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A FAMILY OF HYBRID MAPPINGS AND THEIR FIXED POINT IN CONVEX SPACES UNDER DIAMETRAL δ DISTANCES

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ABSTRACT. We prove some results on coincidence and common fixed points for compatible as well as pointwise R-weakly commuting mappings satisfying a generalized contraction condition on a complete metrically convex metric space that generalize relevant results due to Ćirič and Ume [3], Khan [13, 14], Rhoades [19] and others.

1. INTRODUCTION

So far, there have been many extensions and generalizations of well known fixed point theorems for multi-valued mappings. Many researchers like Assad [1], Assad and Kirk [2], Ćirič and Ume [3], Itoh [10] and others extended fixed point theorems to a more general class of multi-valued mappings while in 1996, Rhoades [19] obtained a generalization of Itoh's fixed point theorem on multi-valued mappings for non-self setting. In this sequence, Huang and Cho [7] proved a common fixed point theorem for sequences of nonself multi-valued mappings in metrically convex metric space. After this wonderful result, many authors applying the same pattern and proved fixed point theorems for a sequence of multi-valued mappings. To mention a few we cite Imdad and Khan ([8], [9]) Khan ([13], [14]) and Khan and Imdad [15] etc.

The purpose of this paper is to prove some coincidence and common fixed point theorems for a sequence of multi-valued and a pair of single valued nonself mappings using diametral delta distance satisfying certain contraction condition. Our result generalizes and extends earlier results due to Khan [13, 14], Rhoades [19], Ćirič and Ume [3], Khan and Imdad [15] and others.

2. PRELIMINARIES

Now, we collect some relevant definitions and results.

Let (X,d) be a metric space. Then following Nadler [17], we recall

(1) $CB(X) = \{A : A \text{ is nonempty closed and bounded subset of } X\}$.

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(2) For nonempty subsets A,B of X and $x \in X$,

$$d(x,A) = \inf\{d(x,a) : a \in A\},\ D(A,B) = \inf\{d(a,b) : a \in A, b \in B\},$$

$H(A,B) = \max\{\sup d(a,B) : a \in A,\ \sup d(A,b) : b \in B\}$
and$d(A,B) = \sup\{d(a,b) : a \in A, b \in B\}$.

Notice that $D(A,B) \leq H(A,B) \leq \delta(A,B)$, it is well known (cf.Kuratowski [12]) that $CB(X)$ is a metric space with the distance function H which is known as the Hausdorff-Pompeiu metric on X.

Definition 2.1. ([9]) *Let K be a nonempty subset of a metric space (X,d), $T : K \to X$ and $F : K \to CB(X)$. The pair (F,T) is said to be pointwise R-weakly commuting on K if for a given $x \in K$ and $Tx \in K$, there exists some $R = R(x) > 0$ such that*

$$d(Ty,FTx) \leq R\, d(Tx,Fx) \text{ for each } y \in F(x) \cap K. \tag{2.1}$$

Moreover, the pair (F,T) will be called R-weakly commuting on K if for each $x \in K, Tx \in K$ and (2.1) holds for some $R > 0$.

If $R = 1$, we get the definition of weak commutativity of (F,T) on K due to Hadžić and Gajić [6]. If $F,T : X \to X$ then Definition 2.1 reduces respectively to pointwise R-weak commutativity and R-weak commutativity for single valued self mappings due to Pant [18].

Definition 2.2. ([5],[6]) *Let K be a nonempty subset of a metric space (X,d), $T : K \to X$ and $F : K \to CB(X)$. The pair (F,T) is said to be weakly commuting (cf.[6]) if for every $x,y \in K$ with $x \in Fy$ and $Ty \in K$, we have*

$$d(Tx,FTy) \leq d(Ty,Fy),$$

whereas the pair (F,T) is said to be compatible (cf.[5]) if for every sequence $\{x_n\} \subset K$ and from the relation

$$\lim_{n\to\infty} d(Fx_n,Tx_n) = 0$$

and $Tx_n \in K$ (for every $n \in \mathbf{N}$) it follows that $\lim_{n\to\infty} d(Ty_n,FTx_n) = 0$, for every sequence $\{y_n\} \subset K$ such that $y_n \in Fx_n, n \in \mathbf{N}$.

For hybrid pairs of self type mappings these definitions were introduced by Kaneko and Sessa [11].

Definition 2.3. ([8]) *Let K be a nonempty subset of a metric space (X,d), $T : K \to X$ and $F : K \to CB(X)$. The pair (F,T) is said to be quasi-coincidentally commuting if for all coincidence points 'x' of (F,T), $T(Fx) \subset F(Tx)$ whenever $Fx \subset K$ and $Tx \in K$ for all $x \in K$.*

Definition 2.4. ([15]) *A mapping $T : K \to X$ is said to be occasionally coincidentally idempotent w.r.t mapping $F : K \to CB(X)$, if there exists a point $z \in K$ such that T is idempotent at the coincidence points of the pair (F,T).*

Definition 2.5. ([16]) *A metric space (X,d) is said to be metrically convex if for any $x,y \in X$ with $x \neq y$ there exists a point $z \in X, x \neq z \neq y$ such that*

$$d(x,z)+d(z,y)=d(x,y).$$

Lemma 2.1. ([4]) *Let $\{A_n\}$ and $\{B_n\}$ be two sequences in $CB(X)$ converging in $CB(X)$ to the sets A and B respectively. Then*

$$\lim_{n\to\infty} \delta(A_n,B_n)=\delta(A,B).$$

3. Main Results

Theorem 3.1. *Let (X,d) be a complete metrically convex metric space and K be a nonempty closed subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $S,T : K \to X$ satisfy the conditions:*

(1) $\delta K \subseteq SK \cap TK, F_i(K)\cap K \subseteq SK, F_j(K)\cap K \subseteq TK$,

(2) $Tx \in \delta K \Rightarrow F_i(x) \subseteq K, Sx \in \delta K \Rightarrow F_j(x) \subseteq K$, *and*

$$\delta(F_i(x),F_j(y)) \leq \alpha\, d(Tx,Sy)+\beta\, max\{d(Tx,F_i(x)),d(Sy,F_j(y))\} \\ +\gamma\, max\{d(Tx,F_i(x))+d(Sy,F_j(y)),d(Tx,F_j(y))+d(Sy,F_i(x))\} \quad (3.1)$$

where $i=2n-1, j=2n,\ (n\in \mathbf{N}), i\neq j$ for all $x,y \in X$ with $x \neq y$, where $\alpha,\beta,\gamma \geq 0$, such that $\alpha+2\beta+3\gamma+\alpha\gamma<1$,

(3) *(F_i,T) and (F_j,S) are compatible pairs,*

(4) *$\{F_n\}$, T and S are continuous on K.*

Then $\{F_n\}, T$ and S have a common coincidence point.

Proof. First, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way. Let $x \in \partial K$. Then since $\partial K \subseteq TK$ there exists a point $x_0 \in K$ such that $x = Tx_0$. From $Tx \in \partial K \Rightarrow F_i(x) \subseteq K$, one concludes that $F_1(x_0) \subseteq F_1(K)\cap K \subseteq SK$. Let $x_1 \in K$ be such that $y_1 = Sx_1 \in F_1(x_0) \subseteq K$. Since $y_1 \in F_1(x_0)$, there exists a point $y_2 \in F_2(x_1)$ such that

$$d(y_1,y_2) \leq \delta(F_1(x_0),F_2(x_1)).$$

Suppose $y_2 \in K$. Then $y_2 \in F_2(K)\cap K \subseteq TK$, which implies that there exists a point $x_2 \in K$ such that $y_2 = Tx_2$. Otherwise, if $y_2 \notin K$, then there exists a point $p \in \partial K$ such that

$$d(Sx_1,p)+d(p,y_2)=d(Sx_1,y_2).$$

Since $p \in \partial K \subseteq TK$, there exists a point $x_2 \in K$ such that $p = Tx_2$ and so

$$d(Sx_1,Tx_2)+d(Tx_2,y_2)=d(Sx_1,y_2).$$

Let $y_3 \in F_3(x_2)$ be such that $d(y_2,y_3) \leq \delta(F_2(x_1),F_3(x_2))$.

Thus, repeating the foregoing arguments, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(1) $y_{2n} \in F_{2n}(x_{2n-1})$, for all $n \in \mathbf{N}$, $y_{2n+1} \in F_{2n+1}(x_{2n})$ for all $n \in \mathbf{N}_0 = \mathbf{N}\cup\{0\}$,

(2) $y_{2n} \in K \Rightarrow y_{2n} = Tx_{2n}$ or $y_{2n} \notin K \Rightarrow Tx_{2n} \in \partial K$, and

$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n})$$

(3) $y_{2n+1} \in K \Rightarrow y_{2n+1} = Sx_{2n+1}$ or $y_{2n+1} \notin K \Rightarrow Sx_{2n+1} \in \partial K$, and

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1})$$

We denote

$P_\circ = \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} = y_{2i}\}$, $P_1 = \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} \neq y_{2i}\}$,
$Q_\circ = \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} = y_{2i+1}\}$ and
$Q_1 = \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} \neq y_{2i+1}\}$.

We note that $(Tx_{2n}, Sx_{2n+1}) \notin P_1 \times Q_1$ and $(Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1$.
Now we distinguish the following three cases.

Case 3.1. If $(Tx_{2n}, Sx_{2n+1}) \in P_\circ \times Q_\circ$, then

$$\begin{aligned}
d(Tx_{2n}, Sx_{2n+1}) &\leq \delta(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})) \\
&\leq \alpha\, d(Tx_{2n}, Sx_{2n-1}) + \beta \max\{d(Tx_{2n}, F_{2n+1}(x_{2n})), d(Sx_{2n-1}, F_{2n}(x_{2n-1}))\} \\
&\quad + \gamma \max\{d(Tx_{2n}, F_{2n+1}(x_{2n})) + d(Sx_{2n-1}, F_{2n}(x_{2n-1})), d(Tx_{2n}, F_{2n}(x_{2n-1})) \\
&\quad + d(Sx_{2n-1}, F_{2n+1}(x_{2n}))\} \\
&\leq \alpha\, d(y_{2n-1}, y_{2n}) + \beta \max\{d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\} \\
&\quad + \gamma \max\{d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n+1})\} \\
&\leq \alpha\, d(y_{2n-1}, y_{2n}) + \beta \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} \\
&\quad + \gamma\, \{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\}
\end{aligned}$$

$$\begin{aligned}
d(Tx_{2n}, Sx_{2n+1}) &\leq (\alpha+\gamma)\, d(y_{2n-1}, y_{2n}) + \beta \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} \\
&\quad + \gamma\, d(y_{2n}, y_{2n+1}).
\end{aligned} \tag{3.2}$$

If we suppose that $d(y_{2n-1}, y_{2n}) \leq d(y_{2n}, y_{2n+1})$, then we obtain

$$d(Tx_{2n}, Sx_{2n+1}) \leq (\alpha+\beta+2\gamma)\, d(y_{2n}, y_{2n+1})$$

which is a contradiction. Therefore from (3.2) we obtain

$$d(Tx_{2n}, Sx_{2n+1}) \leq (\alpha+\beta+\gamma)\, d(y_{2n}, y_{2n-1}) + \gamma\, d(y_{2n}, y_{2n+1})$$

which in turn yields

$$d(Tx_{2n}, Sx_{2n+1}) \leq \left(\frac{\alpha+\beta+\gamma}{1-\gamma}\right) d(Sx_{2n-1}, Tx_{2n}) \tag{3.3}$$

Similarly if $(Sx_{2n-1}, Tx_{2n}) \in Q_\circ \times P_\circ$, then

$$d(Sx_{2n-1}, Tx_{2n}) \leq \left(\frac{\alpha+\beta+\gamma}{1-\gamma}\right) d(Sx_{2n-1}, Tx_{2n-2}) \tag{3.4}$$

Case 3.2. If $(Tx_{2n}, Sx_{2n+1}) \in P_\circ \times Q_1$, then

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1})$$

which in turn yields

$$d(Tx_{2n}, Sx_{2n+1}) \leq d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1})$$

and hence

$$d(Tx_{2n}, Sx_{2n+1}) \leq d(y_{2n}, y_{2n+1}) \leq \delta(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})).$$

Now, proceeding as in Case 1, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq \left(\frac{\alpha+\beta+\gamma}{1-\gamma}\right) d(Sx_{2n-1}, Tx_{2n}).$$

If $(Sx_{2n-1}, Tx_{2n}) \in Q_1 \times P_\circ$, then as earlier, we also obtain

$$d(Sx_{2n-1}, Tx_{2n}) \leq \left(\frac{\alpha+\beta+\gamma}{1-\gamma}\right) d(Sx_{2n-1}, Tx_{2n-2}).$$

Case 3.3. If $(Tx_{2n}, Sx_{2n+1}) \in P_1 \times Q_\circ$, then $Sx_{2n-1} = y_{2n-1}$. As in Case 1, we get

$$\begin{aligned} d(Tx_{2n}, Sx_{2n+1}) = d(Tx_{2n}, y_{2n+1}) &\leq \{d(Tx_{2n}, y_{2n}) + d(y_{2n}, y_{2n+1})\} \\ &\leq d(Tx_{2n}, y_{2n}) + d(y_{2n}, y_{2n+1}) \\ &\leq d(Tx_{2n}, y_{2n}) + \delta(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})) \\ &\leq d(Tx_{2n}, y_{2n}) + \alpha\, d(Tx_{2n}, Sx_{2n-1}) \\ &\quad + \beta \max\{d(Tx_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\} + \gamma \max\{d(Tx_{2n}, y_{2n+1}) \\ &\quad + d(y_{2n-1}, y_{2n}), d(Tx_{2n}, y_{2n}) + d(Sx_{2n-1}, Sx_{2n+1})\}. \end{aligned}$$

Since $\alpha < 1$ and $d(Tx_{2n}, y_{2n}) + d(Tx_{2n}, Sx_{2n-1}) = d(Sx_{2n-1}, y_{2n})$ we obtain

$$d(Tx_{2n}, y_{2n}) + \alpha\, d(Tx_{2n}, Sx_{2n-1}) \leq d(Sx_{2n-1}, y_{2n}).$$

Also, by the triangle inequality we obtain

$$\begin{aligned} d(Tx_{2n}, y_{2n}) + d(Sx_{2n-1}, Sx_{2n+1}) &\leq d(Tx_{2n}, y_{2n}) + d(Sx_{2n-1}, Tx_{2n}) \\ &\quad + d(Tx_{2n}, Sx_{2n+1}) \\ &\leq d(Sx_{2n-1}, y_{2n}) + d(Tx_{2n}, Sx_{2n+1}). \end{aligned}$$

Therefore

$$\begin{aligned} d(Tx_{2n}, Sx_{2n+1}) \leq\ & d(Sx_{2n-1}, y_{2n}) + \beta \max\{d(Tx_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\} \\ & + \gamma\, \{d(Sx_{2n-1}, y_{2n}) + d(Tx_{2n}, y_{2n+1})\}. \end{aligned}$$

If $d(Tx_{2n}, y_{2n+1}) \geq d(y_{2n-1}, y_{2n})$, then we obtain

$$d(Tx_{2n}, Sx_{2n+1}) \leq \left(\frac{1+\gamma}{1-\beta-\gamma}\right) d(Sx_{2n-1}, y_{2n}).$$

Otherwise, if $d(Tx_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n})$, then

$$d(Tx_{2n}, Sx_{2n+1}) \leq \left(\frac{1+\beta+\gamma}{1-\gamma}\right) d(Sx_{2n-1}, y_{2n}) \leq \left(\frac{1+\gamma}{1-\beta-\gamma}\right) d(Sx_{2n-1}, y_{2n}).$$

Now, proceeding as earlier, we also obtain

$$d(Sx_{2n-1}, y_{2n}) \leq \left(\frac{\alpha+\beta+\gamma}{1-\gamma}\right) d(Sx_{2n-1}, Tx_{2n-2}).$$

Therefore combining the above inequalities, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq kd(Sx_{2n-1}, Tx_{2n-2}), \text{ where } k = \left(\frac{1+\gamma}{1-\beta-\gamma}\right)\left(\frac{\alpha+\beta+\gamma}{1-\gamma}\right).$$

Thus in all the cases, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq k \max \{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n-2}, Sx_{2n-1})\} \tag{3.5}$$

whereas

$$d(Sx_{2n+1}, Tx_{2n+2}) \leq k \max \{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Sx_{2n+1})\}. \tag{3.6}$$

Now in the lines of Assad and Kirk [2], it can be shown by induction that for $n \geq 1$, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq k^n q \text{ and } d(Sx_{2n+1}, Tx_{2n+2}) \leq k^{n+\frac{1}{2}} q, \text{ whereas}$$
$$q = k^{\frac{-1}{2}} \max \{d(Tx_0, Sx_1), d(Sx_1, Tx_2)\}.$$

Thus the sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, ..., Sx_{2n-1}, Tx_{2n}, Sx_{2n+1}, ...\}$ is Cauchy and hence converges to the point z in X. Then as noted in [5] there exists at least one subsequence $\{Tx_{2n_k}\}$ or $\{Sx_{2n_k+1}\}$ which is contained in $P_\circ$ or $Q_\circ$ respectively. Suppose that there exists a subsequence $\{Tx_{2n_k}\}$ which is contained in $P_\circ$ for each $k \in \mathbf{N}$, that also converges to z. Using compatibility of (F_j, S), we have

$$\lim_{k\to\infty} d(Sx_{2n_k-1}, F_j(x_{2n_k-1})) = 0 \text{ for any even integer } j \in \mathbf{N},$$

which implies that $\lim_{k\to\infty} d(STx_{2n_k}, F_j(Sx_{2n_k-1})) = 0$.

Using the continuity of S and F_j, one obtains $Sz \in F_j(z)$ for any even integer $j \in \mathbf{N}$. Similarly the continuity of T and F_i implies $Tz \in F_i(z)$ for any odd integer $i \in \mathbf{N}$. Now

$$\begin{aligned}
d(Tz, Sz) &\leq \delta(F_i(z), F_j(z)) \\
&\leq \alpha\, d(Tz, Sz) + \beta \max\{d(Tz, F_i(z)), d(Sz, F_j(z))\} \\
&\quad + \gamma \max\{d(Tz, Fj(z)) + d(Sz, Fi(z)), d(Tz, F_i(z)) + d(Sz, F_j(z))\} \\
&\leq 2\gamma\, d(Tz, Sz)
\end{aligned}$$

implying thereby $Tz = Sz$. Thus z is a common coincidence point of $\{F_n\}, S$ and T.

If one assumes that there exists a subsequence $\{Sx_{2n_k+1}\}$ contained in Q_0, then the foregoing arguments establish the earlier conclusions. This completes the proof. □

Remark 3.1. By setting $F_i = F_j = F$ for all (i and j), $S = T = I_K$, $\beta = 0 = \gamma$ and δ distance is replaced by Hausdorff distance H in Theorem 3.1, and we deduce a theorem due to Khan [13].

Remark 3.2. If δ distance is replaced by Hausdorff distance H in Theorem 3.1, we deduce a theorem due to Khan [14].

Remark 3.3. Theorem 3.1 remains true if we utilize the pointwise $R-$ weak commutative condition.

In the next theorem we utilize the closedness of TK and SK (or $F_i(K)$ and $F_j(K)$) to relax the continuity requirements besides minimizing the commutativity requirements to merely coincidence points.

Theorem 3.2. *Let (X,d) be a metrically convex metric space and K be a nonempty closed subset of X. Let $F_n : K \to CB(X)$ and $S,T : K \to X$ satisfy* (3.1) *and the conditions* (1) *and* (2) *of the Theorem 3.1. Suppose that*

(1) *TK and SK (or $F_i(K)$ and $F_j(K)$) are closed subspaces of X. Then*
(2) *the pair (F_j,S) as well as (F_i,T) has a point of coincidence.*

Moreover, (F_i,T) has a common fixed point if T is quasi-coincidentally commuting and occasionally coincidentally idempotent w.r.t F_i, whereas (F_j,S) has a common fixed point provided S is quasi-coincidentally commuting and occasionally coincidentally idempotent w.r.t F_j.

Proof. Proceeding as in Theorem 3.1, we assume that there exists a subsequence $\{Tx_{2n_k}\}$ which is contained in $P_\circ$ and TK as well as SK are closed subspaces of X. Since $\{Tx_{2n_k}\}$ is Cauchy in TK, it converges to a point $u \in TK$. Let $v \in T^{-1}u$, then $Tv = u$. Since $\{Sx_{2n_k+1}\}$ is a subsequence of a Cauchy sequence, $\{Sx_{2n_k+1}\}$ converges to u as well. Using (3.1) we can write

$$\begin{aligned} d(F_i(v),Tx_{2n_k}) &\leq \delta(F_i(v),F_j(x_{2n_k-1})) \\ &\leq \alpha\, d(Tv,Sx_{2n_k-1}) + \beta\, \max\{d(Sx_{2n_k-1},F_j(x_{2n_k-1})), d(Tv,F_i(v))\} \\ &\quad + \gamma\, \max\{d(Sx_{2n_k-1},F_j(x_{2n_k-1})) + d(Tv,F_i(v)), \end{aligned}$$

$d(Tv,F_j(x_{2n_k-1})) + d(Sx_{2n_k-1},F_i(v))\}$ which on letting $k \to \infty$, reduces to

$$\begin{aligned} d(F_i(v),u) &\leq \beta\, \max\{d(u,F_i(v)),0\} + \gamma\, \max\{d(F_i(v),u), d(F_i(v),u)\} \\ &\leq (\beta+\gamma)\, d(u,F_i(v)) \end{aligned}$$

yielding thereby $u \in F_i(v)$, which implies that $u = Tv \in F_i(v)$ as $F_i(v)$ is closed.

Since the Cauchy sequence $\{Tx_{2n_k}\}$ converges to $u \in K$ and $u \in F_i(v)$, $u \in F_i(K) \cap K \subseteq SK$, there exists $w \in K$ such that $Sw = u$. Again using (3.1) we get

$$\begin{aligned} d(Sw, F_j(w)) = d(Tv, F_j(w)) &\leq \delta(F_i(v), F_j(w)) \\ &\leq \alpha\, d(Tv, Sw) + \beta\, \max\{d(Tv, F_i(v)), d(Sw, F_j(w))\} \\ &\quad + \gamma\, \max\{d(Tv, F_i(v)) + d(Sw, F_j(w)), d(Tv, F_j(w)) + d(Sw, F_i(v))\} \\ &\leq (\alpha + \beta + \gamma)\, d(Sw, F_j(w)) \end{aligned}$$

implying thereby $Sw \in F_j(w)$, that is w is a coincidence point of (S, F_j).

In case $F_i(K)$ and $F_j(K)$ are closed subspaces, then $u \in F_i(K) \cap K \subseteq SK$ or $F_j(K) \cap K \subseteq TK$. The analogous arguments establish the desired conclusions. If we assume that there exists a subsequence $\{Sx_{2n_k+1}\}$ contained in $Q_\circ$ with TK as well as SK closed subspaces of X, then noting that $\{Sx_{2n_k+1}\}$ is Cauchy in SK, the foregoing arguments establish that $Tz \in F_i(z)$ and $Sw \in F_j(w)$.

Since v is a coincidence point of (F_i, T) using the quasi-coincidentally commuting property of (F_i, T) and occasionally coincidentally idempotent property of T w.r.t F_i we have

$$Tv \in F_i(v) \text{ and } u = Tv \Rightarrow Tu = TTv = Tv = u.$$

Therefore $u = Tu = TTv \in TF_i(v) \subset F_i(Tv) = F_i(u)$, which shows that u is the common fixed point of (F_i, T). Similarly using the quasi-coincidentally commuting property of (F_j, S) and occasionally coincidentally idempotent property of S w.r.t F_j we can show that (F_j, S) has a common fixed point as well. This completes the proof. □

Remark 3.4. Theorem 3.2 remains true if we substitute closedness of 'TK and SK' with closedness of '$F_i(K)$ and $F_j(K)'$.

Remark 3.5. By setting $F_i = F_j = F$ for all (i and j), $S = T = I_K$, $\beta = 0 = \gamma$ and if δ distance is replaced by Hausdorff distance H in Theorem 3.2, we deduce a theorem due to Assad and Kirk [2].

Remark 3.6. By setting $F_i = F$ for all i, $F_j = G$ for all j, $S = T = I_K$ and if δ distance is replaced by Hausdorff distance H in Theorem 3.2, we deduce a theorem due to Ćirić and Ume [3].

Remark 3.7. By setting $F_i = F_j = F$ for all (i and j), $S = T = I_K$ and if δ distance is replaced by Hausdorff distance H in Theorem 3.2, we deduce a theorem due to Rhoades [19].

Remark 3.8. We can prove a theorem when the 'closedness of K' is replaced by' compactness of K'.

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REFERENCES

[1] N. A. Assad, *Fixed point theorems for set-valued transformations on compact sets,* Boll. Un Mat. Ital., (4) 1973.

[2] N. A. Assad and W. A. Kirk, *Fixed point theorems for set valued mappings of contractive type,* Pacific J. Math., 43(3) 1972.

[3] Lj. B. Ćirić and J. S. Ume, *Multi-valued non-self mappings on convex metric spaces,* Nonlinear Analysis, (60) 2005.

[4] B. Fisher, *Common fixed point and contraction mappings satisfying a rational inequality,* Math. Sem. Notes, (6) 1978.

[5] O. Hadžić, *On coincidence points in convex metric spaces*, Univ. U. Novom Sadu, Zb. Rad. Prirod. Mat. Fak. Ser. Mat., 19(2) 1986.

[6] O. Hadžić and Lj. Gajić, *Coincidence points for set-valued mappings in convex metric spaces*, Univ. U. Novom Sadu, Zb. Rad. Prirod. Mat. Fak. Ser. Mat., 16(1) 1986.

[7] N. J. Huang and Y. J. Cho, *Common fixed point theorems for a sequence of set-valued mappings*, Korean J. Math. Sci., (4) 1997.

[8] M. Imdad and L. Khan, *Rhoades type fixed point theorems for two hybrid pairs of mappings in metrically convex spaces*, Nonlinear Analysis Hybrid Systems, (4) 2010.

[9] M. Imdad and L. Khan, *Fixed point theorems for a family of hybrid pairs of mappings in metrically convex spaces*, Fixed Point Theory Appl., (3) 2005.

[10] S. Itoh, *Multi-valued generalized contractions and fixed point theorems*, Comment. Math. Univ. Carolinae, (18) 1977.

[11] H. Kaneko and S. Sessa, *Fixed point theorems for compatible multi-valued and single-valued mappings*, Internat. J. Math. Math. Sci., 12(2) 1989.

[12] K. Kuratowski, *Topology*, Academic Press, (1) 1966.

[13] L. Khan, *Hybrid pairs of nonself multi-valued mappings in metrically convex metric spaces*, Global Journal of Pure and Applied Mathematics, 14(11) 2018.

[14] L. Khan, *Hybrid pairs of nonself multi-valued mappings in metrically convex spaces*, Southeast Asian Bulletin of Mathematics, (46) 2022.

[15] L. Khan and M. Imdad, *Rhoades type fixed point theorems for two hybrid pairs of mappings in metrically convex spaces*, Applied Math. Computation, (218) 2012.

[16] K. Menger, *Untersuchungen tiber allgemeine*, Math. Annalen, (100) 1928.

[17] S. B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math., 30(2) 1969.

[18] R. P. Pant, *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl., (188) 1994.

[19] B. E. Rhoades, *A fixed point theorem for a multivalued nonself mapping*, Comment. Math. Univ. Carolinae, 37(2) 1996.

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