SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.20.02.09
Vol.20 (33), No.2 (2024), 263–280

LIPSCHITZ TYPE INEQUALITIES FOR AN INTEGRAL TRANSFORM OF POSITIVE OPERATORS WITH APPLICATIONS

SILVESTRU SEVER DRAGOMIR

ABSTRACT. We introduce the following *integral transform*

$$\mathcal{D}(\mu)(T) := -\int_0^\infty (\lambda + T)^{-1} d\mu(\lambda), \; t > 0,$$

where μ is a positive measure on $(0,\infty)$ and the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

In this paper we show among other results that, if $A \geq m_1 > 0$, $B \geq m_2 > 0$, then

$$\|\mathcal{D}(\mu)(B) - \mathcal{D}(\mu)(A)\| \leq \|B - A\| [m_1, m_2]_{\mathcal{D}(\mu)(\cdot)},$$

where $\mathcal{D}(\mu)(\cdot)$ is a function of t and $[m_1, m_2]_{\mathcal{D}(\mu)(\cdot)}$ is its divided difference. If $f : [0,\infty) \to \mathbb{R}$ is an operator monotone function with $f(0) = 0$, then

$$\left\| f(A)A^{-1} - f(B)B^{-1} \right\| \leq \|B - A\| [m_1, m_2]_{f(\cdot)(\cdot)^{-1}}.$$

Similar inequalities for operator convex functions and some particular examples of interest are also given.

1. INTRODUCTION

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

In 1934, K. Löwner [16] has given a definitive characterization of operator monotone functions as follows, see for instance the recent paper [17, p. 326]:

Theorem 1.1. *A function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda), \tag{1.1}$$

where $b \geq 0$ and μ is a positive measure on $[0,\infty)$.

2020 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. Operator monotone functions, Operator inequalities, Lipschitz type inequalities.

We recall the important fact proved by Löwner and Heinz that states that the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^{\alpha}$ is an operator monotone function for any $\alpha\in[0,1]$, [14]. The function ln is also operator monotone on $(0,\infty)$. For other examples of operator monotone functions, see [10] and [12].

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions, see for instance the recent paper [17, p. 326]:

Theorem 1.2. *A function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ with $f'_{+}(0)\in\mathbb{R}$ if and only if it has the representation*

$$f(t)=f(0)+f'_{+}(0)t+ct^{2}+\int_{0}^{\infty}\frac{t^{2}\lambda}{t+\lambda}d\mu(\lambda), \tag{1.2}$$

where $c\geq 0$ and μ is a positive measure on $[0,\infty)$.

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A|:=(A^{*}A)^{1/2}$.

It is known that [3] in the infinite-dimensional case the map $f(A):=|A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L>0$ such that

$$\||A|-|B|\|\leq L\|A-B\|$$

for any $A,B\in\mathcal{B}(H)$.

However, as shown by Farforovskaya in [7], [8] and Kato in [15], the following inequality holds

$$\||A|-|B|\|\leq\frac{2}{\pi}\|A-B\|\left(2+\log\left(\frac{\|A\|+\|B\|}{\|A-B\|}\right)\right) \tag{1.3}$$

for any $A,B\in\mathcal{B}(H)$ with $A\neq B$.

If the operator norm is replaced with the *Hilbert-Schmidt norm* $\|C\|_{HS}:=(\operatorname{tr}C^{*}C)^{1/2}$ of an operator C, then the following inequality is true [1]

$$\||A|-|B|\|_{HS}\leq\sqrt{2}\|A-B\|_{HS} \tag{1.4}$$

for any $A,B\in\mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is the best possible for a general A and B. If A and B are restricted to being selfadjoint, then the best coefficient is 1.

It has been shown in [3] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A|-|B|\| \leq a_1 \|A-B\| + a_2 \|A-B\|^2 + O\left(\|A-B\|^3\right), \tag{1.5}$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2.$$

In [2] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \leq f'(a) \|A-B\| \tag{1.6}$$

where f is an *operator monotone function* on $(0,\infty)$ and $A, B \geq a > 0$.

One of the problems in perturbation theory is to find bounds for $\|f(A)-f(B)\|$ in terms of $\|A-B\|$ for different classes of measurable functions f for which the function of the operator can be defined. For some results on this topic, see [4], [9] and the references therein.

We have the following integral representation for the power function when $t>0$, $r \in (0,1]$, see for instance [5, p. 145]

$$t^r = \frac{\sin(r\pi)}{\pi} t \int_0^\infty \frac{\lambda^{r-1}}{\lambda+t} d\lambda. \tag{1.7}$$

Observe that for $t>0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)} = \frac{\ln t}{t-1} + \frac{1}{1-t} \ln\left(\frac{u+t}{u+1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.8}$$

for all $t > 0$.

Motivated by these representations, we introduce the following *integral transform*

$$\mathcal{D}(\mu)(t) := -\int_0^\infty \frac{1}{\lambda+t} d\mu(\lambda),\ t>0, \tag{1.9}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.9) exists for all $t>0$.

If $d\mu = w(\lambda)\, d\lambda$, for a continuous and positive function $w(\lambda)$, $\lambda > 0$ where $d\lambda$ is the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := -\int_0^\infty \frac{w(\lambda)}{\lambda+t} d\lambda,\ t>0. \tag{1.10}$$

If we take the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then

$$t^{r-1} = -\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t > 0. \tag{1.11}$$

If we take the kernel $w_{\ln}(\lambda) = (\lambda+1)^{-1}$, $t > 0$, we have the representation

$$\ln t = -(t-1)\mathcal{D}(w_{\ln})(t),\ t > 0. \tag{1.12}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(\mu)(T) := -\int_0^\infty (\lambda+T)^{-1} d\mu(\lambda), \tag{1.13}$$

where μ is as above. Also, for a continuous and positive function $w(\lambda)$, $\lambda > 0$,

$$\mathcal{D}(w)(T) := -\int_0^\infty w(\lambda)(\lambda+T)^{-1} d\lambda, \tag{1.14}$$

for $T > 0$.

Let I an open interval and $\phi : I \to \mathbb{R}$ a differentiable function. Define the *first divided difference* of ϕ to be the function defined on $I \times I$ with the values in $\mathbb{R}$, [13]

$$[x,y]_\phi := \begin{cases} \frac{\phi(x)-\phi(y)}{x-y},\ \text{if } x \neq y,\ x,y \in I, \\ \\ \phi'(x),\ \text{if } x = y \in I. \end{cases}$$

In this paper we show among other results that, if $A \geq m_1 > 0$, $B \geq m_2 > 0$, then

$$\|\mathcal{D}(\mu)(B) - \mathcal{D}(\mu)(A)\| \leq \|B-A\|\,[m_1,m_2]_{\mathcal{D}(\mu)(\cdot)},$$

where $\mathcal{D}(\mu)(\cdot)$ is a function of t.

If $f : [0,\infty) \to \mathbb{R}$ is an operator monotone function with $f(0) = 0$, then

$$\left\|f(A)A^{-1} - f(B)B^{-1}\right\| \leq \|B-A\|\,[m_1,m_2]_{f(\cdot)(\cdot)^{-1}}.$$

Similar inequalities for operator convex functions and some particular examples of interest are also given.

2. Some Preliminary Facts

In the following, whenever we write $\mathcal{D}(\mu)$ we mean that the integral from (1.9) exists and is finite for all $t > 0$.

Lemma 2.1. *For all A, $B > 0$ we have the representation*

$$\mathcal{D}(\mu)(A) - \mathcal{D}(\mu)(B) = \int_0^\infty \int_0^1 (\lambda + (1-t)A + tB)^{-1} \times (A-B)(\lambda + (1-t)A + tB)^{-1} dt\, d\mu(\lambda). \tag{2.1}$$

Proof. Observe that, for all $A, B > 0$

$$\mathcal{D}(\mu)(A) - \mathcal{D}(\mu)(B) = \int_0^\infty \left[(\lambda+B)^{-1} - (\lambda+A)^{-1}\right] d\mu(\lambda). \tag{2.2}$$

Let $T, S > 0$. The function $f(t) = -t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right] = T^{-1}ST^{-1} \tag{2.3}$$

for $T, S > 0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D] := \{(1-t)C+tD,\ t\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t) := f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D) - f(C) = \int_0^1 \frac{d}{dt}(f_{C,D}(t))\,dt = \int_0^1 \nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.4}$$

If we write this equality for the function $f(t) = -t^{-1}$ and $C, D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.5}$$

Now, if we take in (2.5) $C = \lambda + B$, $D = \lambda + A$, then

$$\begin{aligned} &(\lambda+B)^{-1} - (\lambda+A)^{-1} \\ &= \int_0^1 ((1-t)(\lambda+B)+t(\lambda+A))^{-1}(A-B) \\ &\times((1-t)(\lambda+B)+t(\lambda+A))^{-1}\,dt \\ &= \int_0^1 (\lambda+(1-t)B+tA)^{-1}(A-B)(\lambda+(1-t)B+tA)^{-1}\,dt \end{aligned} \tag{2.6}$$

and from (2.2) we derive (2.1). □

Remark 2.1. If $B \geq A > 0$, then by representation (2.1) we derive that

$$\mathcal{D}(\mu)(B) \geq \mathcal{D}(\mu)(A),$$

which shows that $\mathcal{D}(\mu)$ is operator monotone on $(0,\infty)$. For further results related to the operator monotonicity of this integral transform see the recent paper [6].

Remark 2.2. We observe that if A, $B>0$ and $r\in(0,1]$, then by (1.11) we get the identity

$$B^{r-1}-A^{r-1}=\frac{\sin(r\pi)}{\pi}\times\int_0^\infty\lambda^{r-1}\int_0^1(\lambda+(1-t)A+tB)^{-1}\times(A-B)(\lambda+(1-t)A+tB)^{-1}dtd\lambda. \quad (2.7)$$

If A, $B>0$ with $A-1$ and $B-1$ invertible, then

$$(B-1)^{-1}\ln B-(A-1)^{-1}\ln A=\int_0^\infty(\lambda+1)^{-1}\int_0^1(\lambda+(1-t)A+tB)^{-1}\times(A-B)(\lambda+(1-t)A+tB)^{-1}dtd\lambda. \quad (2.8)$$

Corollary 2.1. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function as in (1.1). Then for all A, $B>0$ we have the equality*

$$B^{-1}f(B)-A^{-1}f(A)-f(0)\left(B^{-1}-A^{-1}\right)=\int_0^\infty\lambda\int_0^1(\lambda+(1-t)A+tB)^{-1}(A-B)\times(\lambda+(1-t)A+tB)^{-1}dtd\mu(\lambda). \quad (2.9)$$

If $f(0)=0$, then we have the simpler equality

$$B^{-1}f(B)-A^{-1}f(A)=\int_0^\infty\lambda\int_0^1(\lambda+(1-t)A+tB)^{-1}(A-B)\times(\lambda+(1-t)A+tB)^{-1}dtd\mu(\lambda). \quad (2.10)$$

Proof. From (1.1) we have

$$\frac{f(t)-f(0)}{t}-b=\int_0^\infty\frac{\lambda}{t+\lambda}d\mu(\lambda),$$

namely

$$\frac{f(t)-f(0)}{t}-b=-\mathcal{D}(\ell)(t), \quad (2.11)$$

where $d\ell(\lambda)=\lambda d\mu(\lambda)$, $\lambda>0$. Then for A, $B>0$,

$$\mathcal{D}(\ell)(A)-\mathcal{D}(\ell)(B)=[f(B)-f(0)]B^{-1}-[f(A)-f(0)]A^{-1}=B^{-1}f(B)-A^{-1}f(A)-f(0)\left(B^{-1}-A^{-1}\right)$$

and from (2.1) we derive (2.9). □

Corollary 2.2. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator convex function as in (1.2). Then for all A, $B>0$ we have the equality*

$$\begin{aligned} f(B)B^{-2}-f(A)A^{-2}-f'_{+}(0)\left(B^{-1}-A^{-1}\right)-f(0)\left(B^{-2}-A^{-2}\right) \\ =\int_0^\infty \lambda\int_0^1 (\lambda+(1-t)A+tB)^{-1}(A-B) \\ \times(\lambda+(1-t)A+tB)^{-1}\,dtd\mu(\lambda). \quad (2.12)\end{aligned}$$

If $f(0)=0$, then we have the simpler equality

$$\begin{aligned} f(B)B^{-2}-f(A)A^{-2}-f'_{+}(0)\left(B^{-1}-A^{-1}\right) \\ =\int_0^\infty \lambda\int_0^1 (\lambda+(1-t)A+tB)^{-1}(A-B) \\ \times(\lambda+(1-t)A+tB)^{-1}\,dtd\mu(\lambda). \quad (2.13)\end{aligned}$$

Proof. From (1.2) we have that

$$\frac{f(t)-f(0)-f'_{+}(0)t}{t^2}-c=\int_0^\infty \frac{\lambda}{t+\lambda}d\mu(\lambda),$$

namely

$$\frac{f(t)-f(0)-f'_{+}(0)t}{t^2}-c=-\mathcal{D}(\ell)(t),$$

for $t>0$. Then for A, $B>0$,

$$\begin{aligned}\mathcal{D}(\ell)(A)-\mathcal{D}(\ell)(B)&=f(B)B^{-2}-f(A)A^{-2}-f'_{+}(0)\left(B^{-1}-A^{-1}\right)\\ &\quad-f(0)\left(B^{-2}-A^{-2}\right)\end{aligned}$$

and from (2.1) we derive (2.12). □

Remark 2.3. Let $a>0$ and $f(t)=(t+a)^p$ with $p\in[-1,0)\cup[1,2]$. This function is operator convex and $f(0)=a^p$, $f'(0)=pa^{p-1}$. Then for all A, $B>0$ we have the equality

$$\begin{aligned}(B+a)^pB^{-2}-(A+a)^pA^{-2}-pa^{p-1}\left(B^{-1}-A^{-1}\right)-a^p\left(B^{-2}-A^{-2}\right) \\ =\int_0^\infty \lambda\int_0^1 (\lambda+(1-t)A+tB)^{-1}(A-B) \\ \times(\lambda+(1-t)A+tB)^{-1}\,dtd\mu(\lambda), \quad (2.14)\end{aligned}$$

for some positive measure μ on $(0,\infty)$.

3. Main Results

We have the following Lipschitz type inequality:

Theorem 3.1. *Assume that* $A \geq m_1 > 0$, $B \geq m_2 > 0$. *Then*

$$\|\mathcal{D}(\mu)(B) - \mathcal{D}(\mu)(A)\| \leq \|B - A\| [m_1, m_2]_{\mathcal{D}(\mu)(\cdot)}, \tag{3.1}$$

where $\mathcal{D}(\mu)(\cdot)$ *is a function of* $t > 0$.

Proof. From identity (2.1) by taking the norm we get that

$$\begin{aligned} &\|\mathcal{D}(\mu)(B) - \mathcal{D}(\mu)(A)\| \\ &\leq \int_0^\infty \left\| \int_0^1 (\lambda + (1-t)A + tB)^{-1} (B - A) (\lambda + (1-t)A + tB)^{-1} dt \right\| d\mu(\lambda) \\ &\leq \int_0^\infty \left(\int_0^1 \left\| (\lambda + (1-t)A + tB)^{-1} (B - A) (\lambda + (1-t)A + tB)^{-1} \right\| dt \right) d\mu(\lambda) \\ &\leq \|B - A\| \int_0^\infty \left(\int_0^1 \left\| (\lambda + (1-t)A + tB)^{-1} \right\|^2 dt \right) d\mu(\lambda) \end{aligned} \tag{3.2}$$

for all $A, B > 0$.

Assume that $m_2 > m_1$. Then

$$(1-t)A + tB + \lambda \geq (1-t)m_1 + tm_2 + \lambda,$$

which implies that

$$((1-t)A + tB + \lambda)^{-1} \leq ((1-t)m_1 + tm_2 + \lambda)^{-1},$$

and

$$\left\| ((1-t)A + tB + \lambda)^{-1} \right\|^2 \leq ((1-t)m_1 + tm_2 + \lambda)^{-2} \tag{3.3}$$

for all $t \in [0, 1]$ and $\lambda \geq 0$.

Therefore, by integrating (3.3) we derive

$$\begin{aligned} &\int_0^\infty \left(\int_0^1 \left\| ((1-t)A + tB + \lambda)^{-1} \right\|^2 dt \right) d\mu(\lambda) \\ &\leq \int_0^\infty \left(\int_0^1 ((1-t)m_1 + tm_2 + \lambda)^{-2} dt \right) d\mu(\lambda) \\ &= \frac{1}{m_2 - m_1} \int_0^\infty \left(\int_0^1 ((1-t)m_1 + tm_2 + \lambda)^{-1} \right. \\ &\left. \times (m_2 - m_1)((1-t)m_1 + tm_2 + \lambda)^{-1} dt \right) d\mu(\lambda) \\ &= \frac{\mathcal{D}(\mu)(m_2) - \mathcal{D}(\mu)(m_1)}{m_2 - m_1} \text{ (by (2.1))} \end{aligned}$$

and from (3.2) we deduce

$$\|\mathcal{D}(\mu)(B)-\mathcal{D}(\mu)(A)\| \le \|B-A\| \frac{\mathcal{D}(\mu)(m_2)-\mathcal{D}(\mu)(m_1)}{m_2-m_1}. \tag{3.4}$$

The case $m_2 < m_1$ works in a similar way and we also obtain (3.4).

Let $\varepsilon > 0$. Then $B+\varepsilon \ge m+\varepsilon > m$. From (3.4) we get

$$\begin{aligned}\|\mathcal{D}(\mu)(B+\varepsilon)-\mathcal{D}(\mu)(A)\| &\le \frac{\|B-A\|}{m+\varepsilon-m}[\mathcal{D}(\mu)(m+\varepsilon)-\mathcal{D}(\mu)(m)] \\ &= \|B-A\| \left[\frac{\mathcal{D}(\mu)(m+\varepsilon)-\mathcal{D}(\mu)(m)}{\varepsilon}\right]\end{aligned}$$

and by taking the limit over $\varepsilon \to 0+$, using the continuity and differentiability of $\mathcal{D}(\mu)$ we deduce the second part of (3.1). □

Corollary 3.1. *Assume that $f:[0,\infty) \to \mathbb{R}$ is an operator monotone function. If $A \ge m_1 > 0$, $B \ge m_2 > 0$, then*

$$\begin{aligned}&\left\|f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right)\right\| \\ &\le \|B-A\| \times \begin{cases} \left(\frac{f(m_1)m_1^{-1}-f(m_2)m_2^{-1}}{m_2-m_1}-\frac{f(0)}{m_2m_1}\right) \text{ if } m_1 \ne m_2, \\ \\ \frac{f(m)-f(0)-f'(m)m}{m^2} \text{ if } m_1=m_2=m.\end{cases}\end{aligned} \tag{3.5}$$

If $f(0)=0$, then we have the simpler inequalities

$$\begin{aligned}&\left\|f(A)A^{-1}-f(B)B^{-1}\right\| \\ &\le \|B-A\| \times \begin{cases} \frac{f(m_1)m_1^{-1}-f(m_2)m_2^{-1}}{m_2-m_1} \text{ if } m_1 \ne m_2, \\ \\ \frac{f(m)-f'(m)m}{m^2} \text{ if } m_1=m_2=m.\end{cases}\end{aligned} \tag{3.6}$$

Proof. From (1.1) we have that

$$\frac{f(t)-f(0)}{t}-b=-\mathcal{D}(\ell)(t),$$

where $d\ell(\lambda)=\lambda d\mu(\lambda)$, $\lambda>0$.

Then

$$\begin{aligned}&\frac{1}{m_2-m_1}[\mathcal{D}(w,\mu)(m_2)-\mathcal{D}(w,\mu)(m_1)] \\ &= \frac{f(m_1)m_1^{-1}-f(m_2)m_2^{-1}}{m_2-m_1}-\frac{f(0)}{m_2m_1}\end{aligned}$$

and

$$\mathcal{D}'(\mu)(m)=\frac{f(m)-f(0)-f'(m)m}{m^2}.$$

By making use of (3.1) we derive (3.5). □

Remark 3.1. Let $A \geq m_1 > 0$, $B \geq m_2 > 0$ and $r \in (0,1]$. Then by (3.6) we have the power inequalities

$$\left\|A^{r-1} - B^{r-1}\right\| \leq \|B - A\| \times \begin{cases} \frac{m_1^{r-1} - m_2^{r-1}}{m_2 - m_1} \text{ if } m_1 \neq m_2, \\ \\ \frac{1-r}{m^{2-r}} \text{ if } m_1 = m_2 = m. \end{cases} \tag{3.7}$$

If we take $f(t) = \ln(t+1)$, then we get from (3.6) that

$$\begin{aligned} &\left\|A^{-1}\ln(A+1) - B^{-1}\ln(B+1)\right\| \\ &\leq \|B - A\| \times \begin{cases} \frac{m_1^{-1}\ln(m_1+1) - m_2^{-1}\ln(m_2+1)}{m_2 - m_1} \text{ if } m_1 \neq m_2, \\ \\ \frac{(m+1)\ln(m+1) - m}{m^2(m+1)} \text{ if } m_1 = m_2 = m. \end{cases} \end{aligned} \tag{3.8}$$

Corollary 3.2. *Assume that $f : [0,\infty) \to \mathbb{R}$ is an operator convex function. If $A \geq m_1 > 0$, $B \geq m_2 > 0$, then*

$$\begin{aligned} &\left\|f(B)B^{-2} - f(A)A^{-2} - f'_+(0)\left(B^{-1} - A^{-1}\right) - f(0)\left(B^{-2} - A^{-2}\right)\right\| \\ &\leq \|B - A\| \\ &\times \begin{cases} \frac{f(m_1)m_1^{-2} - f(m_2)m_2^{-2}}{m_2 - m_1} - \frac{f'_+(0)}{m_1 m_2} - f(0)\frac{m_1 + m_2}{m_1^2 m_1^2} \text{ if } m_1 \neq m_2, \\ \\ 2\frac{f(m) - f(0)}{m^3} - \frac{f'(m) + f'_+(0)}{m^2} \text{ if } m_1 = m_2 = m. \end{cases} \end{aligned} \tag{3.9}$$

If $f(0) = 0$, then

$$\begin{aligned} &\left\|f(B)B^{-2} - f(A)A^{-2} - f'_+(0)\left(B^{-1} - A^{-1}\right)\right\| \\ &\leq \|B - A\| \times \begin{cases} \frac{f(m_1)m_1^{-2} - f(m_2)m_2^{-2}}{m_2 - m_1} - \frac{f'_+(0)}{m_1 m_2} \text{ if } m_1 \neq m_2, \\ \\ \frac{2}{m^2}\left[\frac{f(m)}{m} - \frac{f'(m) + f'_+(0)}{2}\right] \text{ if } m_1 = m_2 = m. \end{cases} \end{aligned} \tag{3.10}$$

Proof. From (1.5) we have that

$$\frac{f(t) - f(0) - f'_+(0)t}{t^2} - c = -\mathcal{D}(\ell)(t),$$

for $t > 0$.

Then

$$\begin{aligned} &\frac{\mathcal{D}(\mu)(m_2) - \mathcal{D}(\mu)(m_1)}{m_2 - m_1} \\ &= \frac{1}{m_2 - m_1}\left[\frac{f(m_1) - f(0) - f'_+(0)m_1}{m_1^2} - \frac{f(m_2) - f(0) - f'_+(0)m_2}{m_2^2}\right] \\ &= \frac{f(m_1)m_1^{-2} - f(m_2)m_2^{-2}}{m_2 - m_1} - \frac{f'_+(0)}{m_1 m_2} - f(0)\frac{m_1 + m_2}{m_1^2 m_1^2}. \end{aligned}$$

Since

$$\left(\frac{f(t)-f(0)-f'_{+}(0)t}{t^2}\right)' = \frac{\left(f'(t)-f'_{+}(0)\right)t^2-2t\left(f(t)-f(0)-f'_{+}(0)t\right)}{t^4}$$
$$= \frac{f'(t)+f'_{+}(0)}{t^2}-2\frac{f(t)-f(0)}{t^3},$$
$$\mathcal{D}'(w\mu)(m) = 2\frac{f(m)-f(0)}{m^3}-\frac{f'(m)+f'_{+}(0)}{m^2}$$

and from (3.1) we obtain (3.9). □

Remark 3.2. If we take $f(t) = -\ln(t+1)$ in (3.10), then for $A \geq m_1 > 0$, $B \geq m_2 > 0$ we get

$$\left\|B^{-2}\ln(B+1)-A^{-2}\ln(A+1)-B^{-1}+A^{-1}\right\| \tag{3.11}$$
$$\leq \|B-A\| \times \begin{cases} \frac{m_2^{-2}\ln(m_2+1)-m_1^{-2}\ln(m_1+1)}{m_2-m_1}+\frac{1}{m_1m_2} \text{ if } m_1 \neq m_2, \\ \\ \frac{m+2}{m^2(m+1)}-\frac{2\ln(m+1)}{m^3} \text{ if } m_1 = m_2 = m. \end{cases}$$

Proposition 3.1. *Assume that $A > 0$, $B > 0$. Then*

$$\|\mathcal{D}(\mu)(B)-\mathcal{D}(\mu)(A)\| \leq \|B-A\| \left[\left\|A^{-1}\right\|^{-1}, \left\|B^{-1}\right\|^{-1}\right]_{\mathcal{D}(\mu)(\cdot)}. \tag{3.12}$$

Proof. Since A and B are invertible, then $\|A\|\left\|A^{-1}\right\| \geq \left\|AA^{-1}\right\| = 1$ which yields that $\|A\| \geq \left\|A^{-1}\right\|^{-1}$ and $\|B\| \geq \left\|B^{-1}\right\|^{-1}$. Therefore by Theorem 3.1 written for $m_1 = \left\|A^{-1}\right\|^{-1}$ and $m_2 = \left\|B^{-1}\right\|^{-1}$ we obtain the desired inequality (3.12). □

If $A > 0$, $B > 0$, then by taking $m_1 = \left\|A^{-1}\right\|^{-1}$ and $m_2 = \left\|B^{-1}\right\|^{-1}$ in Corollaries 3.1 and 3.2 one can derive some similar inequalities. However, the details are left to the interested reader.

4. Midpoint and Trapezoid Inequalities

We have the following midpoint type inequalities:

Proposition 4.1. *For all A, $B \geq m > 0$ we have the midpoint inequality*

$$\left\|\int_0^1 \mathcal{D}(\mu)((1-t)A+tB)\,dt-\mathcal{D}(\mu)\left(\frac{A+B}{2}\right)\right\| \tag{4.1}$$
$$\leq \frac{1}{4}\mathcal{D}'(\mu)(m)\|B-A\|.$$

Proof. Since A, $B \geq m$, $\frac{A+B}{2} \geq m > 0$ and $(1-t)A+tB \geq m > 0$ for all $t \in [0,1]$ and by (3.1)

$$\begin{aligned} &\left\| \mathcal{D}(\mu)((1-t)A+tB) - \mathcal{D}(\mu)\left(\frac{A+B}{2}\right) \right\| \\ &\le \mathcal{D}'(\mu)(m) \left\| (1-t)A+tB - \frac{A+B}{2} \right\| \\ &= \mathcal{D}'(\mu)(m) \left| t-\frac{1}{2} \right| \|B-A\| \end{aligned} \tag{4.2}$$

for all $t \in [0,1]$.

Taking the integral in (4.2), we get

$$\begin{aligned} &\left\| \int_0^1 \mathcal{D}(\mu)((1-t)A+tB)\,dt - \mathcal{D}(\mu)\left(\frac{A+B}{2}\right) \right\| \\ &\le \int_0^1 \left\| \mathcal{D}(\mu)((1-t)A+tB) - \mathcal{D}(\mu)\left(\frac{A+B}{2}\right) \right\| dt \\ &\le \mathcal{D}'(\mu)(m)\|B-A\| \int_0^1 \left| t-\frac{1}{2} \right| dt = \frac{1}{4}\mathcal{D}'(\mu)(m)\|B-A\| \end{aligned}$$

and the inequality (4.1) is proved. □

The case of operator monotone functions is as follows:

Corollary 4.1. *Assume that $f : [0,\infty) \to \mathbb{R}$ is an operator monotone function. If A, $B \ge m > 0$, then*

$$\begin{aligned} &\left\| \int_0^1 ((1-t)A+tB)^{-1} f((1-t)A+tB)\,dt - \left(\frac{A+B}{2}\right)^{-1} f\left(\frac{A+B}{2}\right) \right. \\ &\left. -f(0)\left(\int_0^1 ((1-t)A+tB)^{-1}dt - \left(\frac{A+B}{2}\right)^{-1} \right) \right\| \\ &\le \frac{1}{4}\|B-A\| \frac{f(m)-f(0)-f'(m)m}{m^2}. \end{aligned} \tag{4.3}$$

If $f(0)=0$, then

$$\begin{aligned} &\left\| \int_0^1 ((1-t)A+tB)^{-1} f((1-t)A+tB)\,dt - \left(\frac{A+B}{2}\right)^{-1} f\left(\frac{A+B}{2}\right) \right\| \\ &\le \frac{1}{4}\|B-A\| \frac{f(m)-f'(m)m}{m^2}. \end{aligned} \tag{4.4}$$

Proof. From (1.1) we have that

$$\frac{f(t)-f(0)}{t} - b = -\mathcal{D}(\ell)(t),$$

where $d\ell(\lambda) = \lambda d\mu(\lambda)$, $\lambda > 0$.

Then

$$\int_0^1 \mathcal{D}(\ell)((1-t)A+tB)\,dt$$
$$= \int_0^1 ((1-t)A+tB)^{-1} f((1-t)A+tB)\,dt - f(0)\int_0^1 ((1-t)A+tB)^{-1}\,dt - b,$$

and

$$\mathcal{D}(\ell)\left(\frac{A+B}{2}\right) = \left(\frac{A+B}{2}\right)^{-1} f\left(\frac{A+B}{2}\right) - f(0)\left(\frac{A+B}{2}\right)^{-1} - b$$

and from (4.1) we get (4.3). □

From inequality (4.4) we get the following power inequality

$$\left\| \int_0^1 ((1-t)A+tB)^{r-1}\,dt - \left(\frac{A+B}{2}\right)^{r-1} \right\| \le \frac{1-r}{4m^{2-r}} \|B-A\|, \tag{4.5}$$

where $r \in (0,1]$ and $A,\, B \ge m > 0$.

The following logarithmic inequality also holds

$$\begin{aligned} &\left\| \int_0^1 ((1-t)A+tB)^{-1} \ln((1-t)A+tB+1)\,dt \right. \\ &\left. - \left(\frac{A+B}{2}\right)^{-1} \ln\left(\frac{A+B}{2}+1\right) \right\| \\ &\le \frac{(m+1)\ln(m+1)-m}{4m^2(m+1)} \|B-A\|, \end{aligned} \tag{4.6}$$

where $A,\, B \ge m > 0$.

Corollary 4.2. *Assume that $f : [0,\infty) \to \mathbb{R}$ is an operator convex function. If A, $B \ge m > 0$, then*

$$\begin{aligned} &\left\| \int_0^1 ((1-t)A+tB)^{-2} f((1-t)A+tB)\,dt - \left(\frac{A+B}{2}\right)^{-2} f\left(\frac{A+B}{2}\right) \right. \\ &- f(0)\left(\int_0^1 ((1-t)A+tB)^{-2}\,dt - \left(\frac{A+B}{2}\right)^{-2}\right) \\ &\left. - f'_+(0)\left(\int_0^1 ((1-t)A+tB)^{-1}\,dt - \left(\frac{A+B}{2}\right)^{-1}\right) \right\| \\ &\le \frac{1}{2m^2}\|B-A\|\left(\frac{f(m)-f(0)}{m} - \frac{f'(m)+f'_+(0)}{2}\right). \end{aligned} \tag{4.7}$$

If $f(0)=0$, *then*

$$\left\| \int_0^1 ((1-t)A+tB)^{-2} f((1-t)A+tB)\,dt - \left(\frac{A+B}{2}\right)^{-2} f\left(\frac{A+B}{2}\right) \right. \tag{4.8}$$
$$\left. - f'_+(0)\left(\int_0^1 ((1-t)A+tB)^{-1}\,dt - \left(\frac{A+B}{2}\right)^{-1}\right)\right\|$$
$$\le \frac{1}{2m^2}\|B-A\|\left(\frac{f(m)}{m} - \frac{f'(m)+f'_+(0)}{2}\right).$$

Proof. From (1.5) we have that

$$\frac{f(t)-f(0)-f'_+(0)t}{t^2} - c = -\mathcal{D}(\ell)(t),$$

for $t>0$. We have

$$\int_0^1 \mathcal{D}(\ell)((1-t)A+tB)\,dt = \int_0^1 ((1-t)A+tB)^{-2} f((1-t)A+tB)\,dt$$
$$- f(0)\int_0^1 ((1-t)A+tB)^{-2}\,dt$$
$$- f'_+(0)\int_0^1 ((1-t)A+tB)^{-1}\,dt - c$$

and

$$\mathcal{D}(\ell,)\left(\frac{A+B}{2}\right) = \left(\frac{A+B}{2}\right)^{-2} f\left(\frac{A+B}{2}\right) - f(0)\left(\frac{A+B}{2}\right)^{-2}$$
$$- f'_+(0)\left(\frac{A+B}{2}\right)^{-1} - c$$

and from (4.1) we get (4.7). □

From (4.8) we get the logarithmic inequality

$$\left\| \int_0^1 ((1-t)A+tB)^{-2} \ln((1-t)A+tB+1)\,dt \right. \tag{4.9}$$
$$- \left(\frac{A+B}{2}\right)^{-2} \ln\left(\frac{A+B}{2}+1\right)$$
$$\left. - \int_0^1 ((1-t)A+tB)^{-1}\,dt + \left(\frac{A+B}{2}\right)^{-1}\right\|$$
$$\le \frac{1}{4m^2}\|B-A\|\left(\frac{m+2}{m+1} - \frac{2\ln(m+1)}{m}\right)$$

for $A,\, B \ge m > 0$.

We have the following midpoint type inequalities:

Proposition 4.2. *For all* $A, B \geq m > 0$ *we have the trapezoid inequality*

$$\left\| \frac{\mathcal{D}(\mu)(A) + \mathcal{D}(\mu)(B)}{2} - \int_0^1 \mathcal{D}(\mu)((1-t)A + tB)\, dt \right\| \tag{4.10}$$
$$\leq \frac{1}{4} \mathcal{D}'(\mu)(m) \|B - A\| .$$

Proof. Since $A, B \geq m$, $(1-s)A + s\frac{A+B}{2}$, $s\frac{A+B}{2} + (1-s)B \geq m > 0$ for all $s \in [0,1]$ and from (2.5) we get

$$\left\| \mathcal{D}(\mu)(A) - \mathcal{D}(\mu)\left((1-s)A + s\frac{A+B}{2}\right) \right\| \tag{4.11}$$
$$\leq \frac{1}{2} \mathcal{D}'(\mu)(m) \|B - A\| s$$

and

$$\left\| \mathcal{D}(\mu)(B) - \mathcal{D}(\mu)\left(s\frac{A+B}{2} + (1-s)B\right) \right\| \tag{4.12}$$
$$\leq \frac{1}{2} \mathcal{D}'(\mu)(m) \|B - A\| s.$$

From (4.11) and (4.12) we derive by addition, division by 2 and the triangle inequality that

$$\left\| \frac{\mathcal{D}(\mu)(A) + \mathcal{D}(\mu)(B)}{2} - \frac{1}{2}\left[\mathcal{D}(\mu)\left((1-s)A + s\frac{A+B}{2}\right) + \mathcal{D}(\mu)\left(s\frac{A+B}{2} + (1-s)B\right)\right] \right\|$$
$$\leq \frac{1}{2} \mathcal{D}'(\mu)(m) \|B - A\| s$$

for all $s \in [0,1]$.

By taking the integral and using its properties, we derive

$$\left\| \frac{\mathcal{D}(\mu)(A) + \mathcal{D}(\mu)(B)}{2} - \frac{1}{2}\left[\int_0^1 \mathcal{D}(\mu)\left((1-s)A + s\frac{A+B}{2}\right) + \mathcal{D}(\mu)\left(s\frac{A+B}{2} + (1-s)B\right) ds\right] \right\| \tag{4.13}$$
$$\leq \frac{1}{2} \mathcal{D}'(\mu)(m) \|B - A\| \int_0^1 s ds = \frac{1}{4} \mathcal{D}'(\mu)(m) \|B - A\| .$$

Now, using the change of variable $t = 2s$ we have

$$\frac{1}{2} \int_0^1 \mathcal{D}(\mu)\left((1-t)A + t\frac{A+B}{2}\right) dt = \int_0^{1/2} \mathcal{D}(\mu)((1-s)A + sB)\, ds$$

and by the change of variable $t = 1 - v$ we have

$$\frac{1}{2}\int_0^1 \mathcal{D}(\mu)\left(t\frac{A+B}{2} + (1-t)A\right)dt = \frac{1}{2}\int_0^1 \mathcal{D}(\mu)\left((1-v)\frac{A+B}{2} + vB\right)dv.$$

Moreover, if we make the change of variable $v = 2s - 1$ we also have

$$\frac{1}{2}\int_0^1 \mathcal{D}(\mu)\left((1-v)\frac{A+B}{2} + vB\right)dv = \int_{1/2}^1 \mathcal{D}(\mu)((1-s)A + sB)\,ds.$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left[\mathcal{D}(\mu)\left((1-s)A + s\frac{A+B}{2}\right) + \mathcal{D}(\mu)\left(s\frac{A+B}{2} + (1-s)B\right)\right]ds \\
&= \int_0^{1/2} \mathcal{D}(\mu)((1-s)A + sB)\,dt + \int_{1/2}^1 \mathcal{D}(\mu)((1-s)A + sB)\,ds \\
&= \int_0^1 \mathcal{D}(\mu)((1-s)A + sB)\,ds
\end{aligned}$$

and by (4.13) we deduce the desired result (4.10). □

Corollary 4.3. *Assume that $f : [0,\infty) \to \mathbb{R}$ is an operator monotone function with $f(0) = 0$. If $A, B \geq m > 0$, then*

$$\begin{aligned}
&\left\|\frac{f(A)A^{-1} + f(B)B^{-1}}{2} - \int_0^1 ((1-t)A + tB)^{-1} f((1-t)A + tB)\,dt\right\| \qquad (4.14)\\
&\leq \frac{f(m) - f'(m)m}{4m^2}\|B - A\|.
\end{aligned}$$

Assume that $A, B \geq m > 0$. Then by Corollary 4.3 we obtain the following power inequalities

$$\left\|\frac{A^{r-1} + B^{r-1}}{2} - \int_0^1 ((1-t)A + tB)^{r-1}\,dt\right\| \leq \frac{1-r}{4m^{2-r}}\|B - A\|, \qquad (4.15)$$

where $r \in (0,1]$.

We can also state the logarithmic inequality

$$\begin{aligned}
&\left\|\frac{A^{-1}\ln(A+1) + B^{-1}\ln(B+1)}{2}\right. \qquad (4.16)\\
&\left. - \int_0^1 ((1-t)A + tB)^{-1}\ln((1-t)A + tB + 1)\,dt\right\| \\
&\leq \frac{(m+1)\ln(m+1) - m}{4m^2(m+1)}\|B - A\|,
\end{aligned}$$

provided that $A, B \geq m > 0$.

Corollary 4.4. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is an operator convex function with* $f(0)=0$*. If* $A, B\geq m>0$*, then*

$$\left\|\frac{f(A)A^{-2}+f(B)B^{-2}}{2}-\int_0^1((1-t)A+tB)^{-2}f((1-t)A+tB)\,dt\right.\tag{4.17}$$
$$\left.-f'_+(0)\left(\frac{A^{-1}+B^{-1}}{2}-\int_0^1((1-t)A+tB)^{-1}dt\right)\right\|$$
$$\leq\frac{1}{2m^2}\|B-A\|\left(\frac{f(m)}{m}-\frac{f'(m)+f'_+(0)}{2}\right).$$

Assume that $A, B\geq m>0$. Then by Corollary 4.4 we obtain the following logarithmic inequalities

$$\left\|\frac{A^{-2}\ln(A+1)+B^{-2}\ln(B+1)}{2}\right.\tag{4.18}$$
$$-\int_0^1((1-t)A+tB)^{-2}\ln((1-t)A+tB+1)\,dt$$
$$\left.-\left(\frac{A^{-1}+B^{-1}}{2}-\int_0^1((1-t)A+tB)^{-1}dt\right)\right\|$$
$$\leq\frac{1}{4m^2}\|B-A\|\left(\frac{m+2}{m+1}-\frac{2\ln(m+1)}{m}\right).$$

ACKNOWLEDGEMENT

The author would like to thank the anonymous referees for valuable suggestions that have been implemented in the final version of the manuscript.

REFERENCES

[1] H. Araki and S. Yamagami, *An inequality for Hilbert-Schmidt norm*, Commun. Math. Phys. **81** (1981), 89-96.
[2] R. Bhatia, *First and second order perturbation bounds for the operator absolute value*, Linear Algebra Appl. **208/209** (1994), 367-376.
[3] R. Bhatia, *Perturbation bounds for the operator absolute value*. Linear Algebra Appl. **226/228** (1995), 639–645.
[4] R. Bhatia, D. Singh and K. B. Sinha, *Differentiation of operator functions and perturbation bounds,* Comm. Math. Phys. **191** (1998), no. 3, 603–611.
[5] R. Bhatia, *Matrix Analysis*. Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.
[6] S. S. Dragomir, *Operator monotonicity of an integral transform of positive operators in Hilbert spaces with applications*, Preprint RGMIA Res. Rep. Coll. **23** (2020), to appear.
[7] Yu. B. Farforovskaya, *Estimates of the closeness of spectral decompositions of self-adjoint operators in the Kantorovich-Rubinshtein metric* (in Russian), Vesln. Leningrad. Gos. Univ. Ser. Mat. Mekh. Astronom. **4** (1967), 155-156.
[8] Yu. B. Farforovskaya, *An estimate of the norm* $\|f(B)-f(A)\|$ *for self-adjoint operators A and B* (in Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. **56** (1976), 143-162 .

[9] Yu. B. Farforovskaya and L. Nikolskaya, *Modulus of continuity of operator functions,* Algebra i Analiz **20** (2008), no. 3, 224–242; translation in St. Petersburg Math. J. **20** (2009), no. 3, 493–506.

[10] J. I. Fujii, Y. Seo, *On parametrized operator means dominated by power ones*, Sci. Math. **1** (1998) 301–306.

[11] T. Furuta, *Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation*, Linear Algebra Appl. **429** (2008) 972–980.

[12] T. Furuta, *Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function f on* $[0,\infty)$, J. Math. Inequal. **9** (2015), no. 1, 47–52.

[13] F. Hansen, *The fast track to Löwner's theorem*, Linear Algebra Appl.438 (2013), no.11, 4557–4571.

[14] E. Heinz, *Beiträge zur Störungsteorie der Spektralzerlegung*, Math. Ann. **123** (1951) 415–438.

[15] T. Kato, *Continuity of the map $S \to |S|$ for linear operators*, Proc. Japan Acad. **49** (1973), 143-162.

[16] K. Löwner, *Über monotone MatrixFunktionen*, Math. Z. **38** (1934) 177–216.

[17] H. Najafi, M. S. Moslehian, M. Fujii and R. Nakamoto, *Estimates of operator convex and operator monotone functions on bounded intervals*, Hokkaido Mathematical Journal Vol. **45** (2016) p. 325–336

(Received: September 27, 2022)
(Revised: October 14, 2024)

Silvestru Sever Dragomir
Victoria University,
Applied Mathematics Research Group, ISILC,
Melbourne, Australia, PO Box 14428
e-mail: sever.dragomir@vu.edu.au
and
University of the Witwatersrand,
School of Computer Science
& Applied Mathematics,
Johannesburg, South Africa