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ON KILLING MAGNETIC CURVES IN THE HYPERBOLOID MODEL OF $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ GEOMETRY

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ABSTRACT. A Killing magnetic curve is a trajectory of a charged particle on a Riemannian manifold under the action of a Killing magnetic field. In this paper we study Killing magnetic curves in the hyperboloid model of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ geometry.

1. INTRODUCTION

Let F be a closed 2-form on a Riemannian 3-manifold (M,g), called the *magnetic field*. This title comes from the fact that a closed 2-form can be regarded as a generalization of a static magnetic field on the 3-dim Euclidean space [15].

A curve $\gamma(t)$ on a Riemannian 3-manifold (M,g) is called a *magnetic curve* if its velocity vector field satisfies the Lorentz equation

$$\nabla_{\gamma'}\gamma' = \Phi(\gamma'), \tag{1.1}$$

where ∇ is the Levi-Civita connection of g and Φ is $(1,1)$-tensor field on M, called the *Lorentz force*, related to the magnetic field F by

$$g(\Phi(X),Y) = F(X,Y), \quad \forall X,Y \in \mathfrak{X}(M). \tag{1.2}$$

For $\Phi = 0$ the differential equation (1.1) coincides with the geodesic equation. Hence, we can say that magnetic trajectories are generalizations of geodesics.

The vector field V on M is called a *Killing vector field* if it satisfies the Killing equation

$$g(\nabla_Y V,Z) + g(\nabla_Z V,Y) = 0, \quad \forall Y,Z \in \mathfrak{X}(M). \tag{1.3}$$

The Killing vector field can be interpreted as an infinitesimal generator of isometry on the manifold in the sense that the flow generated by this field is a continuous isometry of the manifold.

In particular, Killing vector fields define an important class of magnetic fields called *Killing magnetic fields* and the trajectories corresponding to the Killing magnetic fields are called the *Killing magnetic curves*.

Killing magnetic curves in the Euclidean 3-space $\mathbb{E}^3$ and Minkowski 3-spacetime $\mathbb{E}^3_1$ were studied by Druţă-Romaniuc and Munteanu in [4] and [3], respectively. Munteanu and Nistor considered Killing magnetic curves in the $\mathbb{S}^2 \times \mathbb{R}$ space in

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[13] and Erjavec and Inoguchi in the Sol_3 space in [5]. Killing magnetic curves and magnetic curves using the right half-space model of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ geometry are studied in [8] and [9], respectively.

The goal of this paper is to study the Killing magnetic curves in the hyperboloid model of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ geometry. The unique features of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ geometry make this task seem more complicated than in other homogeneous geometries.

Let us recall that the cross product of two vector fields $X, Y \in \mathfrak{X}(M)$ on Riemannian manifold M is defined as follows

$$g(X \times Y, Z) = dv_g(X,Y,Z), \quad \forall Z \in \mathfrak{X}(M), \tag{1.4}$$

where dv_g denotes a volume form on M.

It is known that in 3-dim space a closed 2-form can be identified with divergence free vector field via the Hodge operator and the volume form of the oriented manifold. If V is a Killing vector field on M, let $F_V = i_V dv_g$ be the corresponding Killing magnetic field, where i denotes the inner product on M. Hence, the Lorentz force Φ_V corresponding to the Killing magnetic field F_V is

$$\Phi_V(X) = V \times X,$$

and then the Lorentz equation (1.1) can be written as

$$\nabla_{\gamma'}\gamma' = V \times \gamma'. \tag{1.5}$$

2. The hyperboloid model of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ geometry

Two models of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ geometry appear in the literature. The first one is the right half-space model of $\mathrm{SL}(2,\mathbb{R})$ geometry ($\mathcal{R}$) and the second one is the hyperboloid model of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ geometry ($\mathcal{H}$). Each of these is useful in a certain context. The isometry between them is constructed in [6]. The hyperboloid model is introduced in [11] and used in [2, 7]. On the other hand, the right half-space model is explained in detail in [14] and used in [1, 10, 12].

In [6], the diffeomorphism $\pi : \mathcal{H} \to \mathcal{R}$ mapping $(r, \vartheta, \varphi) \mapsto (x, y, \theta)$ is given by

$$\begin{aligned}
x &= \frac{\cos\vartheta}{\coth 2r - \sin\vartheta}, \\
y &= \frac{1}{\sinh 2r(\coth 2r - \sin\vartheta)}, \\
\theta &= \arctan\left(\frac{\sin\varphi - \tanh r \cdot \cos(\varphi - \vartheta)}{\cos\varphi + \tanh r \cdot \sin(\varphi - \vartheta)}\right).
\end{aligned}$$

Briefly, we recall the fundamental properties of the hyperboloid model of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ geometry. The idea is to start with the collineation group which acts on the projective 3-space $\mathcal{P}^3(\mathbb{R})$ and the projective sphere $\mathcal{PS}^3(\mathbb{R})$ and preserves a hyperboloid polarity, i.e. a scalar product of signature $(--++)$. Using the one-sheeted hyperboloid solid

$$\mathcal{H}:\ -x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 < 0,$$

with an appropriate choice of a subgroup of the collineation group of $\mathcal{H}$ as an isometry group, the universal covering space $\widetilde{\mathcal{H}}$ of the hyperboloid $\mathcal{H}$ will give us the so-called hyperboloid model of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ geometry.

The Riemannian metric in the hyperboloid model of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ space is given by

$$(ds)^2 = (dr)^2 + \sinh^2 r\cosh^2 r(d\vartheta)^2 + \big(\sinh^2 r(d\vartheta) + (d\varphi)\big)^2, \tag{2.1}$$

where $r \in [0,\infty)$ and $\vartheta \in [-\pi,\pi)$ are polar coordinates of the intersection point of a fiber and the hyperbolic base plane and $\varphi \in (-\frac{\pi}{2},\frac{\pi}{2})$ is a fiber coordinate with an extension to $\mathbb{R}$ for the universal covering. From (2.1) one can easily see that the metric is invariant under rotations about a fiber through the origin and translations along fibers. Therefore, the symmetric metric tensor field g is given by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh^2 r\cosh 2r & \sinh^2 r \\ 0 & \sinh^2 r & 1 \end{pmatrix}.$$

The Euclidean coordinates, corresponding to the hyperboloid coordinates (r,ϑ,φ), are given by

$$\begin{aligned} x &= \tan\varphi, \\ y &= \tanh r\cdot\frac{\cos(\vartheta-\varphi)}{\cos\varphi}, \\ z &= \tanh r\cdot\frac{\sin(\vartheta-\varphi)}{\cos\varphi}. \end{aligned} \tag{2.2}$$

These formulas are important for later visualization of curves and surfaces in E^3, i.e., enabling us to visualize magnetic curves inside of the one-sheeted hyperboloid that represents our ambient space.

The orthonormal coframe field is given by

$$\theta^1 = dr, \quad \theta^2 = \frac{1}{2}\sinh 2r\, d\vartheta, \quad \theta^3 = \sinh^2 r\, d\vartheta + d\varphi,$$

and the associated orthonormal right invariant frame field by

$$e_1 = \frac{\partial}{\partial r}, \quad e_2 = \frac{2}{\sinh 2r}\frac{\partial}{\partial\vartheta} - \tanh r\frac{\partial}{\partial\varphi}, \quad e_3 = \frac{\partial}{\partial\varphi}. \tag{2.3}$$

Hence,

$$\partial_r = e_1, \quad \partial_\vartheta = \sinh r\cosh r\, e_2 + \sinh^2 r\, e_3, \quad \partial_\varphi = e_3. \tag{2.4}$$

In the covariant derivative fashion, the Levi-Civita connection ∇ of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is given by

$$\begin{array}{lll} \nabla_{e_1}e_1 = 0 & \nabla_{e_1}e_2 = -e_3 & \nabla_{e_1}e_3 = e_2 \\ \nabla_{e_2}e_1 = 2\coth 2r\, e_2 + e_3 & \nabla_{e_2}e_2 = -2\coth 2r\, e_1 & \nabla_{e_2}e_3 = -e_1 \\ \nabla_{e_3}e_1 = e_2 & \nabla_{e_3}e_2 = -e_1 & \nabla_{e_3}e_3 = 0. \end{array} \tag{2.5}$$

Hence, we have the following commutation relations of the basis

$$[e_1,e_2] = -2\coth 2r\, e_2 - 2\, e_3, \quad [e_1,e_3] = [e_2,e_3] = 0.$$

The non-vanishing components of the Riemannien curvature tensor defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

up to symmetry properties, are

$$R(e_1,e_2)e_1 = 7e_2, \quad R(e_2,e_3)e_2 = -e_3, \quad R(e_1,e_3)e_1 = -e_3.$$

Moreover, if we put $R_{ijkl} = -g(R_{ijk},e_l)$, where $R_{ijk} = R(e_i,e_j)e_k$, we obtain

$$R_{1212} = -7, \quad R_{1313} = 1, \quad \text{and} \quad R_{2323} = 1.$$

3. Killing vector fields in the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ geometry

In this section we recall basic facts on Killing vector fields and determine the Killing vector fields in the hyperboloid model of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ geometry.

Recall that the vector field V on M is a Killing vector field if it satisfies the Killing equation

$$g(\nabla_Y V,Z) + g(\nabla_Z V,Y) = 0, \quad \forall Y,Z \in \mathfrak{X}(M). \tag{3.1}$$

Let us assume that the Killing vector field has the form

$$V = a(r,\vartheta,\varphi)\cdot e_1 + b(r,\vartheta,\varphi)\cdot e_2 + c(r,\vartheta,\varphi)\cdot e_3.$$

Substituting V in (3.1) and taking $Y = e_i,\ Z = e_j$ for all $i,j \in \{1,2,3\}$, we obtain the following system of differential equations for Killing vector fields in the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ space,

$$\begin{aligned}
&\partial_r a = 0, && \partial_r b + \frac{2}{\sinh 2r}\partial_\vartheta a - \tanh r\partial_\varphi a - 2b\coth 2r = 0,\\
&\partial_r c + \partial_\varphi a - 2b = 0, && \frac{2}{\sinh 2r}\partial_\vartheta b - \tanh r\partial_\varphi b + 2a\coth 2r = 0,\\
&\partial_\varphi c = 0, && \frac{2}{\sinh 2r}\partial_\vartheta c - \tanh r\partial_\varphi c + \partial_\varphi b + 2a = 0.
\end{aligned} \tag{3.2}$$

Solving of (3.2) is a real challenge. We have been unable to solve (3.2) in the general case. However, we found two simpler solutions. The computer programm Maple also could find only these two solutions. The third and the fourth solution were found by luck through the study of the Killing vector fields of the disc model of the hyperbolic plane.

By long but straightforward computation it is possible to check that the solutions of the (3.2) are the following vector fields

$$\begin{aligned}
&V_1 = e_3, \qquad V_2 = \sinh 2r\, e_2 + \cosh 2r\, e_3,\\
&V_3 = \sin\vartheta\, e_1 + \cosh 2r\cos\vartheta\, e_2 + \sinh 2r\cos\vartheta\, e_3,\\
&V_4 = \cos\vartheta\, e_1 - \cosh 2r\sin\vartheta\, e_2 - \sinh 2r\sin\vartheta\, e_3.
\end{aligned}$$

Usually, the Killing vector fields are given in the canonical basis, instead of in the basis of an ambient space. Hence, using the relations (2.3), the Killing vector fields in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ are given by

$$\begin{aligned}
&V_1 = \partial_\varphi, \qquad V_2 = 2\partial_\vartheta + \partial_\varphi,\\
&V_3 = \sin\vartheta\,\partial_r + 2\coth 2r\cos\vartheta\,\partial_\vartheta + \tanh r\cos\vartheta\,\partial_\varphi,\\
&V_4 = \cos\vartheta\,\partial_r - 2\coth 2r\sin\vartheta\,\partial_\vartheta - \tanh r\sin\vartheta\,\partial_\varphi.
\end{aligned}$$

Remark 3.1. Theoretically, it is possible to express the vectors of canonical basis (the partial derivatives) of one model, e.g. $\mathcal{R}$, as a functions depending on vectors of the canonical basis of the second model $\mathcal{H}$, but transformation formulas are generally complicated. Namely, in our previous work we gave the transformation of canonical 1-forms between the two models.

$$dx = \overbrace{\frac{2\cos\vartheta}{\sinh^2 2r(\coth 2r - \sin\vartheta)^2}}^{A}dr + \overbrace{\frac{1-\coth 2r\sin\vartheta}{(\coth 2r - \sin\vartheta)^2}}^{B}d\vartheta,$$

$$dy = \overbrace{\frac{2(-1+\coth 2r\sin\vartheta)}{\sinh 2r(\coth 2r - \sin\vartheta)^2}}^{C}dr + \overbrace{\frac{\cos\vartheta}{\sinh 2r(\coth 2r - \sin\vartheta)^2}}^{D}d\vartheta,$$

$$d\theta = \overbrace{\frac{-\cos\vartheta}{\cosh^2 2r(1-2\sin\vartheta\tanh r + \tanh^2 r)}}^{E}dr + \overbrace{\frac{\tanh r(\sin\vartheta - \tanh r)}{1-2\sin\vartheta\tanh r+\tanh^2 r}}^{F}d\vartheta + d\varphi.$$

Hence, we can uniquely determine transformations of the partial derivatives

$$\begin{aligned}
\partial_x &= \frac{D}{AD-BC}\partial_r - \frac{C}{AD-BC}\partial_\vartheta - \frac{DE+FC}{AD-BC}\partial_\varphi,\\
\partial_y &= \frac{-B}{AD-BC}\partial_r + \frac{A}{AD-BC}\partial_\vartheta + \frac{BE-AF}{AD-BC}\partial_\varphi,\\
\partial_\theta &= \partial_\varphi.
\end{aligned}$$

Although some parts of these ratios will be reduced, it is not easy to compare Killing vector fields in the two models using these transformations.

4. KILLING MAGNETIC CURVES IN THE $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ GEOMETRY

The goal of this section is to find magnetic curves corresponding to the Killing magnetic fields in the hyperboloid model of the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ space.

4.1. **Case A**

In this subsection we consider Killing magnetic curves which correspond to the Killing vector field $V_1 = \partial_\varphi$.

Our first task is to deduce the magnetic curve equation (1.5) for a regular curve $\gamma(s) = (r(s), \vartheta(s), \varphi(s))$ in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. We have

$$\gamma'(s) = r'(s)\frac{\partial}{\partial r} + \vartheta'(s)\frac{\partial}{\partial \vartheta} + \varphi'(s)\frac{\partial}{\partial \varphi},$$

and from (2.4)

$$\gamma' = r'e_1 + \frac{1}{2}\sinh 2r\,\vartheta'\,e_2 + \left(\sinh^2 r\,\vartheta' + \varphi'\right)e_3. \tag{4.1}$$

Next we compute the covariant derivative $\nabla_{\gamma'}\gamma'$. Taking (2.5) into account, we obtain

$$\begin{aligned}\nabla_{\gamma'}\gamma' = &\left(r'' - \frac{1}{2}\sinh 2r\,\vartheta'\left(\left(1+4\sinh^2 r\right)\vartheta' + 2\varphi'\right)\right)e_1\\ &+\left(\frac{1}{2}\sinh 2r\,\vartheta'' + 2\left(1+3\sinh^2 r\right)r'\vartheta' + 2r'\varphi'\right)e_2\\ &+\left(\varphi'' + \sinh^2 r\,\vartheta'' + \sinh 2r\,r'\vartheta'\right)e_3.\end{aligned} \tag{4.2}$$

Using relation (4.1) and formula (1.4) we have

$$V_1 \times \gamma' = -\frac{1}{2}\sinh 2r\,\vartheta'\,e_1 + r'e_2. \tag{4.3}$$

Remark 4.1. (4.3) can be obtained in another way. Let

$$dv_g = \sinh r\cosh r(dr\wedge d\vartheta\wedge d\varphi)$$

be the volume element of the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. The Killing vector field $V_1 = \partial_\varphi$ by $F_V = i_V dv_g$, defines the magnetic field

$$F_{V_1}(X,Y) = dv_g(X,Y,\partial_\varphi) = \frac{1}{2}\sinh 2r(dr\wedge d\vartheta)(X,Y). \tag{4.4}$$

From (1.2) and (4.4) we get

$$\Phi_{V_1}(\partial_r) = \frac{2}{\sinh 2r}\partial_\vartheta - \tanh r\partial_\varphi, \quad \Phi_{V_1}(\partial_\vartheta) = -\frac{1}{2}\sinh 2r\partial_r, \quad \Phi_{V_1}(\partial_\varphi) = 0.$$

Hence, from (2.4) the Lorentz force Φ_{V_1} acts on the basis vectors of the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ as

$$\Phi_{V_1}(e_1) = e_2, \quad \Phi_{V_1}(e_2) = -e_1, \quad \Phi_{V_1}(e_3) = 0.$$

Finally we obtain the right hand side of the relation (4.3)

$$\begin{aligned}\Phi_{V_1}(\gamma') &= \Phi_{V_1}\left(r'e_1 + \frac{1}{2}\sinh 2r\,\vartheta'\,e_2 + \left(\sinh^2 r\,\vartheta' + \varphi'\right)e_3\right)\\ &= r'e_2 - \frac{1}{2}\sinh 2r\,\vartheta'\,e_1.\end{aligned}$$

Further, equalizing the right hand sides of (4.2) and (4.3), we obtain the following system of differential equations

$$\begin{aligned}r'' - \frac{1}{2}\sinh 2r\,\vartheta'\left[\left(1+4\sinh^2 r\right)\vartheta' + \left(2\varphi'-1\right)\right] &= 0,\\ \frac{1}{2}\sinh 2r\,\vartheta'' + r'\left[2\left(1+3\sinh^2 r\right)\vartheta' + \left(2\varphi'-1\right)\right] &= 0,\\ \varphi'' + \sinh^2 r\,\vartheta'' + \sinh 2r\,r'\vartheta' &= 0.\end{aligned} \tag{4.5}$$

Using (4.1), the arc length condition is given by

$$(r')^2+\left(\frac{1}{2}\sinh 2r\,\vartheta'\right)^2+\left(\sinh^2 r\,\vartheta'+\varphi'\right)^2=1. \tag{4.6}$$

Next, we try to solve the obtained system (4.5). The third equation can be rewritten as

$$\frac{d}{ds}\left(\varphi'+\sinh^2 r\,\vartheta'\right)=0.$$

Hence, $\varphi'+\sinh^2 r\,\vartheta'=c,\ c\in\mathbb{R}$ and therefore

$$\varphi'(s)=c-\sinh^2 r(s)\,\vartheta'(s). \tag{4.7}$$

Substituting (4.7) in the first two equations of (4.5) and in the equation (4.6), we get

$$4r''-\sinh 4r\left(\vartheta'\right)^2-2\sinh 2r\,\vartheta'\,(2c-1)=0,$$

$$\sinh 2r\,\vartheta''+4\cosh 2r\,r'\vartheta'+2r'\,(2c-1)=0, \tag{4.8}$$

$$(r')^2+\left(\frac{1}{2}\sinh 2r\,\vartheta'\right)^2=1-c^2,\quad -1\le c\le 1. \tag{4.9}$$

Next, using (4.9) we consider some particular solutions of (4.5) and (4.8).
If we assume $r'=0$, we consider two possibilities $r=0$ and $r=r_0\neq 0$. In the first case, when $r=0$, (4.9) implies $c=\pm 1$ and $\varphi(s)=\pm s+\varphi_0$. Hence, we have the first Killing magnetic curve

$$\gamma(s)=\left(0,\ 0,\ \pm s+\varphi_0\right). \tag{4.10}$$

In the second case, from (4.9) and (4.7) it follows that

$$\vartheta'=\frac{2\sqrt{1-c^2}}{\sinh 2r_0}=\text{const.},\quad \text{and}\quad \varphi'=c-\sqrt{1-c^2}\tanh r_0=\text{const.}, \text{respectively}.$$

By the first equation of (4.5), the constant c is given as a solution of the equation

$$2c+2\sqrt{1-c^2}\coth 2r_0-1=0. \tag{4.11}$$

Examining (4.11), it can be proved that $c\in[-1,1],\ \forall r_0\in\mathbb{R}$. Thus, we obtain the second Killing magnetic curve

$$\gamma(s)=\left(r_0,\ \frac{2\sqrt{1-c^2}}{\sinh 2r_0}s+\vartheta_0,\ \left(c-\tanh r_0\sqrt{1-c^2}\right)s+\varphi_0\right). \tag{4.12}$$

Further, if we assume $\vartheta'=0$, then by (4.9) $r'=\sqrt{1-c^2}$. From (4.7) and the second equation of (4.5) it follows $\varphi(s)=\frac{1}{2}s+\varphi_0,\ \varphi_0\in\mathbb{R}$. Next, from the first equation of (4.5) and the equation (4.9) we get $r(s)=\pm\frac{\sqrt{3}}{2}s+r_0,\ r_0\in\mathbb{R}$. Thus, the third Killing magnetic curve is given by

$$\gamma(s)=\left(\pm\frac{\sqrt{3}}{2}s+r_0,\vartheta_0,\frac{1}{2}s+\varphi_0\right). \tag{4.13}$$

To find exact solutions for the system of differential equations (4.8) is a true challenge. Notice that if we try to eliminate $(2c-1)$ from the system (4.8), multiplying the first equation by $\frac{1}{2}r'$ and the second by $\sinh 2r\,\vartheta'$, after adding obtained expressions, we get an equation which is exactly the derivative of (4.6).

Even though we are unable to fully solve the system in the general case, we briefly describe our try. From (4.9) we can assume $r'(s) = \sqrt{1-c^2}\cos f(s)$ and $\vartheta'(s) = \frac{2\sqrt{1-c^2}}{\sinh 2r}\sin f(s)$, where $f = f(s)$ is an arbitrary function. By (4.7),

$$\varphi'(s) = c - \sqrt{1-c^2}\tanh r \sin f(s). \tag{4.14}$$

Substituting expressions for r', ϑ' and φ' into the first and the second equations of (4.5), we obtain a complicated nonlinear system of differential equations for f which we are unable to solve.

However, it seems reasonable try to find a solution of (4.8) assuming $c = \frac{1}{2}$. In this case from (4.8) we have

$$4r'' - \sinh 4r\, \left(\vartheta'\right)^2 = 0, \tag{4.15}$$

and

$$\vartheta'' + 4\coth 2r\; r'\; \vartheta' = 0. \tag{4.16}$$

Integrating (4.16) by separation of variables we obtain

$$\vartheta'(s) = \frac{k}{\sinh^2 2r(s)}, \;\; k \in \mathbb{R}. \tag{4.17}$$

Substituting (4.17) in (4.23) and (4.15) yields a contradiction. Thus, there is no Killing magnetic curve parameterized by arc length for $c = \frac{1}{2}$.

However, we proved the following theorem.

Theorem 4.1. *The Killing magnetic curves in the hyperboloid model of* $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ *geometry parameterized by arc length, corresponding to the Killing vector field* $V = \partial_\varphi$ *are solutions of the system of differential equations* (4.5)*. In particular, some analytical solutions of the system* (4.5) *are*

(a) *vertical geodesics given by*

$$\gamma(s) = (0,\ 0,\ \pm s + \varphi_0)\,.$$

(b) *curves given by*

$$\gamma(s) = \left(r_0,\ \frac{2\sqrt{1-c^2}}{\sinh 2r_0}s + \vartheta_0,\ \left(c - \sqrt{1-c^2}\tanh r_0\right)s + \varphi_0 \right).$$

(c) *curves given by*

$$\gamma(s) = \left(\pm\frac{\sqrt{3}}{2}s + r_0, \vartheta_0, \frac{1}{2}s + \varphi_0 \right),$$

where $r_0, \vartheta_0, \varphi_0 \in \mathbb{R}$ *and* $c \in [-1,1]$ *satisfying* (4.11).

Using coordinate transformation (2.2), we can visualise Killing magnetic curves in the hyperboloid model of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. Fig. 1 presents the Killing magnetic curves in cases (a) (green), (b) (red), (c) (blue) of Theorem 4.1 for $r_0 = 0.3$, $\vartheta_0 = \varphi_0 = 0$, $s \in [-5,5]$.

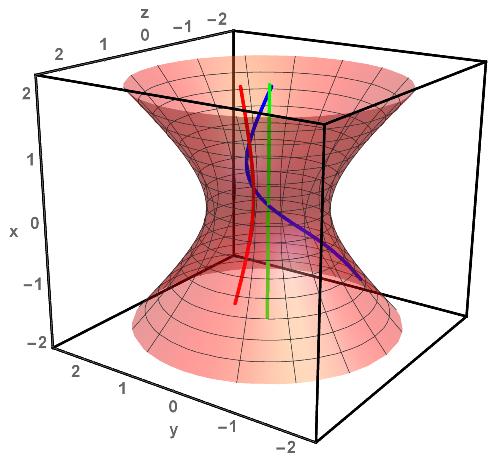

FIGURE 1. *Killing magnetic curves in Case A*

Remark 4.2. As mentioned in the Introduction, the magnetic curve coincides with the geodesic when the Lorentz force is zero. Particularly, the Killing magnetic curves, which correspond to the Killing magnetic field ∂_φ, are geodesics if the right hand side of the equation (4.3) vanishes, i.e. if $r = 0$ or $r = \text{const}$ and $\vartheta = \text{const}$. Hence, curves given in the last two cases of Theorem 4.1 are not geodesics.

4.2. **Case B**

Here, by analogy to the previous case, we consider magnetic curves which correspond to the Killing vector field $V_2 = 2\partial_\theta + \partial_\varphi$.
Using relation (4.1) and formula (1.4) we have

$$V_2 \times \gamma' = \frac{1}{2}\sinh 2r\left(2\varphi' - \vartheta'\right) e_1 + r'\cosh 2r\, e_2 - r'\sinh 2r\, e_3. \tag{4.18}$$

Remark 4.3. We could obtain (4.18) in another way. Let

$$dv_g = \sinh r \cosh r (dr \wedge d\vartheta \wedge d\varphi)$$

be the volume element of $\mathrm{SL}(2,\mathbb{R})$. The Killing vector field $V_2 = 2\partial_\theta + \partial_\varphi$ by $F_V = i_V dv_g$, defines the magnetic field

$$F_{V_2}(X,Y) = -\sinh 2r (dr \wedge d\varphi)(X,Y) - \frac{1}{2}\sinh 2r (dr \wedge d\vartheta)(X,Y). \tag{4.19}$$

From (1.2) and (4.19) we get

$$\Phi_{V_2}(\partial_r) = \coth 2r \partial_\vartheta - \tanh r (1 + 3\cosh^2 r)\partial_\varphi,$$

$$\Phi_{V_2}(\partial_\vartheta) = -\frac{1}{2}\sinh 2r \partial_r, \quad \Phi_{V_2}(\partial_\varphi) = \sinh 2r \partial_r.$$

Hence, from (2.4) the Lorentz force Φ_{V_2} acts on the basis vectors of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ as

$$\begin{aligned}\Phi_{V_2}(e_1) &= \cosh 2r\, e_2 - \sinh 2r\, e_3,\\ \Phi_{V_2}(e_2) &= -\cosh 2r\, e_1,\\ \Phi_{V_2}(e_3) &= \sinh 2r\, e_1.\end{aligned}$$

Finally we obtain the right hand side of the relation (4.18)

$$\begin{aligned}\Phi_{V_2}(\gamma') &= \Phi_{V_2}\left(r' e_1 + \frac{1}{2}\sinh 2r\, \vartheta'\, e_2 + \left(\sinh^2 r\, \vartheta' + \varphi'\right) e_3\right)\\ &= \frac{1}{2}\sinh 2r \left(2\varphi' - \vartheta'\right) e_1 + r' \cosh 2r\, e_2 - r' \sinh 2r\, e_3.\end{aligned}$$

Further, equalizing the right hand sides of the equations (4.2) and (4.18), we obtain the following system of differential equations

$$\begin{aligned} r'' - \frac{1}{2}\sinh 2r\left[\left(1+4\sinh^2 r\right)(\vartheta')^2 + \vartheta'\left(2\varphi' - 1\right) + 2\varphi'\right] &= 0,\\ \frac{1}{2}\sinh 2r\, \vartheta'' + r'\left[2\left(1+3\sinh^2 r\right)\vartheta' + 2\varphi' - \cosh 2r\right] &= 0,\\ \varphi'' + \sinh^2 r\, \vartheta'' + \sinh 2r\, r'\, (\vartheta' + 1) &= 0.\end{aligned} \tag{4.20}$$

Next, we try to solve the system (4.20).

The third equation can be rewritten as

$$\frac{d}{ds}\left(\varphi' + \sinh^2 r\, (\vartheta' + 1)\right) = 0.$$

Hence, $\varphi' + \sinh^2 r\, (\vartheta'+1) = C$, $C \in \mathbb{R}$ and

$$\varphi'(s) = C - \sinh^2 r(s)\, (\vartheta'(s) + 1). \tag{4.21}$$

Substituting (4.21) in (4.20) and (4.6), we respectively get

$$4r'' - \sinh 4r\left(\vartheta'\right)^2 - 2\sinh 2r\left(\vartheta'\left(2C - 1 - 4\sinh^2 r\right) + 2(C - \sinh^2 r)\right) = 0,$$

$$\sinh 2r\, \vartheta'' + 4\cosh 2r\, r'\vartheta' + 2r'\left(2C - 1 - 4\sinh^2 r\right) = 0, \tag{4.22}$$

$$(r')^2 + \left(\frac{1}{2}\sinh 2r\, \vartheta'\right)^2 + \left(C - \sinh^2 r\right)^2 = 1. \tag{4.23}$$

Next, using (4.23) we consider some particular solutions of (4.22).

Analogously to Case A, if we assume $r' = 0$, two possibilities $r = 0$ and $r = r_0 \neq 0$ are considered. In case $r = 0$, (4.23) implies $C = \pm 1$ and hence $\varphi(s) = \pm s + \varphi_0$. Therefore, we obtain the already known Killing magnetic curve

$$\gamma(s) = (0,\ 0,\ \pm s + \varphi_0)\,. \tag{4.24}$$

In case $r = r_0 \neq 0$, from (4.23) we obtain

$$\vartheta' = \frac{2\sqrt{1 - (C - \sinh^2 r_0)^2}}{\sinh 2r_0} = const,$$

and from (4.21)

$$\varphi' = C - \sinh^2 r_0 - \sqrt{1-(C-\sinh^2 r_0)^2}\,\tanh r_0 = const.$$

By the first equation of (4.20), the constant C is given as a solution of the equation

$$2\coth 2r_0\left(1-(C-\sinh^2 r_0)^2\right)+(2C-1-4\sinh^2 r_0)\sqrt{1-(C-\sinh^2 r_0)^2}+\sinh 2r_0(C-\sinh^2 r_0)=0.$$

where $C\in[-1,1]$, $\forall r_0\in\mathbb{R}$.

Thus, we obtain the Killing magnetic curve

$$\gamma(s)=\left(r_0,\ \frac{2\sqrt{1-(C-\sinh^2 r_0)^2}}{\sinh 2r_0}s+\vartheta_0,\ \left(C-\sinh^2 r_0-\sqrt{1-(C-\sinh^2 r_0)^2}\,\tanh r_0\right)s+\varphi_0\right).$$

If we assume $\vartheta' = 0$, then by the second equation of (4.22) we get $r' = 0$ (or $r = const.$ which is equivalent). This leads to the already known Killing magnetic curve (4.24).

Also, it seems reasonable try to find a solution of (4.20) assuming $\vartheta' = -1$. In this case, from (4.21), it follows $\varphi' = C$. By the second equation of (4.22) we again obtain $r' = 0$.

Unfortunately, we couldn't solve the system for the general case.

Hence, we proved the following theorem.

Theorem 4.2. *The Killing magnetic curves in the hyperboloid model of* $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ *geometry parameterised by arc length, corresponding to the Killing vector field* $V = 2\partial_\vartheta + \partial_\varphi$ *are solutions of the system of differential equations* (4.20). *In particular, some analytical solutions of the system* (4.20) *are*

(a) *geodesic given by*

$$\gamma(s) = (0,0,\pm s+\varphi_0)\,,$$

(b) *curves given by*

$$\gamma(s)=\left(r_0,\ \frac{2\sqrt{1-(C-\sinh^2 r_0)^2}}{\sinh 2r_0}s+\vartheta_0,\ \left(C-\sinh^2 r_0-\sqrt{1-(C-\sinh^2 r_0)^2}\,\tanh r_0\right)s+\varphi_0\right).$$

where $r_0,\vartheta_0,\varphi_0\in\mathbb{R}$ *and* $C\in[-1,1]$ *satisfying* (4.2).

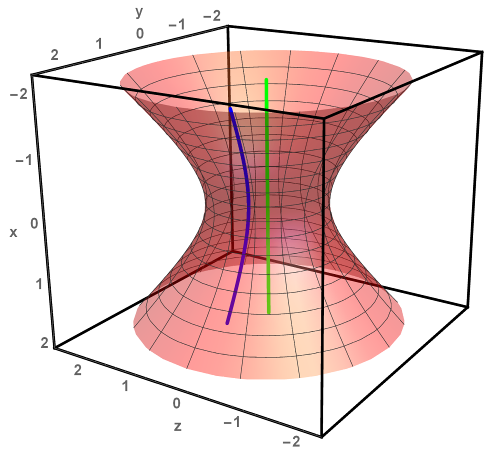

FIGURE 2. *Killing magnetic curves in Case B*

Using coordinate transformation (2.2), we can visualise Killing magnetic curves in the hyperboloid model of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. Fig. 2 presents the Killing magnetic curves in cases (a) (green) and (b) (blue) of Theorem 4.2 for $r_0 = 0.5$, $\vartheta_0 = \varphi_0 = 0$, $s \in [-5,5]$.

Remark 4.4. Notice that the Killing magnetic curves, corresponding to the Killing magnetic field $2\partial_\vartheta + \partial_\varphi$ are geodesics if the right hand side of the equation (4.18) vanishes, i.e. $r = 0$ or $r =$ const. and $2\varphi' = \vartheta'$. The first assumption leads to the already known geodesics and the second one to a very complicated system of differential equations.

4.3. **Case C**

In this subsection we consider Killing magnetic curves which correspond to the Killing vector field

$$V_3 = \sin\vartheta\, e_1 + \cosh 2r \cos\vartheta\, e_2 + \sinh 2r \cos\vartheta\, e_3.$$

Using relation (4.1) and the formula (1.4) we have

$$\begin{aligned} V_3 \times \gamma' = \cos\vartheta \left(\cosh 2r\, \varphi' - \sinh^2 r\, \vartheta'\right) e_1 + \\ + \left(\cos\vartheta\, r' \sinh 2r - \sin\vartheta (\sinh^2 r\, \vartheta' + \varphi')\right) e_2 \\ + \left(-\cos\vartheta\, r' \cosh 2r + \sin\vartheta \sinh r \cosh r\, \vartheta'\right) e_3. \end{aligned} \tag{4.25}$$

Remark 4.5. We could obtain (4.25) in another way. Let

$$dv_g = \sinh r \cosh r (dr \wedge d\vartheta \wedge d\varphi)$$

be the volume element of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. The Killing vector field

$$V_3 = \sin\vartheta\, \partial_r + 2\coth 2r \cos\vartheta\, \partial_\vartheta + \tanh r \cos\vartheta\, \partial_\varphi$$

by $F_V = i_V dv_g$, defines the magnetic field

$$\begin{aligned} F_{V_3}(X,Y) = \sinh^2 r \cos\vartheta\ (dr \wedge d\vartheta)(X,Y) - \cosh 2r \cos\vartheta\ (dr \wedge d\varphi)(X,Y) \\ + \sinh r \cosh r \sin\vartheta\ (d\vartheta \wedge d\varphi)(X,Y). \end{aligned} \tag{4.26}$$

From (1.2) and (4.26) we get

$$\begin{aligned} \Phi_{V_3}(\partial_r) &= 2\cos\vartheta \partial_\vartheta - \cos\vartheta (\cosh^2 r + 3\sinh^2 r)\partial_\varphi, \\ \Phi_{V_3}(\partial_\vartheta) &= -\sinh^2 r \cos\vartheta \partial_r - \tanh r \sin\vartheta \partial_\vartheta + \tanh r \cosh 2r \sin\vartheta \partial_\varphi, \\ \Phi_{V_3}(\partial_\varphi) &= \cosh 2r \cos\vartheta \partial_r - \frac{1}{\sinh r \cosh r} \sin\vartheta \partial_\vartheta + \tanh r \sin\vartheta \partial_\varphi. \end{aligned}$$

Hence, from (2.4) the Lorentz force Φ_{V_3} acts on the basis vectors of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ as

$$\begin{aligned} \Phi_{V_3}(e_1) &= \sinh 2r \cos\vartheta\, e_2 - \cosh 2r \cos\vartheta\, e_3, \\ \Phi_{V_3}(e_2) &= -\sinh 2r \cos\vartheta\, e_1 + \sin\vartheta\, e_3 \\ \Phi_{V_3}(e_3) &= \cosh 2r \cos\vartheta\, e_1 - \sin\vartheta\, e_2. \end{aligned}$$

Finally we obtain the right hand side of the relation (4.25)

$$\begin{aligned}\Phi_{V_3}(\gamma') &= \Phi_{V_3}\left(r'e_1 + \frac{1}{2}\sinh 2r\,\vartheta'\,e_2 + \left(\sinh^2 r\,\vartheta' + \varphi'\right)e_3\right)\\ &= \left(\cosh 2r\cos\vartheta\,\varphi' - \sinh^2 r\cos\vartheta\,\vartheta'\right)e_1 +\\ &\quad + \left(\sinh 2r\cos\vartheta\,r' - \sin\vartheta\left(\sinh^2 r\,\vartheta' + \varphi'\right)\right)e_2 +\\ &\quad + \left(-\cosh 2r\cos\vartheta\,r' + \sinh r\cosh r\sin\vartheta\,\vartheta'\right)e_3.\end{aligned}$$

Further, equalizing the right hand sides of the equations (4.2) and (4.25), we obtain the following system of differential equations

$$\begin{aligned}&r'' - \frac{1}{2}\sinh 2r\,\vartheta'\left(\left(1+4\sinh^2 r\right)\vartheta' + 2\varphi'\right) = \cosh 2r\cos\vartheta\,\varphi' - \sinh^2 r\cos\vartheta\,\vartheta',\\ &\frac{1}{2}\sinh 2r\,\vartheta'' + 2\left(1+3\sinh^2 r\right)r'\vartheta' + 2r'\varphi' = \sinh 2r\cos\vartheta\,r' - \sin\vartheta\left(\sinh^2 r\,\vartheta' + \varphi'\right),\\ &\varphi'' + \sinh^2 r\,\vartheta'' + \sinh 2r\,r'\,\vartheta' = -\cosh 2r\cos\vartheta\,r' + \frac{1}{2}\sinh 2r\sin\vartheta\,\vartheta'. \qquad (4.27)\end{aligned}$$

Next, we try to solve the system (4.27). The third equation can be rewritten as

$$\frac{d}{ds}\left(\varphi' + \sinh^2 r\,\vartheta'\right) = \frac{d}{ds}\left(-\frac{1}{2}\sinh 2r\cos\vartheta\right).$$

Hence, $\varphi' + \sinh^2 r\,\vartheta' = C_1 - \frac{1}{2}\sinh 2r\cos\vartheta$, $C_1 \in \mathbb{R}$ and

$$\varphi'(s) = C_1 - \sinh^2 r(s)\,\vartheta'(s) - \frac{1}{2}\sinh 2r(s)\cos\vartheta(s). \qquad (4.28)$$

Substituting (4.28) in the first two equations of (4.27) and in (4.6), we have respectively

$$\begin{aligned}&4r'' - \sinh 4r\left(\vartheta'\right)^2 + 4\sinh 2r\left(\sinh 2r\cos\vartheta - C_1\right)\vartheta' = 2\cosh 2r\cos\vartheta\left(2C_1 - \sinh 2r\cos\vartheta\right),\\ &\sinh 2r\,\vartheta'' + 4\cosh 2r\,r'\vartheta' + 4r'\left(C_1 - \sinh 2r\cos\vartheta\right) = -\sin\vartheta\left(2C_1 - \sinh 2r\cos\vartheta\right), \qquad (4.29)\end{aligned}$$

$$(r')^2 + \left(\frac{1}{2}\sinh 2r\,\vartheta'\right)^2 + \left(C_1 - \frac{1}{2}\sinh 2r\cos\vartheta\right)^2 = 1. \qquad (4.30)$$

Next, using (4.30) we try to find some particular solutions of (4.29).

First, notice that if $r' = \vartheta' = 0$, then (4.30) implies $C_1 = \pm 1 + \frac{1}{2}\sinh 2r_0\cos\vartheta_0$. Substituting this expression to (4.29), we get a contradiction. Next, we try to find a solution such that only one coordinate function r or ϑ is a constant.

If we assume $r' = 0$, we consider two possibilities $r = 0$ and $r = r_0 \neq 0$. In case $r = 0$, (4.29) implies $C_1 = 0$ which contradicts to (4.30). In case $r = r_0 \neq 0$, from (4.30) we get

$$\vartheta' = \frac{\sqrt{4 - \left(2C_1 - \sinh 2r_0\cos\vartheta\right)^2}}{\sinh 2r_0}. \qquad (4.31)$$

Equalizing the right hand sides of (4.29) we obtain

$$\cos\vartheta\,\vartheta'' - \sin\vartheta\,(\vartheta')^2 - \frac{2\sin\vartheta}{\cosh 2r_0}\,(C_1 - \sinh 2r_0\cos\vartheta)\,\vartheta' = 0. \tag{4.32}$$

Combining (4.31) and (4.32) we obtain $C_1 \neq$ const. which implies a contradiction.

If we assume $\vartheta' = 0$, then from (4.30)

$$r' = \sqrt{1 - \left(C_1 - \frac{1}{2}\sinh 2r\cos\vartheta_0\right)^2}. \tag{4.33}$$

Equalizing the right hand sides of (4.29) we obtain

$$\sin\vartheta_0\, r'' + 2\cos\vartheta_0\,\cosh 2r\,(C_1 - \sinh 2r\cos\vartheta_0)\,r' = 0. \tag{4.34}$$

Combining (4.33) and (4.34) we obtain $C_1 \neq$ const. which again leads to a contradiction.

Unfortunately, we couldn't solve the system in the general case, even though we tried different approaches. However, we proved that there are no solutions of (4.27) with at least one linear component function.

Hence, we proved the following proposition.

Proposition 4.1. The Killing magnetic curves in the hyperboloid model of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ geometry parameterized by arc length, corresponding to the Killing vector field $V = \sin\vartheta\,\partial_r + 2\coth 2r\cos\vartheta\,\partial_\vartheta + \tanh r\cos\vartheta\,\partial_\varphi$ are solutions of the system of differential equations (4.27). In particular, there is no Killing magnetic curve with at least one linear component function that corresponds to the Killing vector field V.

4.4. **Case D**

In this subsection we consider Killing magnetic curves which correspond to the Killing vector field

$$V_4 = \cos\vartheta\, e_1 - \cosh 2r\sin\vartheta\, e_2 - \sinh 2r\sin\vartheta\, e_3.$$

Using relation (4.1) and the formula (1.4) we have

$$\begin{aligned} V_4 \times \gamma' = {} & \sin\vartheta\left(\sinh^2 r\,\vartheta' - \cosh 2r\,\varphi'\right) e_1 + \\ & + \left(-\sin\vartheta\sinh 2r\, r' - \cos\vartheta(\sinh^2 r\,\vartheta' + \varphi')\right) e_2 \\ & + \left(\sin\vartheta\cosh 2r\, r' + \cos\vartheta\sinh r\cosh r\,\vartheta'\right) e_3. \end{aligned} \tag{4.35}$$

Remark 4.6. We could obtain (4.35) in another way. Let

$$dv_g = \sinh r\cosh r(dr \wedge d\vartheta \wedge d\varphi)$$

be the volume element of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. The Killing vector field

$$V_4 = \cos\vartheta\,\partial_r - 2\coth 2r\sin\vartheta\,\partial_\vartheta - \tanh r\sin\vartheta\,\partial_\varphi$$

by $F_V = i_V dv_g$, defines the magnetic field

$$F_{V_4}(X,Y) = -\sinh^2 r \sin\vartheta\,(dr\wedge d\vartheta)(X,Y) + \cosh 2r \sin\vartheta\,(dr\wedge d\varphi)(X,Y) + \sinh r\cosh r\cos\vartheta\,(d\vartheta\wedge d\varphi)(X,Y). \tag{4.36}$$

From (1.2) and (4.36) we get

$$\Phi_{V_4}(\partial_r) = -2\sin\vartheta\,\partial_\vartheta + \sin\vartheta(1+4\sinh^2 r)\,\partial_\varphi,$$
$$\Phi_{V_4}(\partial_\vartheta) = \sinh^2 r\sin\vartheta\,\partial_r - \tanh r\cos\vartheta\,\partial_\vartheta + \tanh r\cosh 2r\cos\vartheta\,\partial_\varphi,$$
$$\Phi_{V_4}(\partial_\varphi) = -\cosh 2r\sin\vartheta\,\partial_r - \frac{1}{\sinh r\cosh r}\cos\vartheta\,\partial_\vartheta + \tanh r\cos\vartheta\,\partial_\varphi.$$

Hence, from (2.4) the Lorentz force Φ_{V_4} acts on the basis vectors of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ as

$$\Phi_{V_4}(e_1) = -\sinh 2r\sin\vartheta\, e_2 + \cosh 2r\sin\vartheta\, e_3,$$
$$\Phi_{V_4}(e_2) = \sinh 2r\sin\vartheta\, e_1 + \cos\vartheta\, e_3$$
$$\Phi_{V_4}(e_3) = -\cosh 2r\sin\vartheta\, e_1 - \cos\vartheta\, e_2.$$

Finally we obtain the right hand side of the relation (4.35)

$$\begin{aligned}\Phi_{V_4}(\gamma') &= \Phi_{V_4}\Big(r'e_1 + \frac{1}{2}\sinh 2r\,\vartheta'\,e_2 + \big(\sinh^2 r\,\vartheta' + \varphi'\big)\,e_3\Big)\\ &= \big(-\cosh 2r\sin\vartheta\,\varphi' + \sinh^2 r\sin\vartheta\,\vartheta'\big)\,e_1 +\\ &\quad + \big(-\sinh 2r\sin\vartheta\, r' - \cos\vartheta\big(\sinh^2 r\,\vartheta' + \varphi'\big)\big)\,e_2 +\\ &\quad + \big(\cosh 2r\sin\vartheta\, r' + \sinh r\cosh r\cos\vartheta\,\vartheta'\big)\,e_3.\end{aligned}$$

Further, equalizing the right hand sides of the equations (4.2) and (4.35), we obtain the following system of differential equations

$$\begin{aligned}&r'' - \frac{1}{2}\sinh 2r\,\vartheta'\,\big(\big(1+4\sinh^2 r\big)\,\vartheta' + 2\varphi'\big) = -\cosh 2r\sin\vartheta\,\varphi' + \sinh^2 r\sin\vartheta\,\vartheta',\\ &\frac{1}{2}\sinh 2r\,\vartheta'' + 2\,\big(1+3\sinh^2 r\big)\,r'\vartheta' + 2r'\varphi' = -\sinh 2r\sin\vartheta\, r' - \cos\vartheta\,\big(\sinh^2 r\,\vartheta' + \varphi'\big),\\ &\varphi'' + \sinh^2 r\,\vartheta'' + \sinh 2r\, r'\,\vartheta' = \cosh 2r\sin\vartheta\, r' + \sinh r\cosh r\cos\vartheta\,\vartheta'.\end{aligned} \tag{4.37}$$

Next, we try to solve the system (4.37). The third equation can be rewritten as

$$\frac{d}{ds}\big(\varphi' + \sinh^2 r\,\vartheta'\big) = \frac{d}{ds}\Big(\frac{1}{2}\sinh 2r\sin\vartheta\Big).$$

Hence, $\varphi' + \sinh^2 r\,\vartheta' = C_2 + \frac{1}{2}\sinh 2r\sin\vartheta$, $C_2\in\mathbb{R}$ and

$$\varphi'(s) = C_2 - \sinh^2 r(s)\,\vartheta'(s) + \frac{1}{2}\sinh 2r(s)\sin\vartheta(s). \tag{4.38}$$

Substituting (4.38) in the first two equations of (4.37) and in (4.6), we have respectively

$$\begin{aligned}&4r'' - \sinh 4r\,\big(\vartheta'\big)^2 - 4\sinh 2r\,(\sinh 2r\sin\vartheta + C_2)\,\vartheta' = -2\cosh 2r\sin\vartheta\,(2C_2 + \sinh 2r\sin\vartheta),\\ &\sinh 2r\,\vartheta'' + 4\cosh 2r\, r'\vartheta' + 4r'\,(C_2 + \sinh 2r\sin\vartheta) = -\cos\vartheta\,(2C_2 + \sinh 2r\sin\vartheta),\end{aligned} \tag{4.39}$$

$$(r')^2 + \Big(\frac{1}{2}\sinh 2r\,\vartheta'\Big)^2 + \Big(C_2 + \frac{1}{2}\sinh 2r\sin\vartheta\Big)^2 = 1. \tag{4.40}$$

Next, using (4.40) we try to find some particular solutions of (4.39).

Analogously to Case C, notice that if $r' = \vartheta' = 0$, then (4.40) implies $C_2 = \pm 1 + \frac{1}{2}\sinh 2r_0 \cos\vartheta_0$. Substituting this expression to (4.39), we get a contradiction. Next, we try to find a solution such that only one coordinate function r or ϑ is a constant.

If we assume $r' = 0$, we consider two possibilities $r = 0$ and $r = r_0 \neq 0$. In case $r = 0$, (4.39) implies $C_2 = 0$ which contradicts to (4.40). In case $r = r_0 \neq 0$, from (4.40) we get

$$\vartheta' = \frac{\sqrt{4 - (2C_2 + \sinh 2r_0 \sin\vartheta)^2}}{\sinh 2r_0}. \tag{4.41}$$

Equalizing the right hand sides of (4.39) we obtain

$$\sin\vartheta\,\vartheta'' + \cos\vartheta\,(\vartheta')^2 + \frac{2\cos\vartheta}{\cosh 2r_0}\,(C_2 + \sinh 2r_0 \sin\vartheta)\,\vartheta' = 0. \tag{4.42}$$

Combining (4.41) and (4.42) we obtain $C_2 \neq$ const. which implies a contradiction.

If we assume $\vartheta' = 0$, then from (4.40)

$$r' = \sqrt{1 - \left(C_2 + \frac{1}{2}\sinh 2r \sin\vartheta_0\right)^2}. \tag{4.43}$$

Equalizing the right hand sides of (4.39) we obtain

$$\cos\vartheta_0\, r'' - 2\sin\vartheta_0 \cosh 2r\,(C_2 + \sinh 2r \sin\vartheta_0)\,r' = 0. \tag{4.44}$$

Combining (4.43) and (4.44) we obtain $C_2 \neq$ const. which again leads to a contradiction.

Unfortunately, we couldn't solve the system in the general case. However, we proved the following proposition.

Proposition 4.2. The Killing magnetic curves in the hyperboloid model of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ geometry parameterized by arc length, corresponding to the Killing vector field $V = \cos\vartheta\,\partial_r - 2\coth 2r \sin\vartheta\,\partial_\vartheta - \tanh r \sin\vartheta\,\partial_\varphi$ are solutions of the system of differential equations (4.37). In particular, there is no Killing magnetic curve with at least one linear component function that corresponds to the Killing vector field V.

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