a;1)$.

(2) This is immediate from Corollary 2.1(3).

(3) Take $\lambda = 0$ in Statement (2). □

The next result gives all possible relations among $V_A(a), V_{A^{op}}(a), V_{A_e^{op}}(a+\lambda 1)$, and $V_{A_e^1}(a+\lambda 1)$. It could be viewed as a summary of this paper.

Theorem 2.5. *Let A be a faithful normed algebra, let $a \in A$, and let $\lambda \in \mathbb{C}$. Then*
(1) $\overline{co}V_A(a) = V_{A_e^{op}}(a;1)$*;*
(2) $V_{A^{op}}(a) \subseteq V_{A_e^{op}}(a;1)$*;*
(3) $V_{A^{op}}(a) \subseteq \overline{co}V_A(a)$*;*
(4) $\overline{co}(V_A(a) \cup \{0\}) \subseteq V_{A_e^1}(a;1)$*;*
(5) $V_{A^{op}}(a) \subseteq V_{A_e^1}(a;1)$*;*
(6) $V_{A_e^{op}}(a+\lambda 1;1) \subseteq V_{A_e^1}(a+\lambda 1;1)$.

Proof. (1) This is Theorem 2.3(1) above.

(2) A is a subalgebra of A_e and the norms on A and A_e are same. Therefore by Theorem 2.2(4), we get $V_{A^{op}}(a) \subseteq V_{A_e^{op}}(a;1)$.

(3) This follows from Statement (1) and (2) above.

(4) This is Theorem 2.4(1) above.

(5) This follows from Statement (3) and (4) above.

(6) It is true that $V_{A_e^{op}}(a+\lambda 1) = V_{A_e^{op}}(a;1)+\lambda$. So by Statement (1), we get $V_{A_e^{op}}(a+\lambda 1;1) = \overline{co}V_A(a)+\lambda$. Therefore by Statement (3) above, we have $V_{A_e^{op}}(a+\lambda 1;1) \subseteq V_{A_e^1}(a;1)+\lambda = V_{A_e^1}(a+\lambda 1;1)$. □

3. Counter Examples

In this section, we intend to give several examples showing that various conditions in our main results cannot be omitted.

Let $1 \leq p \leq \infty$. Let $A = (\mathbb{C}^n, \|\cdot\|_p)$ be a Banach space with pointwise linear operations. Then the dual of A is $A^* = (\mathbb{C}^n, \|\cdot\|_q)$, where $\frac{1}{p}+\frac{1}{q} = 1$. Therefore, if $\varphi \in A^*$, then there exists a unique $y \in A^*$ such that $\varphi = \phi_y$ and $\|\varphi\| = \|y\|_q$, where $\phi_y(x) = \langle x,y\rangle = \sum_{k=1}^n x_k y_k\,(x \in A)$.

Example 3.1. *Let* $A = (\mathbb{C}^2, \|\cdot\|_p)$ *with the product* $xy = (x_1y_1, x_1y_2) = x_1y$. *Then* $(A, \|\cdot\|_p)$ *is a non-unital Banach algebra. Then, for* $a = (a_1, a_2) \in A$ *and* $x = (x_1, x_2) \in S(A)$,

$$V_A(a;x) = \{a_1\} \text{ and } V_A(a) = \{a_1\}. \tag{3.1}$$

(I) Take $a = (1,0)$, $a_1 = (0,0)$ and $a_n = a\,(n \geq 2)$. Then clearly $a_n \longrightarrow a$. So we have $\overline{V_A(a;x)} = \{1\}$ and $\overline{\cup_{n=1}^{\infty} V_A(a_n;x)} = \{1,0\}$. Hence the inclusions in Theorems 2.1(5) and 2.2(5) are proper for the sequence $\{a_n\}$.

(II) Let $A = (\mathbb{C}^2, \|\cdot\|_\infty)$. Let $a = (1,1)$, $x = (1,0)$ and $y = (0,1)$. Then x and y are linearly independent, but $V_A(a,x) = V_A(a,y) = \{1\}$. So, the converse of Theorem 2.1(3) is not true.

(III) Let $A = (\mathbb{C}^2, \|\cdot\|_1)$. Then $A_e^1 \cong \mathbb{C}^3$ is a Banach algebra with the product $xy = (x_1y_1 + x_1y_3 + x_3y_1, x_1y_2 + x_2y_3 + x_3y_2, x_3y_3)$. The dual of A_e^1 is $(\mathbb{C}^3, \|\cdot\|_\infty)$. Let $\varphi \in (A_e^1)^*$ and let $a \in A_e^1$, i.e. $(a_1, a_2, 0) \in A_e^1$, because $a \in A$ and A is viewed as a subalgebra of A_e^1. Then there exists $y = (y_1, y_2, y_3) \in \mathbb{C}^3$ such that $\varphi = \phi_y$ and $\|\varphi\| = \|y\|$. So, we have $a_3y_3 = 0$. Now

$$\begin{aligned} V_{A_e^1}(a;1) &= \{\varphi(a) : \varphi \in D_{A_e^1}(1)\} \\ &= \{\phi_y(a) : y \in \mathbb{C}^3 \text{ and } \max\{|y_1|, |y_2|, |y_3|\} = 1 = y_3\} \\ &= \{a_1y_1 + a_2y_2 : \max\{|y_1|, |y_2|\} \leq 1\} \\ &= \{\|a\|_1 z : |z| \leq 1\}. \end{aligned}$$

In particular, taking $a = (1,0)$, we get $\overline{co}(V_A(a) \cup \{0\}) = [0,1]$ and $V_{A_e^1}(a) = \{z \in \mathbb{C} : |z| \leq 1\}$. Thus the inclusion in Theorem 2.4(1) may be strict.

(IV) Let $A = (\mathbb{C}^2, \|\cdot\|_1)$. Then $\|\cdot\|_1$ is not regular. Let $a = (1,0) \in A$. Since $V_A(a) = \{1\}$, it follows that $0 \notin \overline{co}V_A(a)$. Hence Theorem 2.3(6) is not true if the norm is not regular.

Example 3.2. *Let* $A = (\mathbb{C}^2, \|\cdot\|_p)$ *with the product* $xy = (x_1y_1, x_2y_1) = xy_1$. *Then* $(A, \|\cdot\|_p)$ *is a non-unital Banach algebra and* $\|\cdot\|_p$ *is a regular norm on* A.

(I) Let $A = (\mathbb{C}^2, \|\cdot\|_\infty)$. Let $x \in S(A)$ and $\varphi = \phi_y \in D_A(x)$. Then

$$\max\{|x_1|, |x_2|\} = 1 = x_1y_1 + x_2y_2 \leq |x_1y_1| + |x_2y_2| \leq |y_1| + |y_2| = 1.$$

Therefore $x_1y_1 + x_2y_2 = |x_1y_1| + |x_2y_2| = 1$ and hence $x_1y_1, x_2y_2 \in [0,1]$. Take $x_1y_1 = r$ and so, $x_2y_2 = 1 - x_1y_1 = 1 - r$ for some $r \in [0,1]$. Assume that $y_1 \neq 0$. Since $|x_1y_1| + |x_2y_2| = 1 = |y_1| + |y_2|$, we get $|x_1| = 1$ i.e. $x_1 = e^{i\theta}$ for some $\theta \in (-\pi, \pi]$. Therefore by $x_1y_1 = r, x_2y_2 = 1 - r$ and $|y_1| + |y_2| = 1$, we get $y_1 = re^{-i\theta}$, $y_2 = (1-r)e^{-i\theta_1}$ and $x_2 = e^{i\theta_1}$ for some $\theta, \theta_1 \in (-\pi, \pi]$. Now $\varphi(ax) = \phi_y(ax_1) = a_1x_1y_1 + a_2x_1y_2 = a_1r + a_2(1-r)e^{i\theta'}$ for some $r \in [0,1]$ and $\theta' \in (-\pi, \pi]$. If $y_1 = 0$, then $\varphi(ax) = \phi_y(ax_1) = a_2x_1y_2 = a_2re^{i\theta}$ for some $r \in [0,1]$ and $\theta \in (-\pi, \pi]$. Since $x \in S(A)$ is arbitrary,

$$V_A(a) \subset \{a_1r + a_2(1-r)e^{i\theta} : r \in [0,1] \text{ and } \theta \in (-\pi, \pi]\}.$$

Conversely, let $r \in [0,1]$ and $\theta \in (-\pi, \pi]$. Take $x = (e^{i\theta}, 1)$ and $y = (re^{-i\theta}, 1-r)$.

Then $\|x\|_\infty = 1$, $\phi_y \in D_A(x)$, and $a_1 r + a_2(1-r)e^{i\theta} = \phi_y(ax) \in V_A(a;x) \subset V_A(a)$. Thus $V_A(a) = \{a_1 r + a_2(1-r)e^{i\theta} : r \in [0,1], \text{ and } \theta \in (-\pi,\pi]\}$.

(II) Let $A = (\mathbb{C}^2, \|\cdot\|_1)$. Then, as per the similar arguments in (I) above,

$$V_A(a) = \{a_1 r + a_2 r e^{i\theta} : r \in [0,1], \text{ and } \theta \in (-\pi,\pi]\}.$$

(III) It follows from (I) and (II) that $V_A(a)$ is different in $(\mathbb{C}^2, \|\cdot\|_\infty)$ and $(\mathbb{C}^2, \|\cdot\|_1)$ even though the two norms are equivalent.

(IV) Consider the subalgebra $B = \mathbb{C} \times \{0\}$ of A and take $a = (1,0) \in B$ which is the identity of B and so $V_B(a) = \{1\}$. But $V_A(a) = [0,1]$ as in (II) above. So $V_B(a) \subsetneq V_A(a)$. Thus the set inclusion in Theorem 2.2(4) may be strict.

Example 3.3. *Let A be a unital Banach algebra with identity 1 such that $A \neq \{0\}$. Take $a = 1$. Then $V_A(1;1) = \{1\}$. So, $0 \notin \overline{co}V_A(1;1)$. Thus Theorem 2.3(6) is true only for the non-unital case.*

Example 3.4. *Let $A = (\ell^1, \|\cdot\|_1)$ with pointwise operations. Then $\|\cdot\|_1$ is not regular. Define $f_n(k) = 1\,(k \leq n)$ and $f_n(k) = 0\,(k > n)$. Then, by Theorem 2.3(4), $\nu_A(f_n) \leq \|f_n\|_{op} = \|f_n\|_\infty = 1$. But $\|f_n\|_1 = n$ $(n \in \mathbb{N})$. Hence Theorem 2.3(5) is not true, if the norm on A is not regular.*

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